EECS 16B
Designing Information Devices and Systems II

Prof. Ali Niknejad and Prof. Kannan Ramchandran
Department of Electrical Engineering and Computer Sciences, UC Berkeley,
niknejad@berkeley.edu

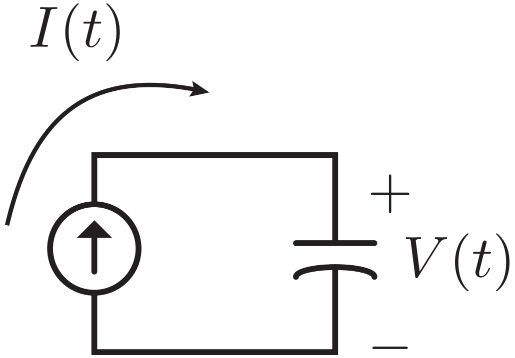
Module 2: RC Circuits, Digital Logic Gates, and Transfer Functions

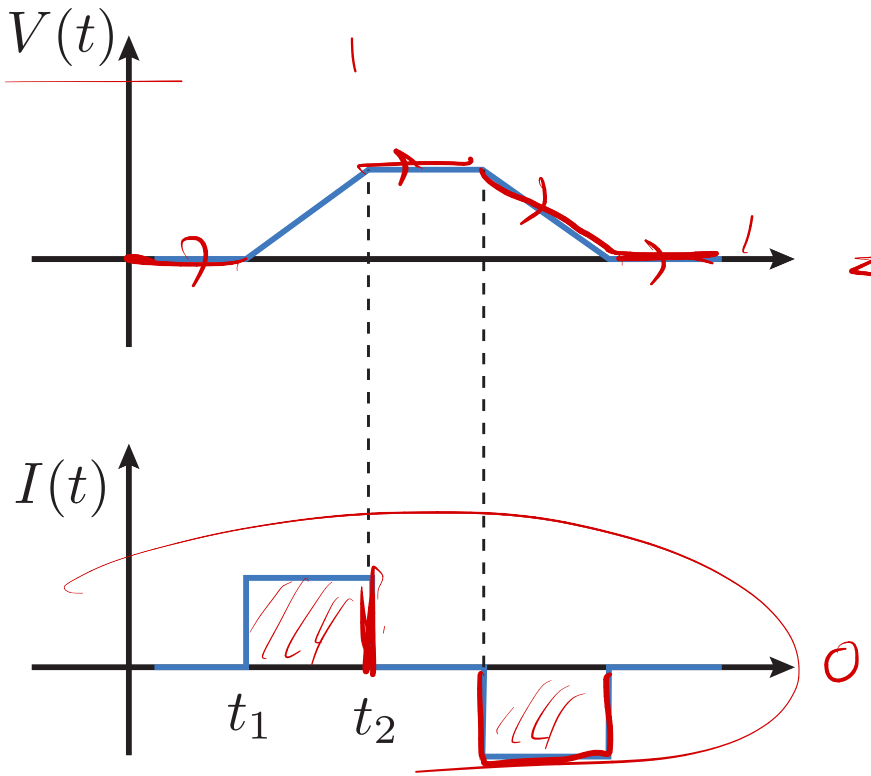
EECS 16B

RC Circuits

- Solve RC with and without inputs
  - "Natural" response and "forced" response
  - "Moving Average" (Convolution Integral)
- Step Response
- Pulse Response
- Preview: Sinusoidal Steady-State Response
- Low Pass vs High Pass
- Introduction to Transistors

Capacitor Discharge: Linear





$K = 0$

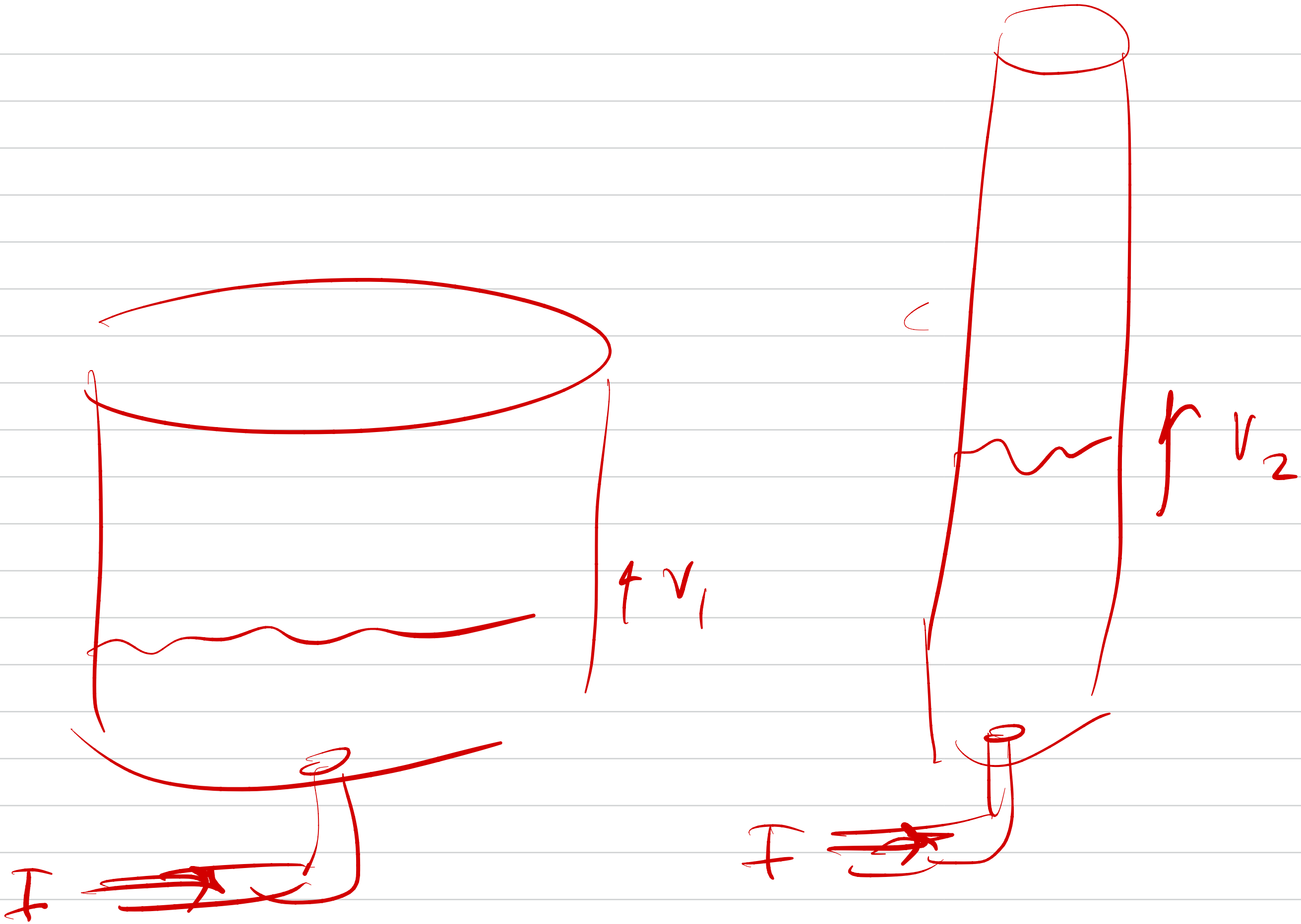
$V(0) = 0$

$$I(t) = C\,\frac{dV}{dt}$$

$$\int \frac{1}{C} I(t)\,dt = \int \frac{dV}{dt}\,dt$$

$$= V(t) + K$$

$$V(t) = K + \frac{1}{C}\int I(t)\,dt$$



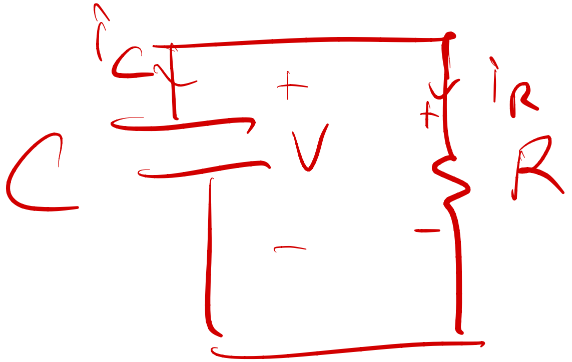
I
V_1
I
V_2

Capacitor Discharge

- Water analogy: Rate of outflow is not constant but depends on the height of the water in the tank (the pressure)

RC Circuit

- Can’t solve by “inspection”, need to solve differential equation



$v(t) = ?$

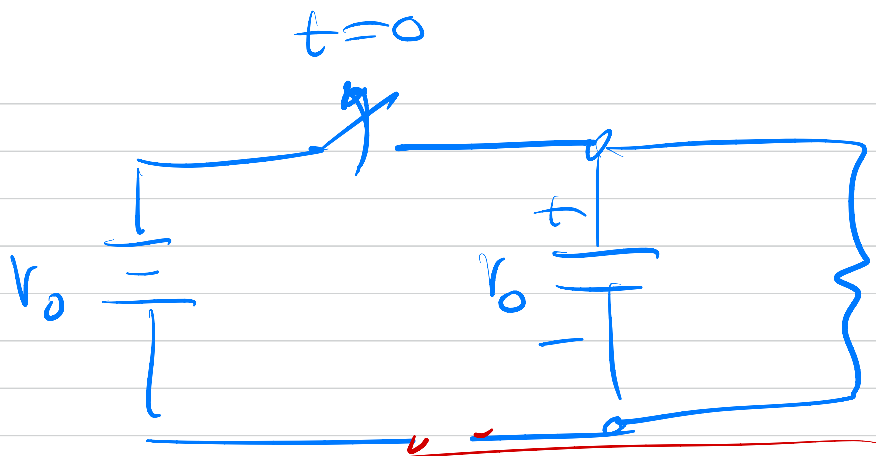
$$i = C\frac{dv}{dt} = -\left(\frac{v}{R}\right)$$

$$\frac{v}{R} + C\frac{dv}{dt} = 0$$

$$\boxed{v + RC\frac{dv}{dt} = 0}$$

Initial Condition

$$v(0) = V_0$$



$$\boxed{v + RC\frac{dv}{dt} = 0}$$

$$1 + RC\frac{1}{\tau} = 0$$

$$\tau = -RC$$

$$v(t) = K e^{t/\tau}$$

$$\cancel{K}\,\cancel{e^{t/\tau}} + RC\frac{1}{\tau}\cancel{K}\,\cancel{e^{t/\tau}} = 0$$

RC Solution

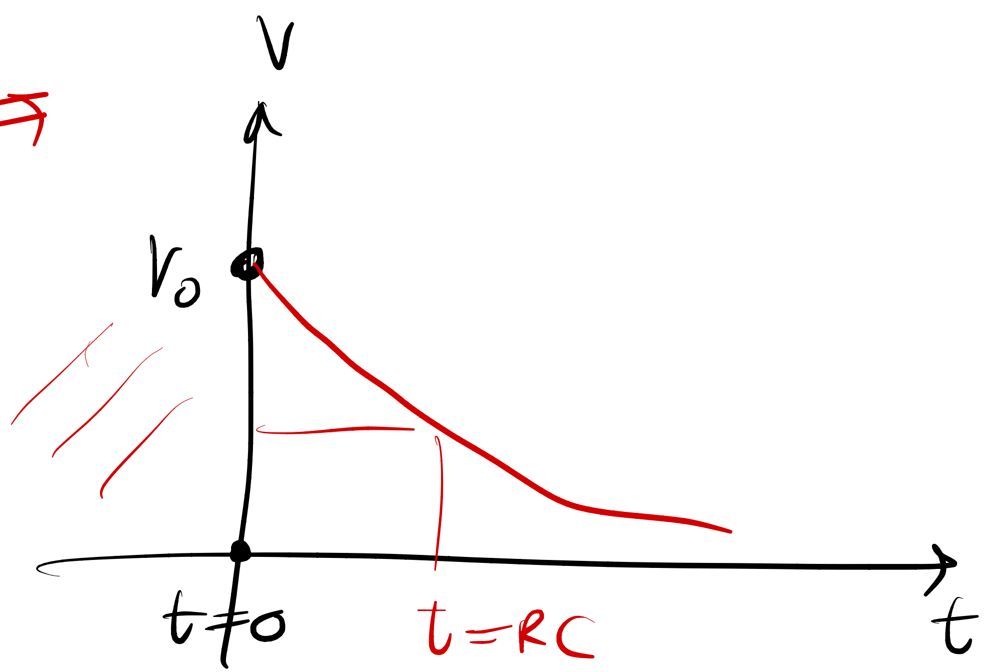
- Easy to guess for first-order equation

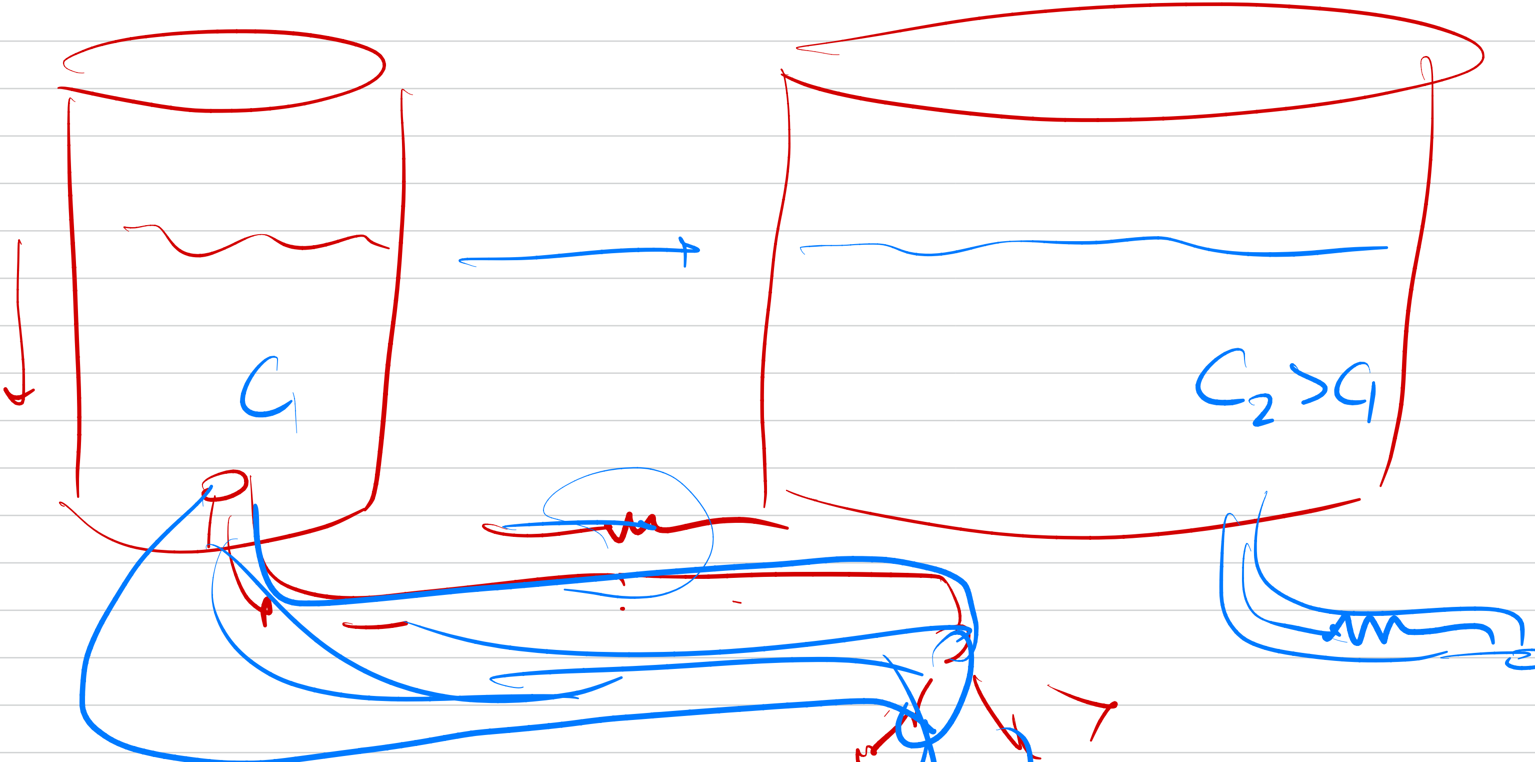
$$v(t) = K e^{-t/\tau'}$$

$$\tau' = RC$$

$$v(0) = V_0 = K \Rightarrow$$

$$\boxed{v(t) = V_0 e^{-t/\tau'}}$$

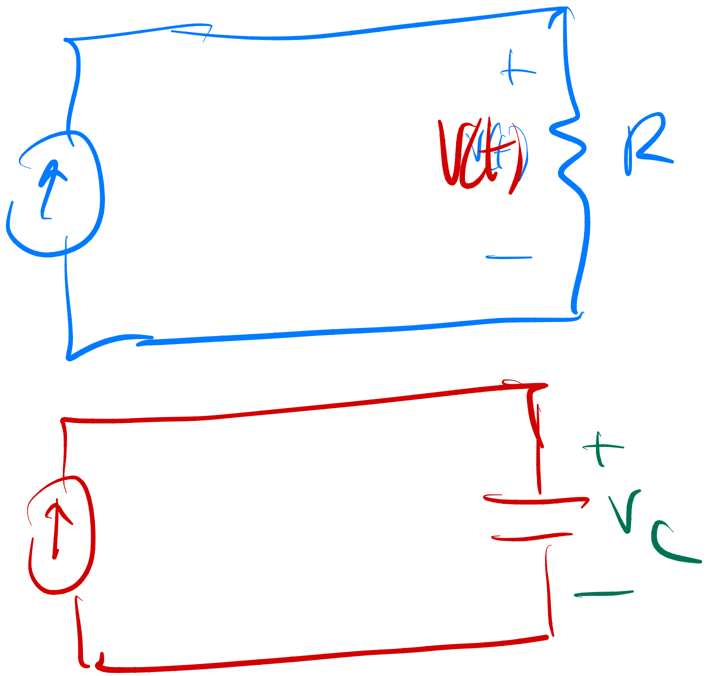


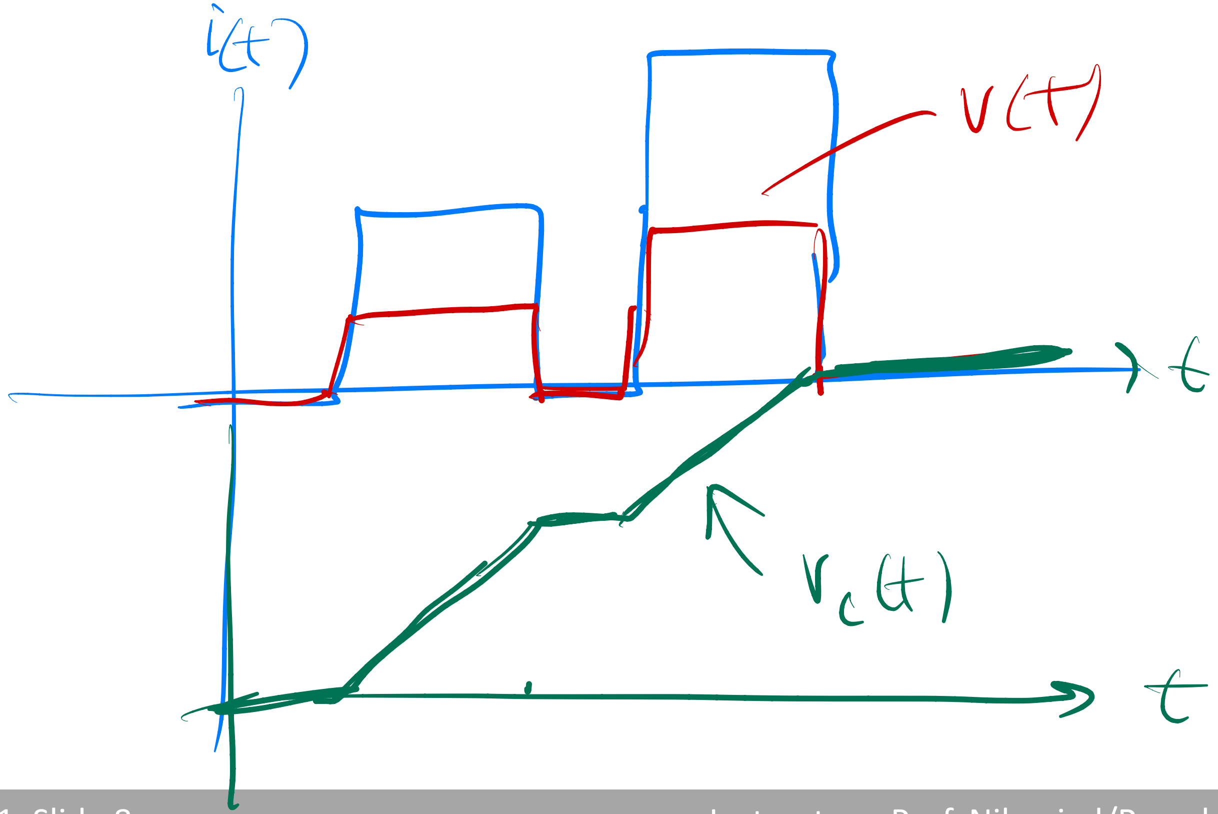


$$\tau = RC$$

Capacitor Has "Memory"

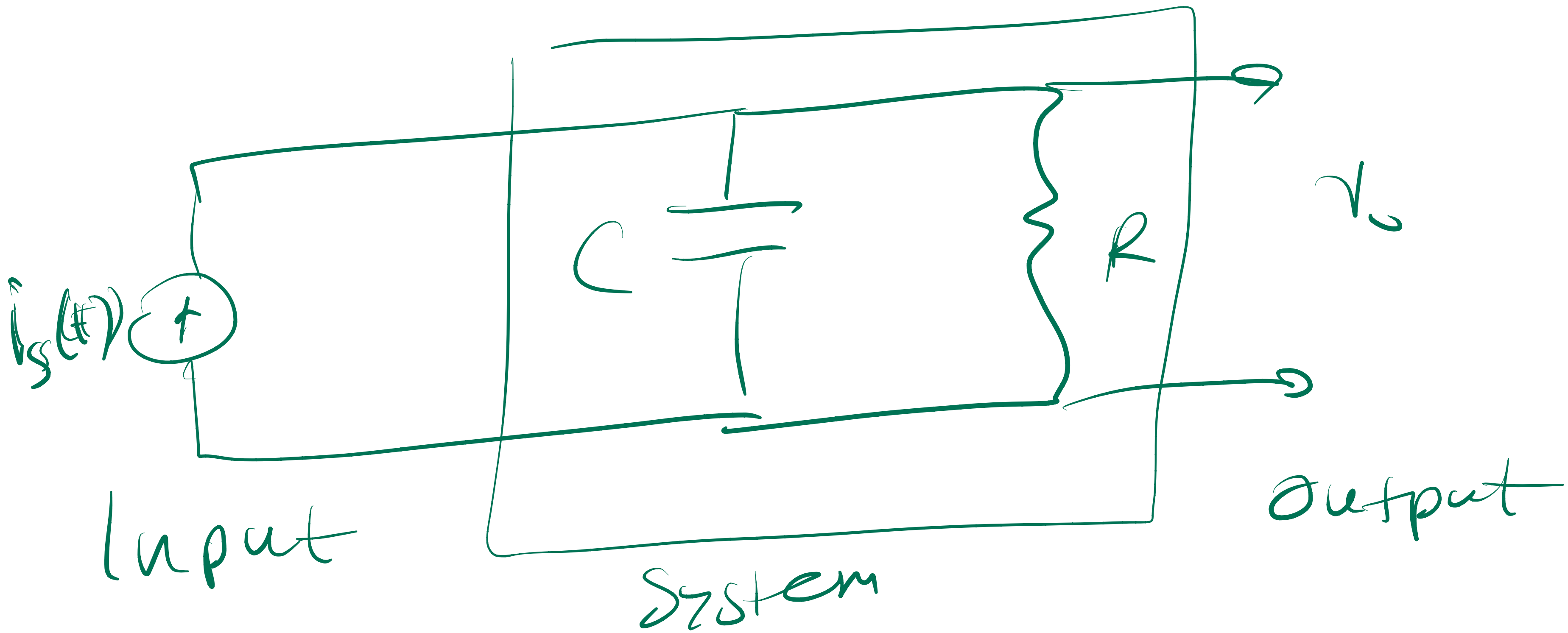
- A resistive circuit without capacitors has to memory, the transfer function is instantaneous and if the input goes to zero, the output immediately goes to zero.





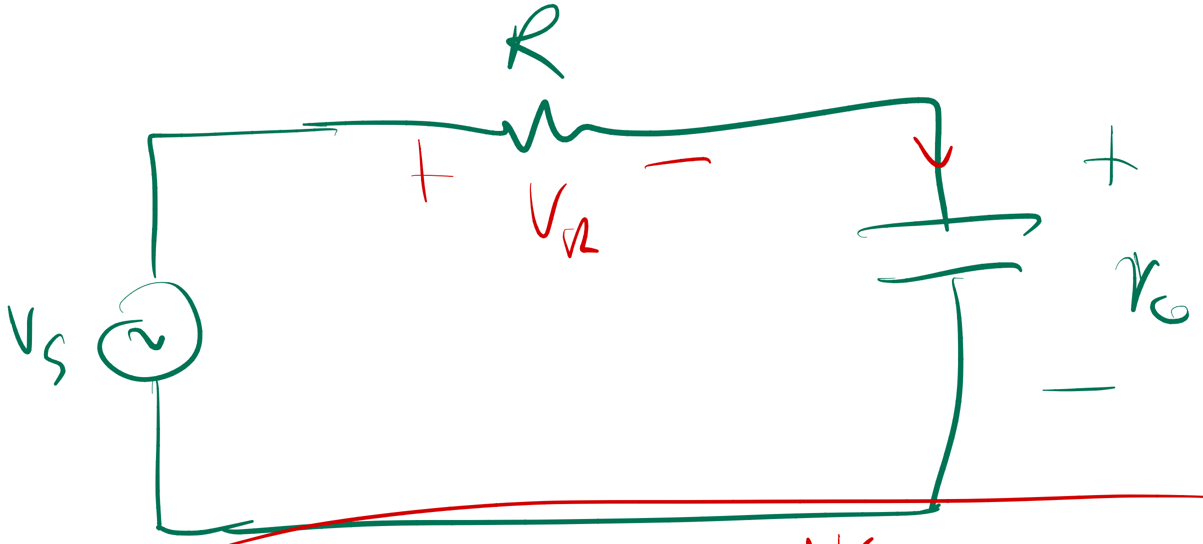
Circuit as a System

- Consider inputs (independent voltage/current sources) and outputs (voltages or currents at any node or in any branch of the circuit). Dependent sources are not inputs.



Linear Constant Coefficient Differential Eq.

- Homogenous solution: No inputs. Sometimes called the "natural" solution
- Forced response or a *particular* solution



$$V_s = V_R + V_o$$

$$i_c = C\frac{dV_o}{dt}$$

$$V_R = RC\frac{dV_o}{dt}$$

$$\boxed{V_s = RC\frac{dV_o}{dt} + V_o}$$

Superposition

- Equation is linear, so a superposition of solutions is also a solution.

$$0 = \tau \frac{dV_o}{dt} + V_o$$

$$0 = \tau \frac{d\psi_1}{dt} + \psi_1 \qquad \psi_1(0) = c_1$$

$$0 = \tau \frac{d\psi_2}{dt} + \psi_2 \qquad \psi_2(0) = c_2$$

$$\psi = \alpha_1 \psi_1 + \alpha_2 \psi_2$$

$$0 = \tau \frac{d(\alpha_1 \psi_1 + \alpha_2 \psi_2)}{dt} + \alpha_1 \psi_1 + \alpha_2 \psi_1$$

$$= \underbrace{\left(\tau \frac{d\psi_1}{dt} + \psi_1\right)}_{0} \alpha_1 + \underbrace{\left(\tau \frac{d\psi_2}{dt} + \psi_2\right)}_{0} \alpha_2$$

Uniqueness

- Note that for a first-order differential equation, there can be at most one unique solution for the homogenous case

$$\psi_{diff} = \psi_1 - \psi_2$$

$$\psi_1(0) = C$$
$$\psi_2(0) = C$$

$$\tau \frac{d\psi_{diff}}{dt} + \psi_{diff} = 0 \qquad \psi_{diff}(0) = 0$$

$$\psi_{diff} \equiv 0 \Rightarrow \psi_1 = \psi_2$$

General Solution

$$v = v_0 + v_p$$

homogeneous solution

particular solution

$$v_s = \tau \frac{dv}{dt} + v$$

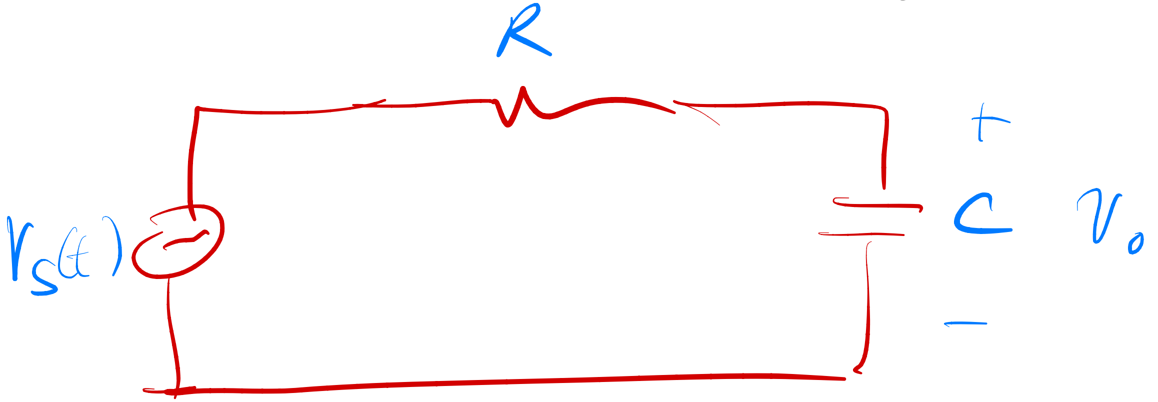
$$= \tau \frac{d}{dt}(v_0 + v_p) + (v_0 + v_p)$$

$$= \left(\tau \frac{dv_0}{dt} + v_0\right) + \left(\tau \frac{dv_p}{dt} + v_p\right)$$

$= 0$

RC Circuit with Inputs

- Now consider an arbitrary source connected to an RC circuit.



$$V_h = K e^{-t/\tau}$$

$$\underline{e^{st} v_s(t)} = \left(\tau \frac{dv}{dt} + v\right) e^{st}$$

$$(V e^{st})' = V' e^{st} + s V e^{st}$$

$$= \tau e^{st} \frac{dv}{dt} + v e^{st}$$

$$s = \frac{1}{\tau}$$

$$= \tau \left(e^{st} v' + \frac{1}{\tau} v e^{st}\right)$$

$$= \tau \left(e^{st} v\right)'$$

Solution with Inputs (Integrating Factor)

$$e^{st} v_s(t) = \tau \left(e^{st} v(t) \right)'$$

$$\int e^{st} v_s(t)\, dt = \tau \left(e^{st} \underline{v(t)} + K \right)$$

$$v(t) = \underbrace{\frac{1}{\tau} e^{-st}}_{s = 1/\tau} \int_{-\infty}^{t} e^{sx} v_s(x)\, dx \quad + \underbrace{K e^{-st}}_{\text{Homogeneous Solution}}$$

Complete Solution

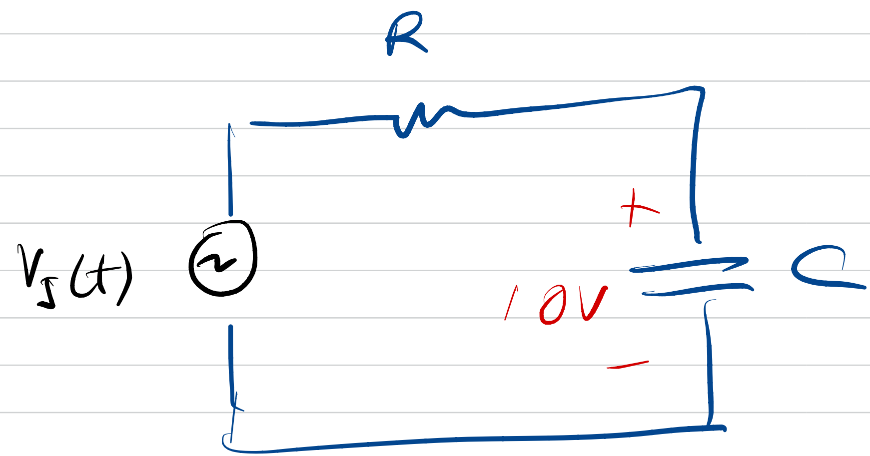
$$v(t) = \underbrace{\frac{1}{\tau} e^{-st} \int_{-\infty}^{t} e^{sx} v_s(x)\, dx}_{} + k e^{-st}$$

$$v_p = \frac{1}{\tau} e^{-st} \int_0^t e^{sx} v_s \, dx$$

$$= \frac{1}{\tau} e^{-st} v_s \int_0^t e^{sx} dx$$

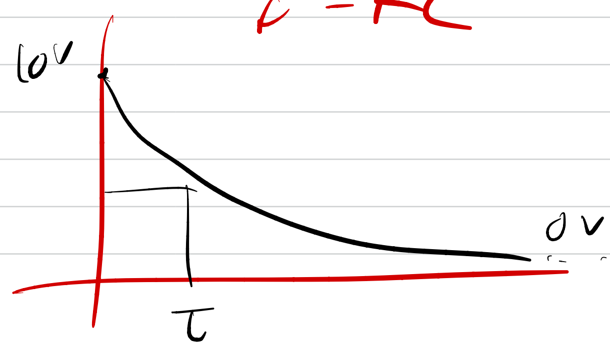
$$= \cancel{\frac{1}{\tau}} \cancel{e^{-st}} v_s \cancel{\frac{1}{s}} \cancel{e^{st}} \Big|_0^t = v_{\cancel{s}}$$

Homogeneous Solution

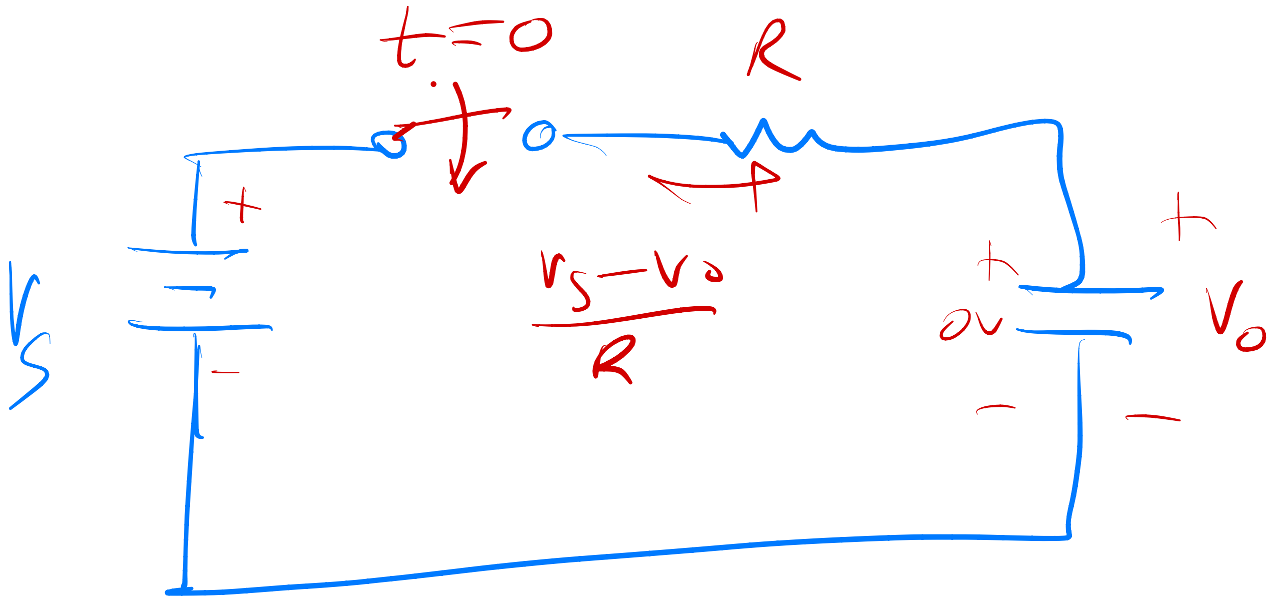


$\tau = RC$

$$v_c(t) = v(0)\, e^{-t/\tau}$$



RC Circuits with DC Inputs



$$V_o(0) = 0 = V_s + K$$

$$K = -V_s$$

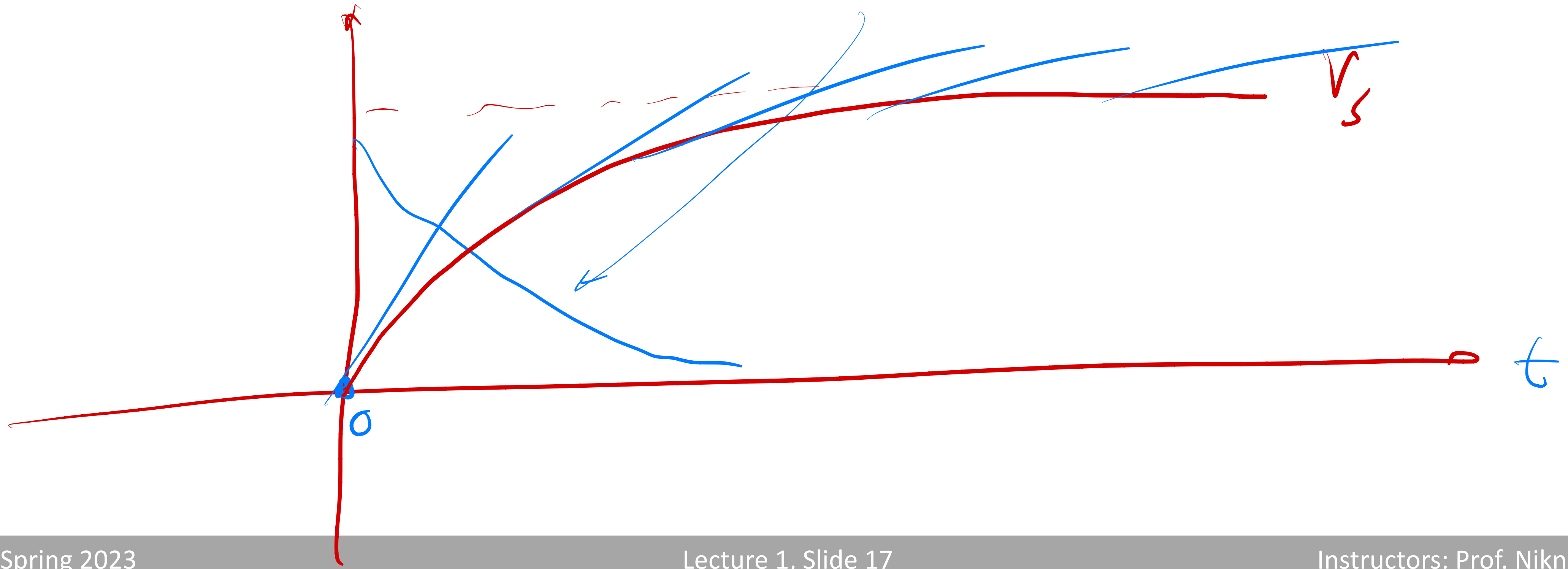
$$V_o(0) = 0V$$

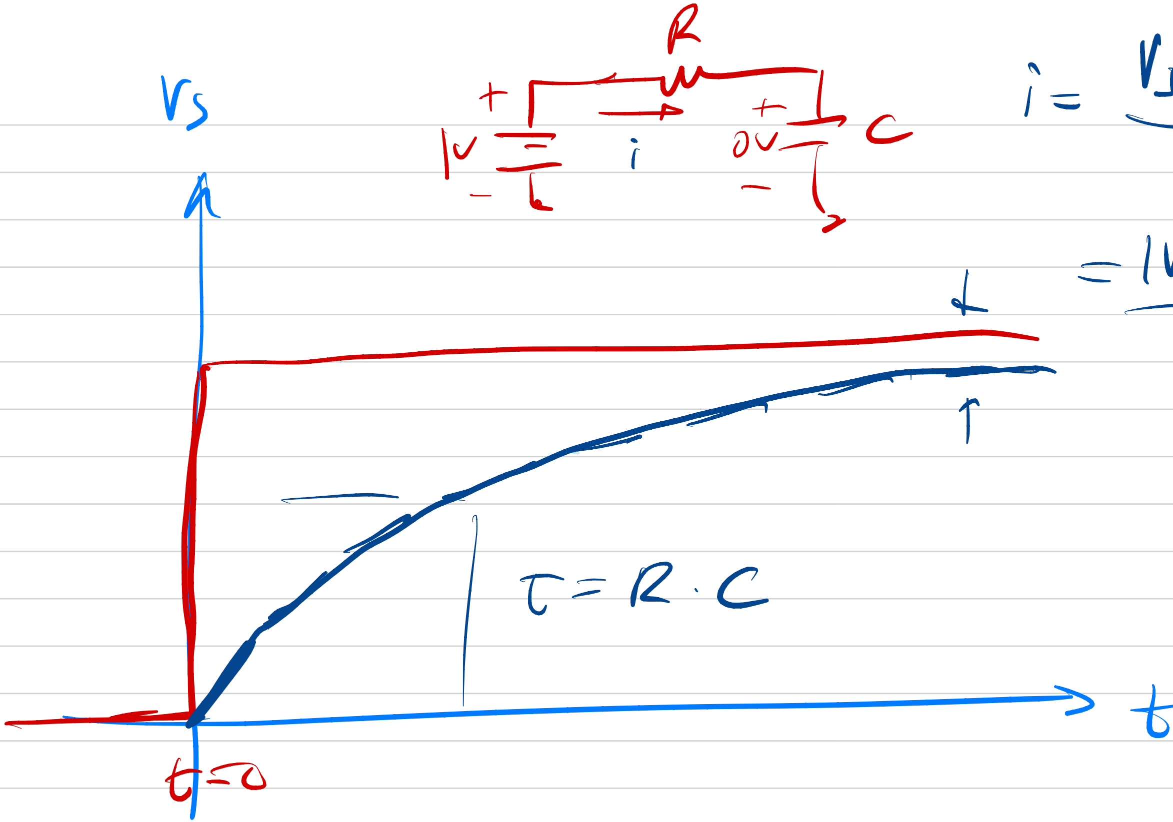
$$V_o(t) = \underbrace{V_s}_{\text{particular solution}} + Ke^{-t/\tau} = 0$$

Switching Circuits

$$v(t) = V_s + K e^{-t/\tau}$$

$$= V_s \left(1 - \underbrace{e^{-t/\tau}}\right)$$





$$i = \frac{V_s - V_c}{R}$$

$$= \frac{1V - V_c(t)}{R}$$

C $q = CV$

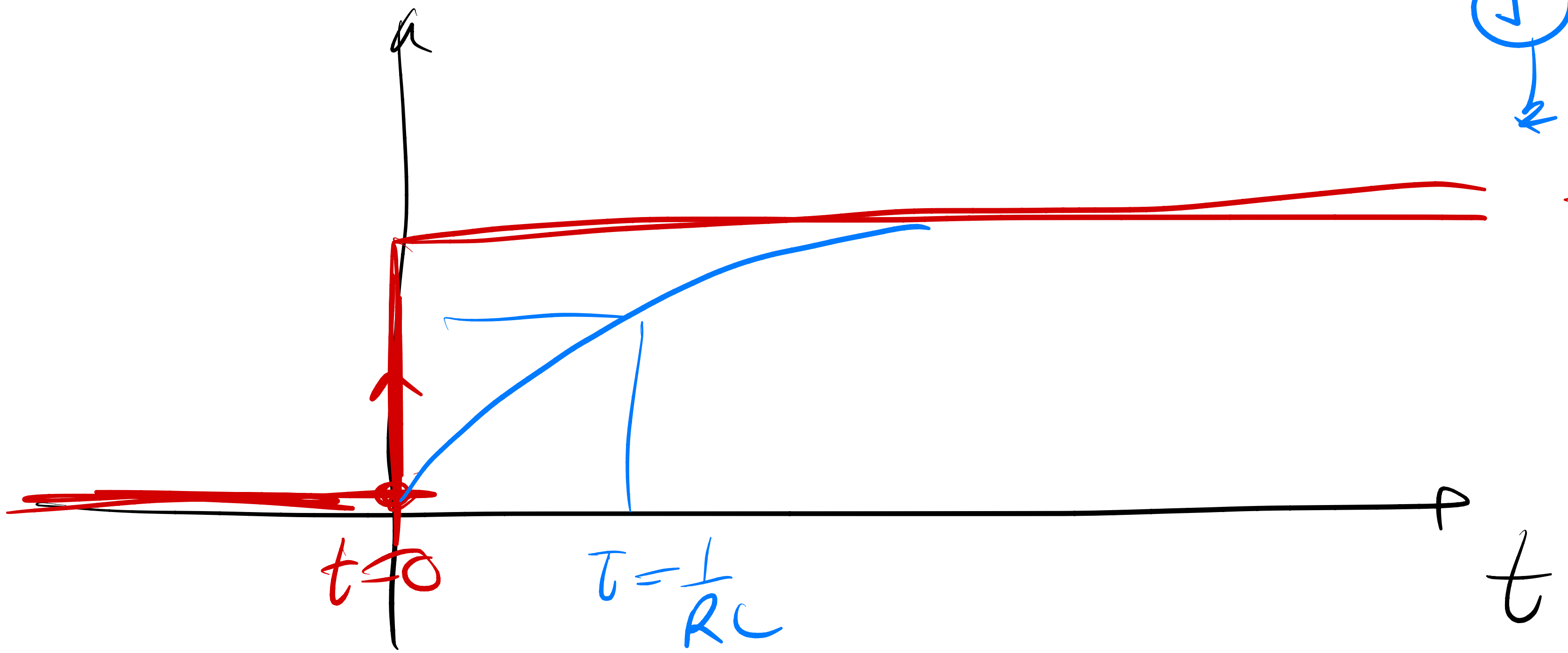
$$i = C\frac{dv}{dt}$$

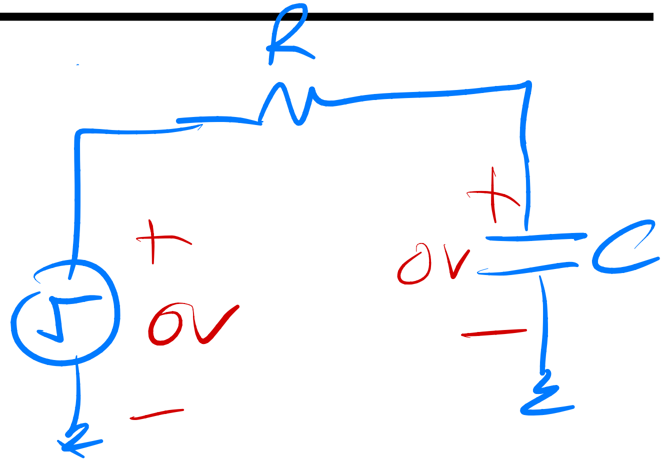
$$v(t) = \frac{1}{C}\int_0^t i(x)\,dx + v(0)$$

Step Response

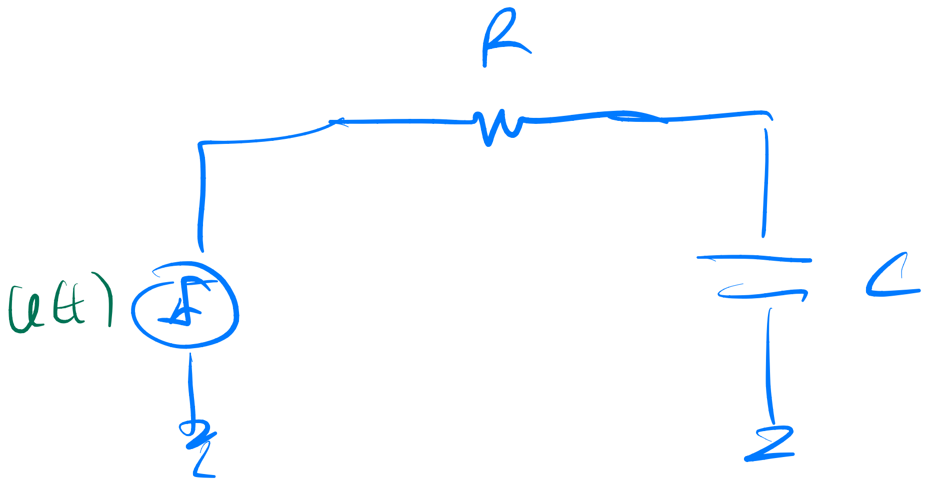
$V_s(t) = u(t)$ ← For Input

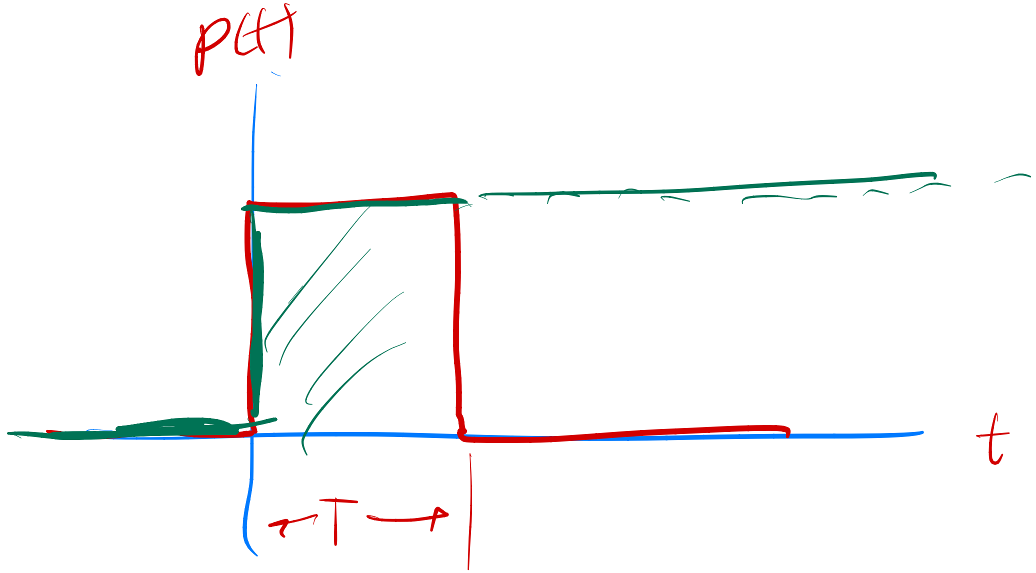
$u(t)$

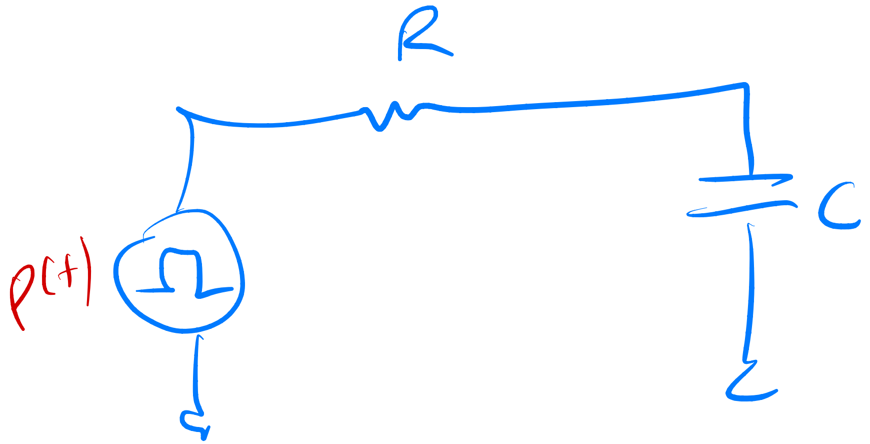




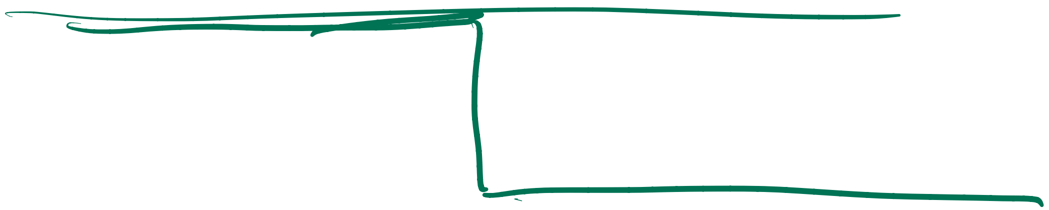
Source Superposition







$$p(t) = u(t) - u(t - T)$$



delayed step func

Pulse Response

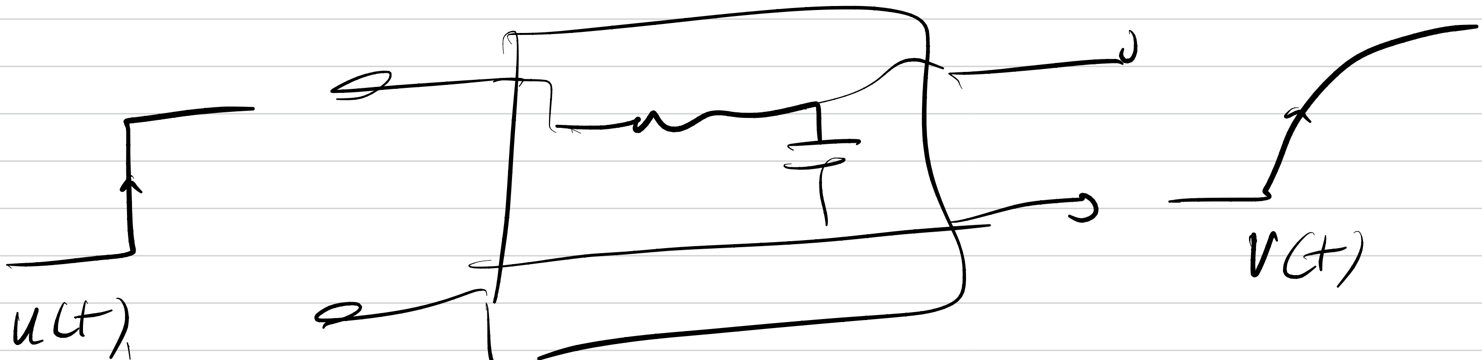
$$p(t) = \tau \frac{dv_c}{dt} + v_c \qquad \tau = RC$$

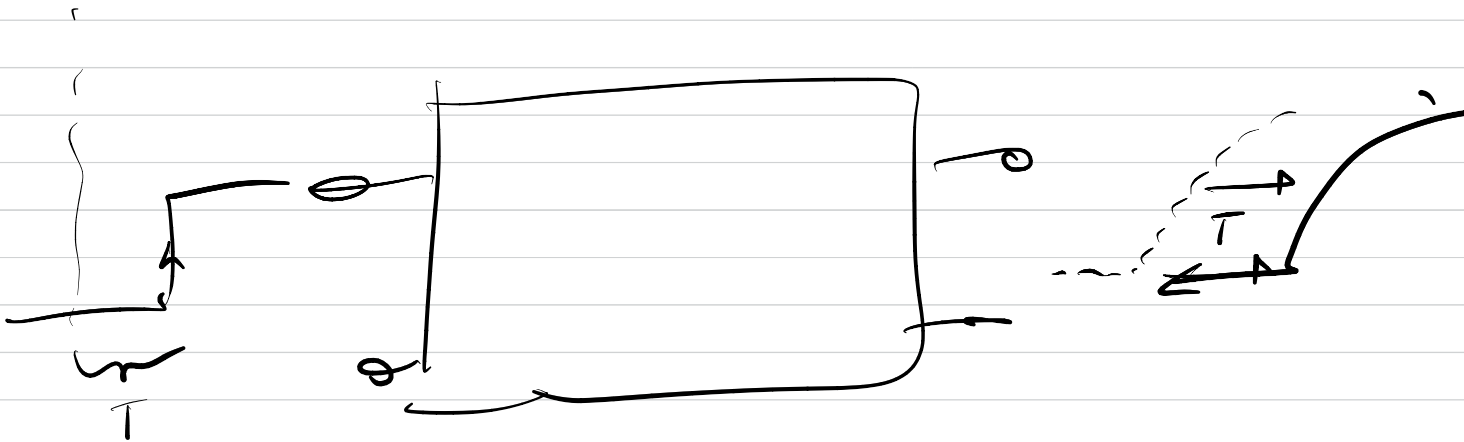
$$u(t) - u(t-T) = \tau \frac{dv_c}{dt} + v_c$$

$$= \tau \frac{d(\tilde{v}_c + \tilde{\tilde{v}}_c)}{dt} + (\tilde{v}_c + \tilde{\tilde{v}}_c)$$

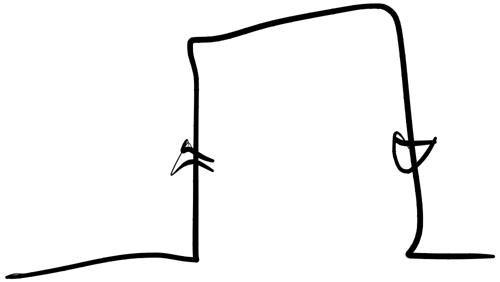
$$u(t) = \tau \left(\frac{d\tilde{v}_c}{dt} + \tilde{v}_c\right)$$ Input #1 → solve

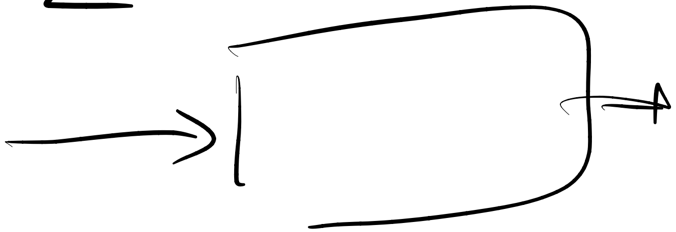
$$u(t-T) = \tau \left(\frac{d\tilde{\tilde{v}}_c}{dt} + \tilde{\tilde{v}}_c\right)$$ Input #2 → solve

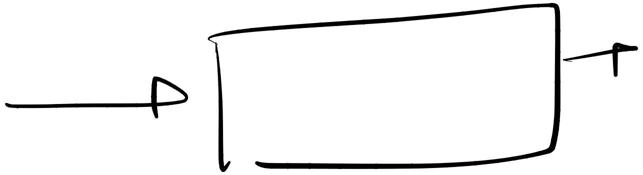

u(t)
V(t)


T
T

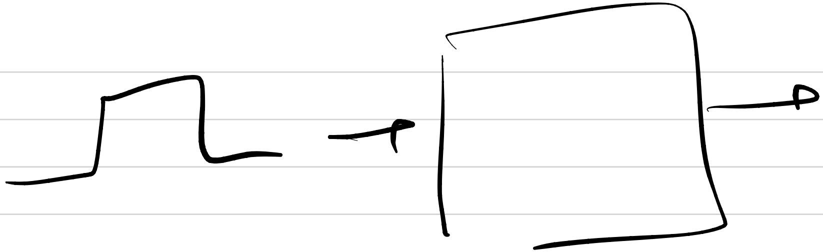
Smearing Out Pulses



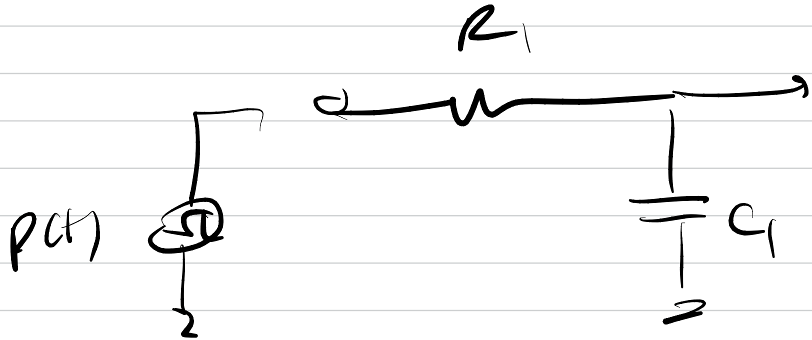
$u(t)$  $v(t) = \left(1 - e^{-t/\tau}\right) V_s \qquad t \geq 0\,s$

$-u(t-T)$  $v(t) = \begin{cases} \left(-1 + e^{-(t-T)/\tau}\right) V_s & t \geq T \\ 0 & t \leq T \end{cases}$

$p(t) = u(t) - u(t-T)$

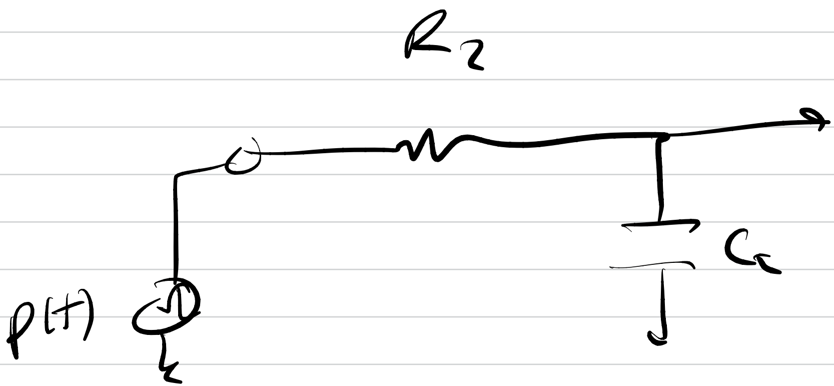



v(t)
1
τ
T
v(t − T)
−1



$T = 1\,ms$

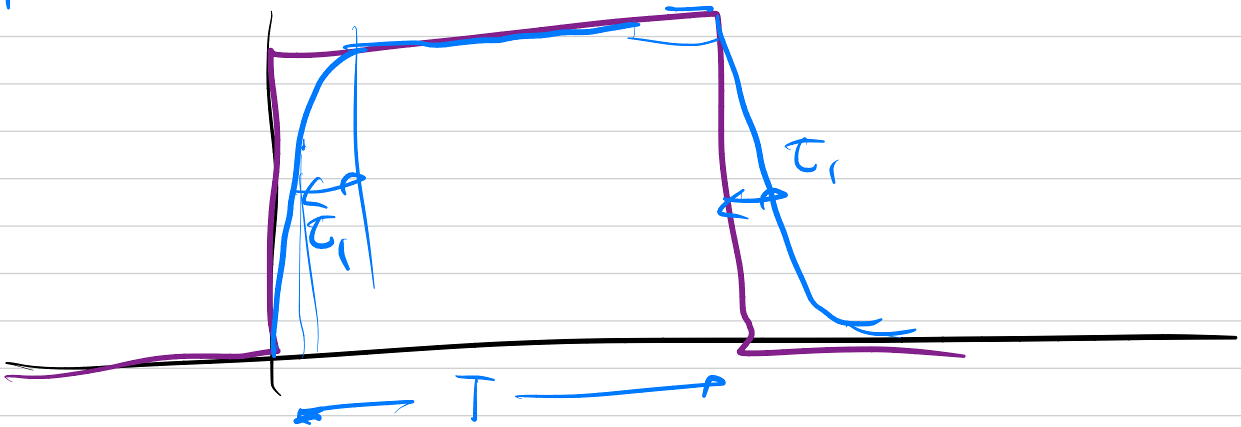
$\tau_1 = R_1 C_1 = 0.1\,ms$



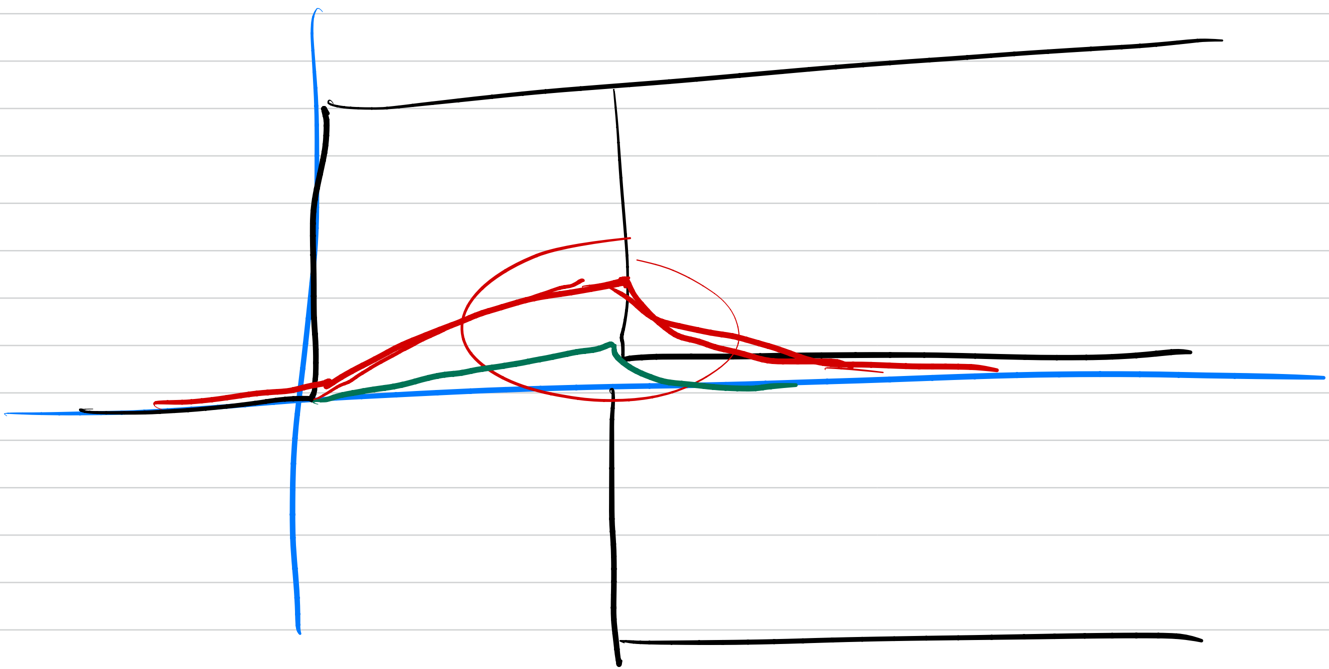
$\tau_2 = R_2 C_2 = 10\,ms$

$\tau_1 = R_1 C_1$

"FAST CIRCUIT"



$$\tau_2 = R_2 C_2 \gg T$$




Ω
?
R
C

Reformulation of General Solution

$$v(t) = \frac{1}{\tau} e^{-t/\tau} \int_{-\infty}^{t} e^{x/\tau} v_s(x)\, dx$$

input

$$= \frac{1}{\tau} \int_{-\infty}^{t} e^{-t/\tau} e^{x/\tau} v_s(x)\, dx$$

$$= \frac{1}{\tau} \int_{-\infty}^{t} e^{(x-t)/\tau} v_s(x)\, dx$$

weighted "~~Moving~~ Average" Interpretation

$$= \frac{1}{\tau} \int_{-\infty}^{t} e^{(x-t)/\tau} \, v_s(x) \, dx$$

$y = t - x$

$dy = -dx$

$$= \frac{1}{\tau} \int_{+\infty}^{0} e^{-y/\tau} \, v_s(t-y) \, (-dy)$$

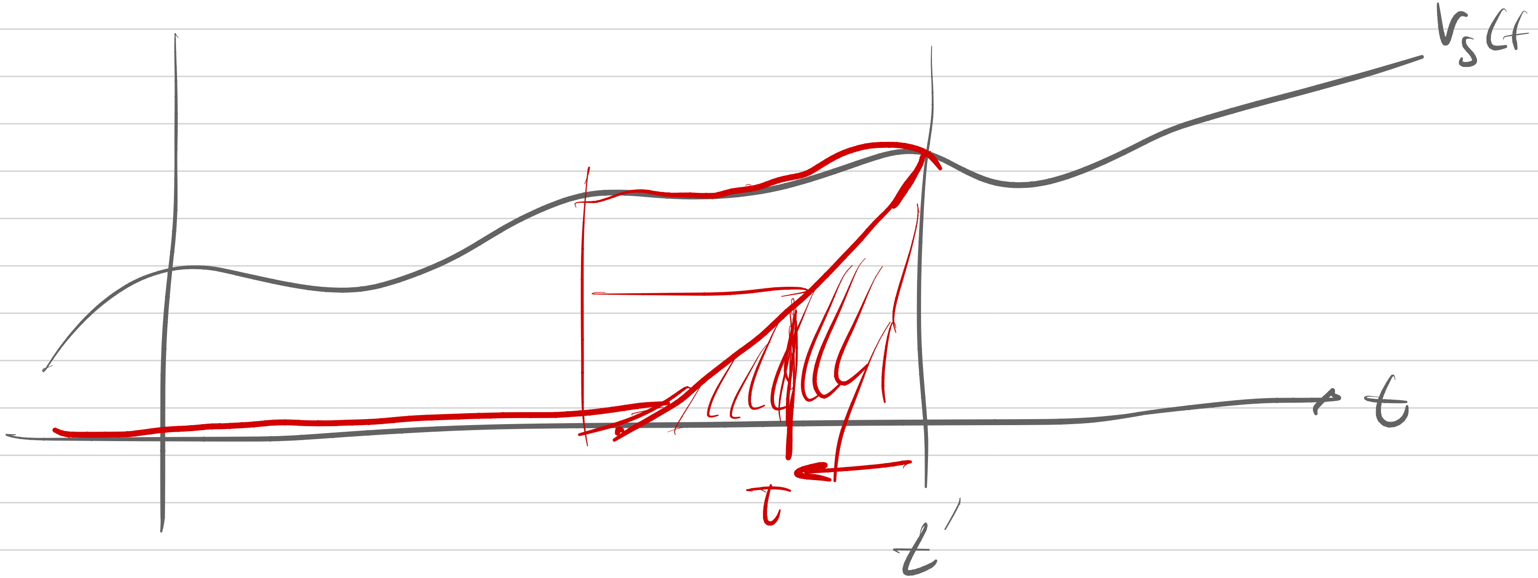
input source

past value

$$= \frac{1}{\tau} \int_{0}^{\infty} e^{-y/\tau} \, v_s(t-y) \, dy$$

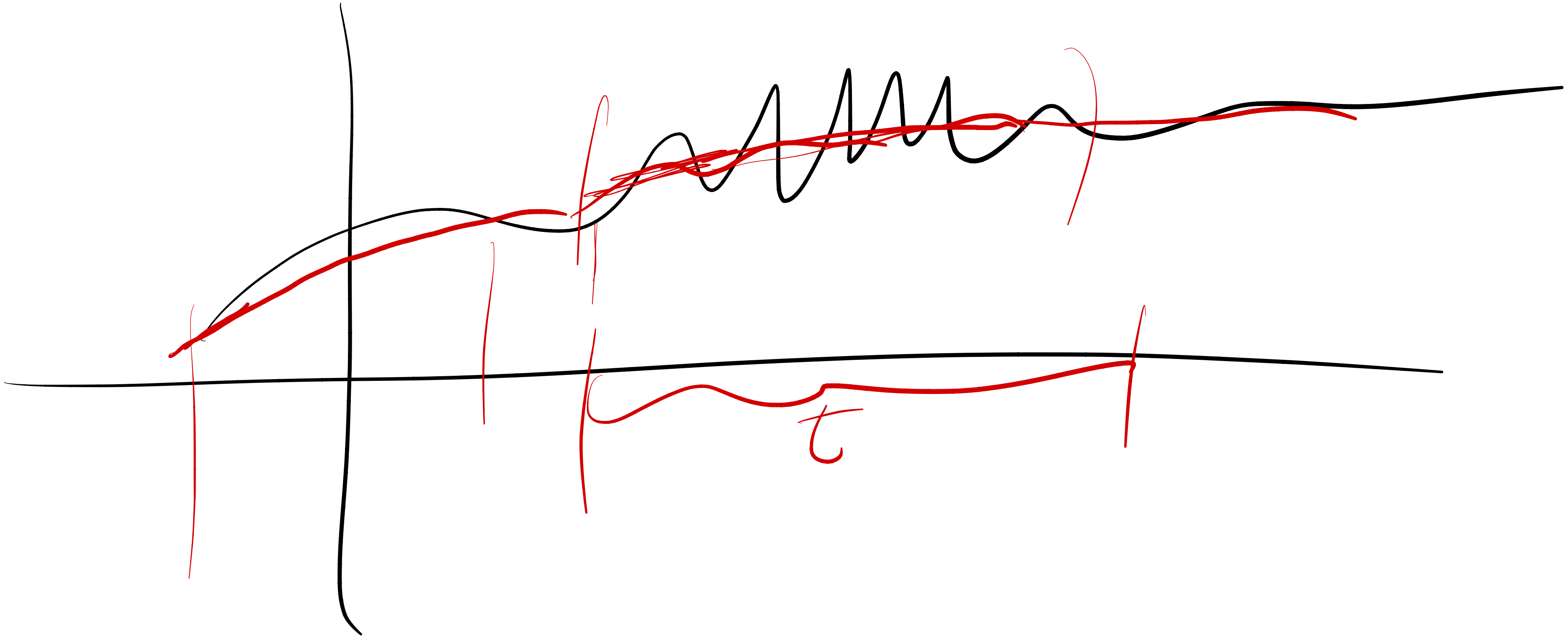
weight

$$= \frac{1}{\tau}\int_0^\infty e^{-y/\tau}\, v_s(t-y)\, dy$$



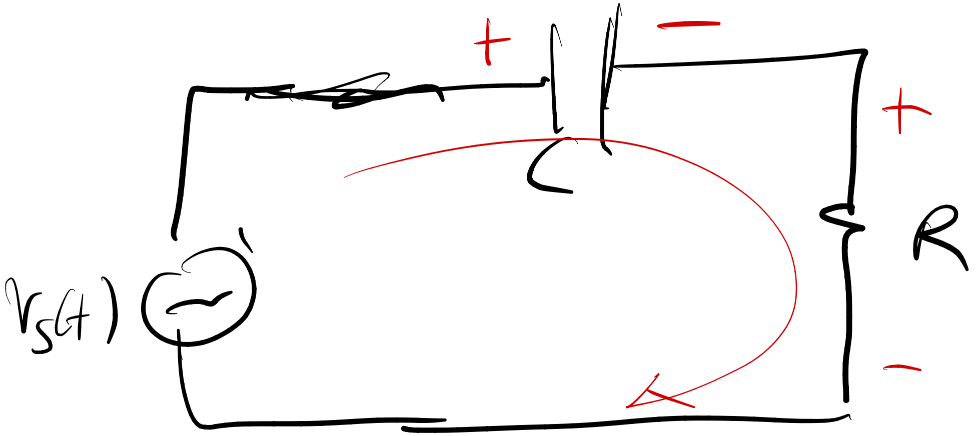
“Slow Pass” Circuit (Low Pass Filter)

- Suppose a function does not change much on the RC time constant scale



"Fast Pass" Circuit (High Pass)

- Now take the output across the resistor. The transfer function can be written as:

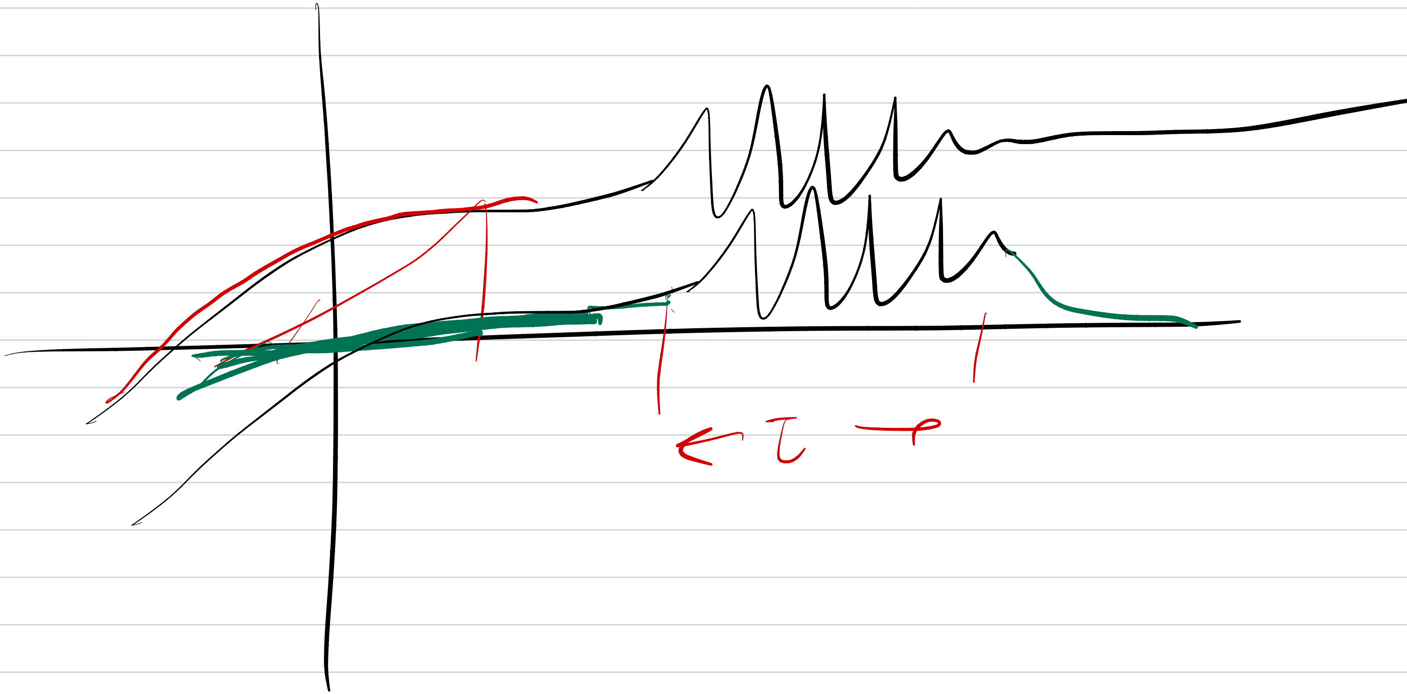


$$v_c(t) = \underbrace{\int_0^\infty e^{-y/\tau} v_s(t-y)\, dy}_{\text{particular solution}}$$

$$v_s = v_c + v_R \qquad v_R = v_s - v_c$$

$$v_R = v_s - v_c$$

$$= v_s - (\text{weighted average})$$


τ

Preview: Sinusoidal Steady State

- Suppose we inject a sinusoidal tone. It's much easier to work with complex exponentials and take the real / imag. part later.

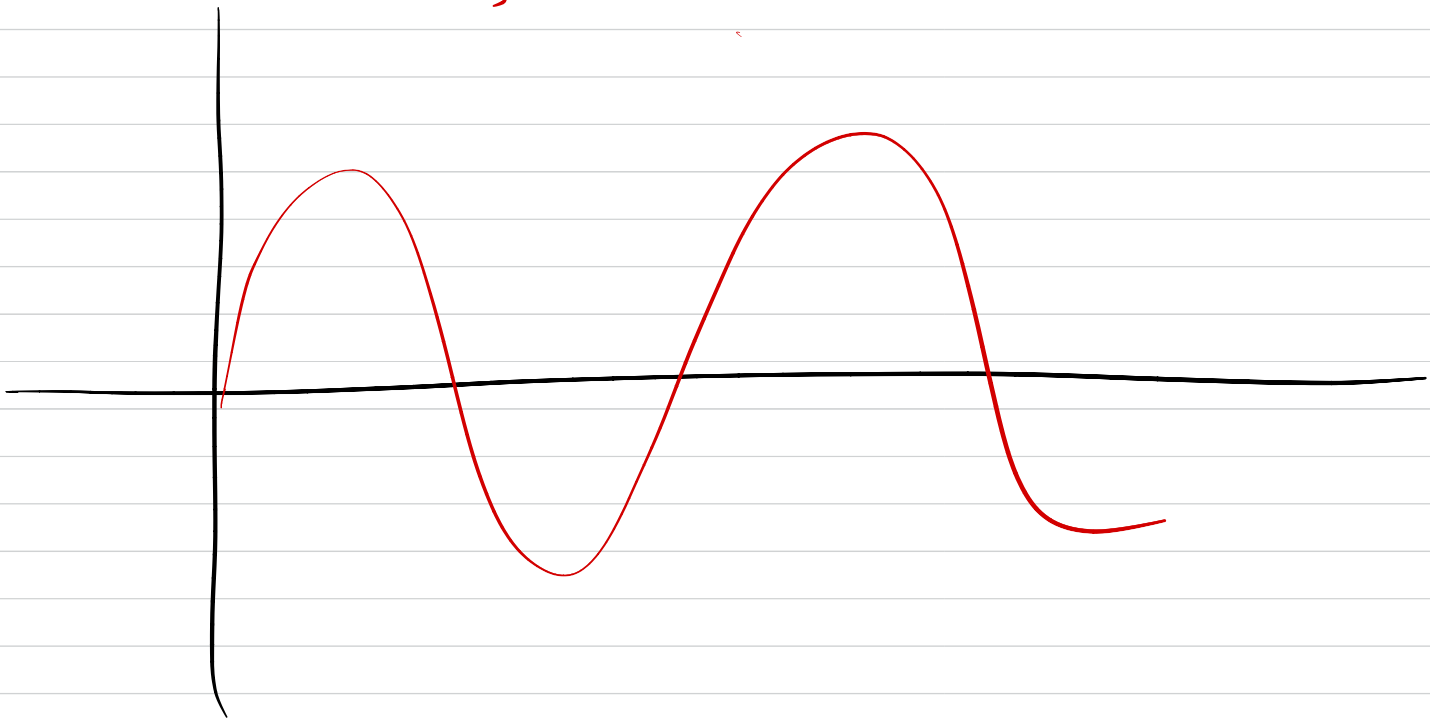
$$v_c(t) = \frac{1}{\tau}\int_0^\infty e^{-y/\tau} v_s(t-y)\, dy$$

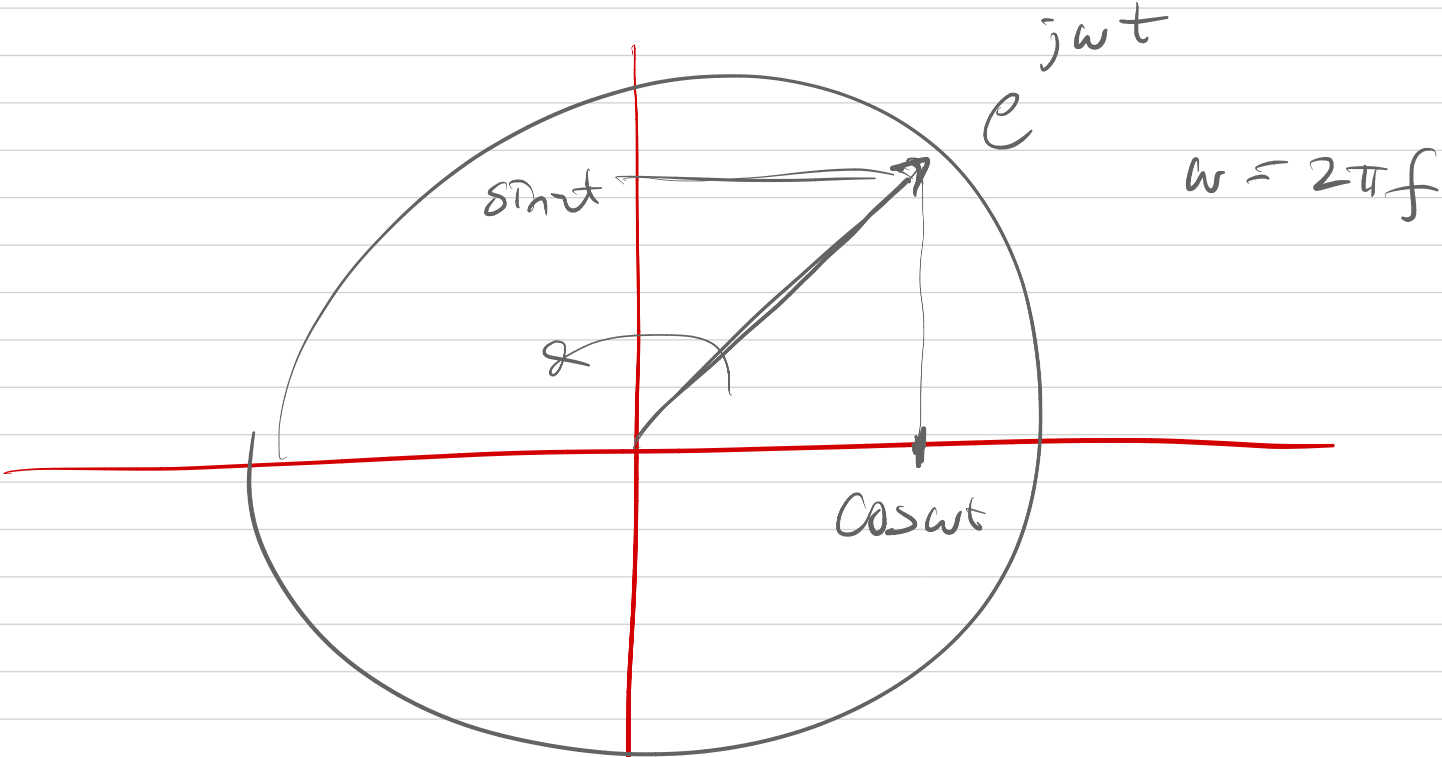
$$v_s(t) = e^{j\omega t}$$

$$= \int_0^\infty \frac{e^{-y/\tau}}{\tau} \times e^{j\omega(t-y)}\, dy = \int_0^\infty \frac{1}{\tau} e^{j\omega t - j\omega y - y/\tau}\, dy$$

$$= e^{j\omega t}\int_0^\infty \frac{1}{\tau} e^{-(j\omega + \frac{1}{\tau})y}\, dy$$

$$r_s(t) = A \sin \omega t$$





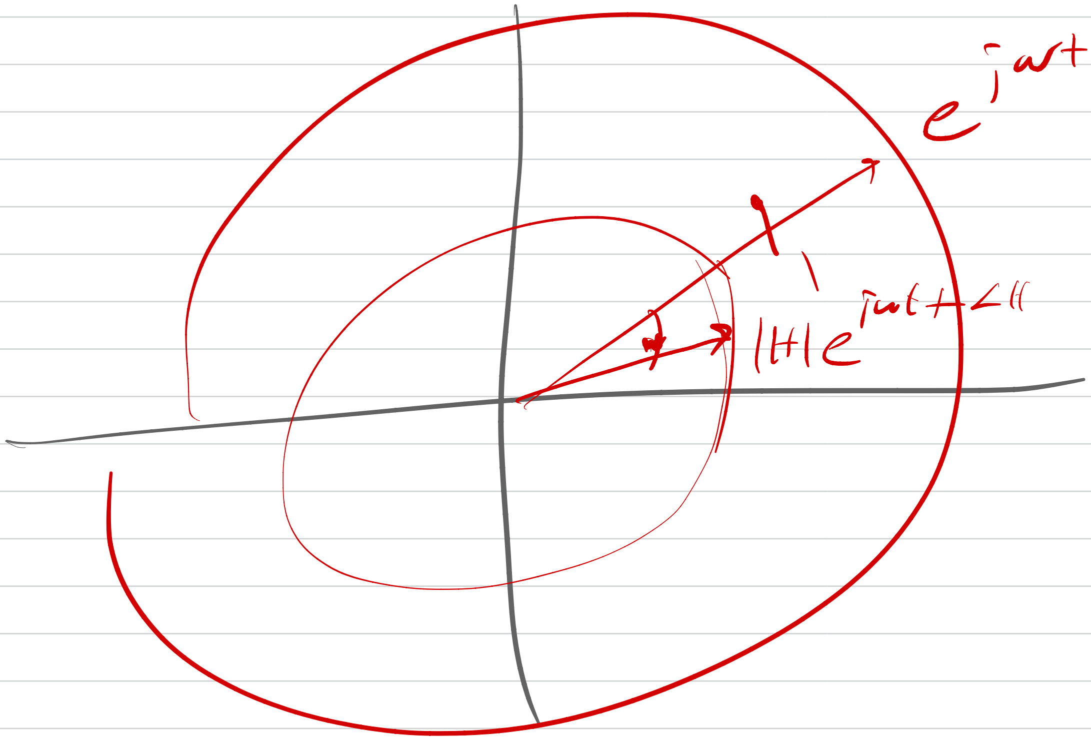
$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

Complex Exponential Solution

$$= e^{j\omega t} \int_0^\infty \frac{1}{\tau} e^{-(j\omega + \frac{1}{\tau})y} \, dy$$

$$= \frac{e^{j\omega t}}{\tau} \left(\frac{-e^{-(j\omega + \frac{1}{\tau})y}}{(j\omega + \frac{1}{\tau})} \right) \Bigg|_0^\infty = \frac{e^{j\omega t}}{\tau} \, \frac{+1}{j\omega + \frac{1}{1/\tau}}$$

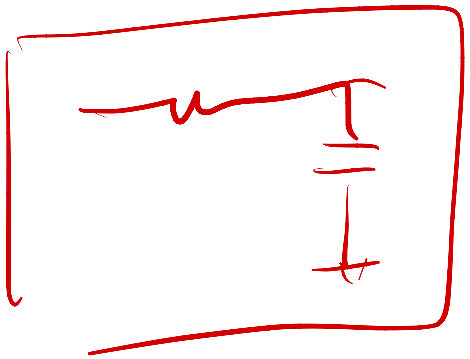
$$= e^{j\omega t} \underbrace{\left(\frac{+1}{1 + j\omega\tau} \right)}_{H}$$



$e^{j\omega t}$
$|H|e^{j\omega t + \angle H}$

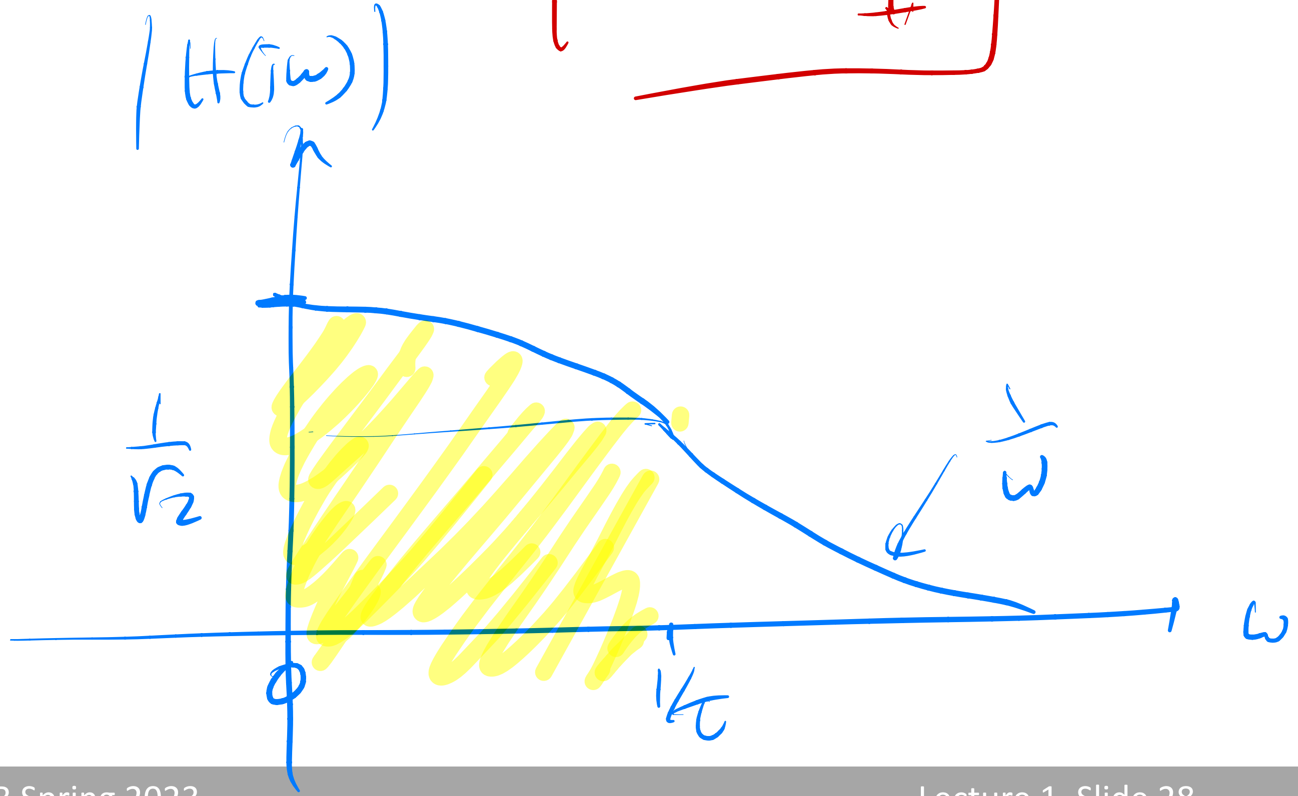
Final "Transfer" Function

$e^{j\omega t} \longrightarrow$



$\longrightarrow e^{j\omega t} \, H(j\omega)$

$|H(j\omega)| \, e^{j\omega t + j\angle H}$



$$H(j\omega) = \frac{1}{1 + j\omega\tau}$$

$$\sim \frac{1}{j\omega\tau}$$

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

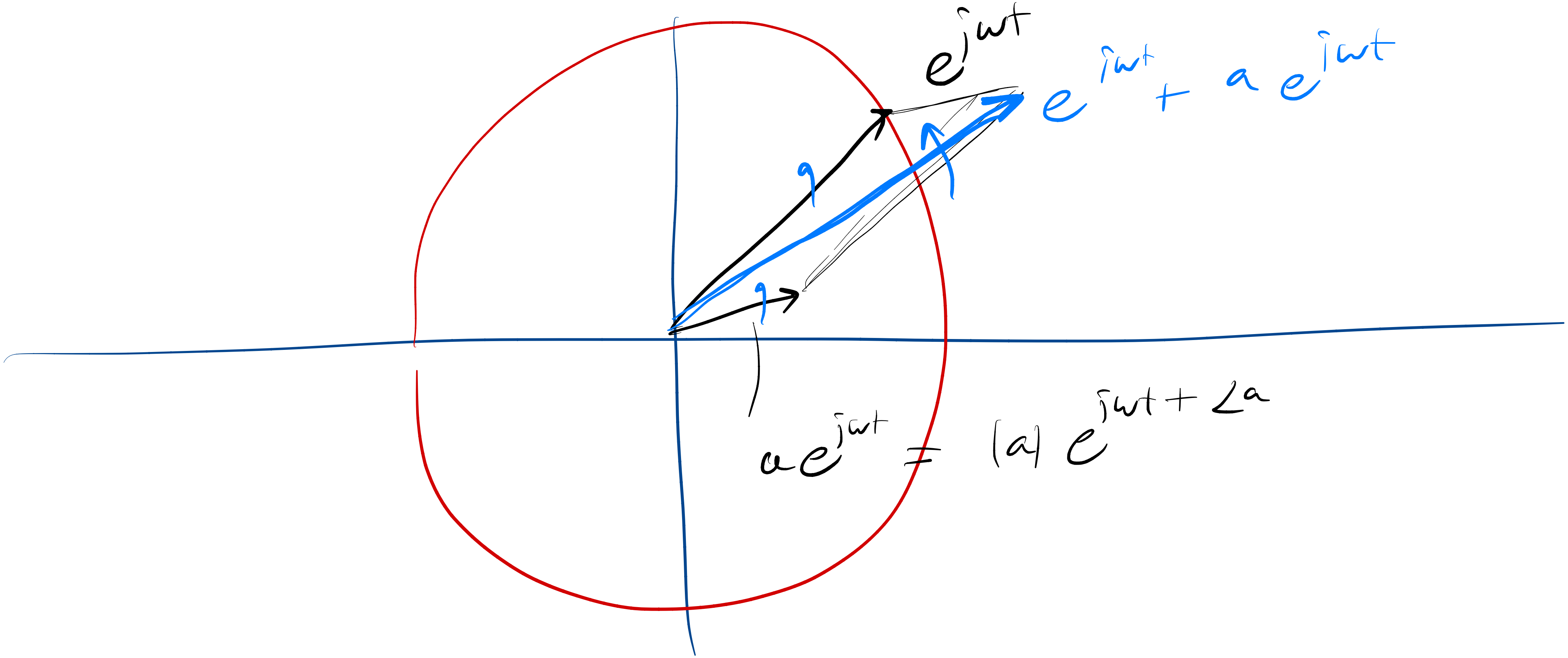
$$e^{j\omega t} + e^{-j\omega t} = 2\cos\omega t$$

Why It Worked: Going Around a Circle

- Summing many delayed copies of a complex exponential function still results in a complex exponential. Only magnitude and phase changes.

See demo

"Natural" Eigenfunction Solution



Low Pass for Sinusoidal Inputs

Application: Filtering out Noise

- Listen to an audio signal and note that while speech and music has a lot of distinct tones, noise is random, with many high pitched and low-pitched parts.
- What if we use a "low pass" filter to get rid of high-pitched parts?