Lecture 3 Key Concepts

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KC: Key Concepts.

1 Lecture 3, Module 2

1.1 Slide 16-17

In the last lecture we derived the general solution to a first-order differential equation with constant coefficients and an arbitrary input. We solved the problem using the integrating factor method and essentially we solved a math problem, not a physics or engineering problem. Now we'd like to understand the solution as an engineering system. So, returning to the RC circuit with a switch, let's walk through the solution intuitively. **KC:** The current in the circuit depends on the difference between the source input (constant) and the capacitor voltage, divided by R.

Initially the capacitor is discharged and the current flow is V_s/R, where V_s is the source voltage. But as the capacitor charges, the current decreases. **KC:** The slope of the output voltage over time decreases because the rate of flow of charge onto the capacitor decreases. In steady-state, the capacitor is charged to V_s and zero current flows.

1.2 Slide 18

KC: The step response is very common in electrical engineering. Basically we want to understand how the circuit behaves when the input goes from zero to 1V when the initial capacitors are discharged. This is useful as we'll see later.

1.3 Slide 19-20

Here we show the concept of source superposition. **KC:** If you can break your source into two or three simpler components as a sum, then you can use superposition to reconstruct the total solution as a sum of component solutions, where each component solution solves the original system for a given simpler input.

A perfect example is the pulse response. **KC:** The pulse response can be de-composed into two step responses.

1.4 Slide 21

KC: The system is time-invariant, meaning that if you know then solution to $u(t)$, then you know the solution to $u(t-T)$, which is just a delayed step. If $y(t)$ is the solution to $u(t)$, the solution to $u(t-T)$ is simply $y(t-T)$.

This allows us to find the complete pulse response of the RC circuit.

We also can see why the RC time constant is important for digital circuits. **KC:** if the $\tau = RC$ time constant is greater than the period of a pulse, the output pulse will be heavily attenduated and distorted. This limits how fast a circuit can transmit pulses and is ultimately the reason we run computers and communication systems at a given speed and not faster.

1.5 Slide 22-23

Here we reformulate the general solution as "weighted sum", a form that we can analyze. This is important when the input is an arbitrary function. You may recognize this integral as a "convolution integral", something you'll learn about in 120.

KC: The weighted sum integral is saying that The RC circuits response is due to not just the current input, but past values of the input as well. The past values are weighted exponentially so that only the most recent τ or so seconds really matter. So the circuit has a memory of about $\sim \tau$ seconds.

Even though this is a simple RC circuit, we can generalize this result quite a bit. First notice that any single capacitor circuit can be reduced to the same form by using Thevenin/Norton. You'll get some practice in HW ! Then later on we'll show that even if you have 100 capacitors (and inductors), you can still analyze the system using the same form through some linear algebra magic.

1.6 Slide 24

Why is this "weighted sum" perspective useful? We can already guess that if a circuit has fast variations, faster than τ, they will get averaged out and not appear at the output. We may be tempted to call this circuit a "slow pass" circuit.

1.7 Slide 25

KC: If we take the output across the resistor, we get a different response. We can use the "weighted sum" to understand the behavior. Because the voltage across the resistor is the difference between the input source and the capacitor voltage, which tends to average the input, the resistor voltage is the input minus its average. In other words, it's a "fast pass" circuit. Slow variations are averaged and cancelled out.

1.8 Slide 26

KC: Sinusoidal response is important as we'll see later. Sinusoids play an extremely important role in linear systems.

KC: Sinusoids can be decomposed into the sum (difference) of complex exponentials. For example:

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

Using source superposition, we can analyze the response to $e^{j\omega t}$ and then use superposition to find the sine or cosine response.

KC: The complex exponential $e^{j\omega t}$ is extremely simple when we view it on the complex plane. It simply traces a circle, perhaps the simplest and most symmetric shape possible.

KC: The sum of complex exponentials at the same frequency is always another complex exponential, only the phase is altered.

Interesting but out of scope: Perhaps as a preview to later courses, you will see that the complex exponential is the best function to represent the behavior of a linear time-invariant system because it has the property that if you delay a complex exponential, it's behavior does not change. If you walking on a circle starting from time $t = 0$s or at time $t = 100$s the trace will be the same, only the phase will differ.

1.9 Slide 28

KC: After a detailed calculation, we find the steady-state response to $e^{j\omega t}$ is simply $H \cdot e^{j\omega t}$, where H is a complex number. H is only a function of ω and is usually written as $H(j\omega)$.
KC: $H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$ which means the output has an amplitude that is attenuated by $|H(j\omega)|$ and a phase that is retarded by $\angle H(j\omega)$.

KC: For our RC circuit, we find that H has a very nice form. The response is given by

$$H(j\omega) = \frac{1}{1 + j\omega\tau}$$

Preview to a later part of the course: We call this a "low-pass" response because low frequencies pass:

$$|H(\omega \ll 1/\tau)| \approx 1$$

whereas high frequencies are attenuated:

$$|H(\omega \gg 1/\tau)| \approx 1/\omega\tau$$

We call the frequency τ the cut-off frequency. Later in the course we'll call it the 3-dB frequency.

Note that up to a frequency of $\omega = 1/\tau$, half the energy of the input passes to the output. That's because $|H(j\omega)| = 0.5$ when $\omega = 1/\tau$.