EECS 16B

Designing Information Devices and Systems II

Prof. Ali Niknejad and Prof. Kannan Ramchandran

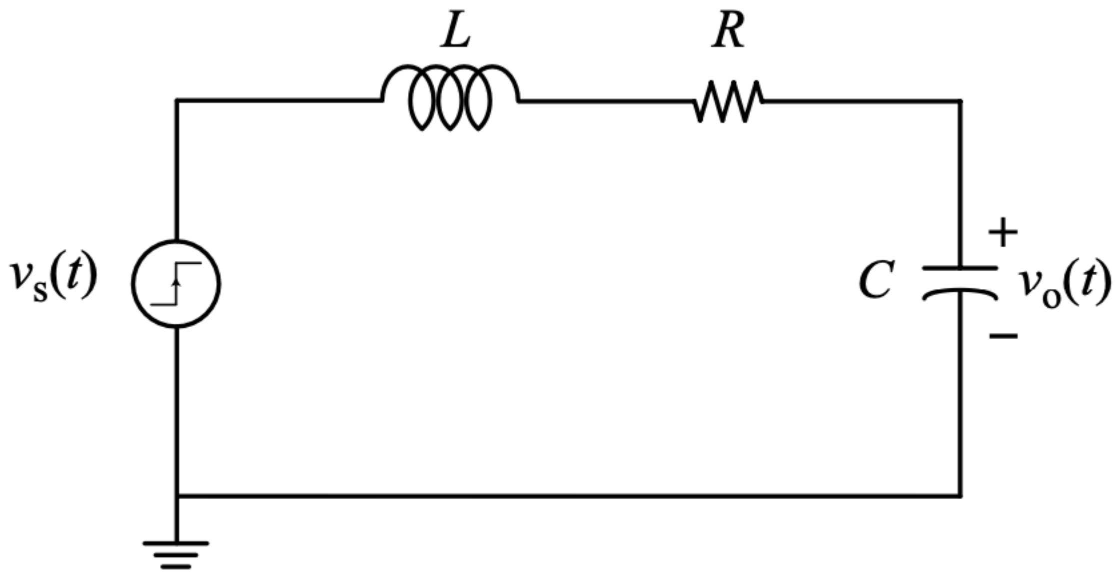
Department of Electrical Engineering and Computer Sciences, UC Berkeley,

niknejad@berkeley.edu

Module 5: RLC Circuits

EECS 16B

Series RLC Circuit



- This is a very important circuit and we'll spend some time understanding the behavior of the circuit.

KVL For Series RLC Circuit

$$v_s(t) = v_C(t) + RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2}$$

$$v_0(t) = v_C(t) = 0\text{V}$$

$$i(0) = i_L(0) = 0\text{A}$$

- Due to interaction between current in inductor and voltage on capacitor, we end up with a 2nd order differential equation
- We must specify two initial conditions, the voltage (or charge) on the capacitor and the current (or flux) in the inductor

Solution for Constant Inputs

- We'll solve the situation when we apply a constant input to the circuit at some time.
- Note the final value of the state of the circuit is predictable based on DC steady-state:

$$V_{dd} = v_C(t) + RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2}$$

$$V_{dd} = v_C(\infty)$$

Steady-State Solution

- Let's simply plug in the steady-state solution and solve for the unknown *transient*solution, which is the solution to the homogeneous differential equation:

$$v_C(t) = V_{dd} + v(t)$$

$$V_{dd} = V_{dd} + v(t) + RC\frac{dv}{dt} + LC\frac{d^2v}{dt^2}$$

Homogeneous Solution

- Try an exponential solution as before to satisfy the homogeneous equation:

$$0 = v(t) + RC\frac{dv}{dt} + LC\frac{d^2v}{dt^2}$$

$$0 = 1 + RCs + LCs^2 = 1 + (s\tau)2\zeta + (s\tau)^2$$

$$\tau = \frac{1}{\omega_0}$$

$$s\tau = -\zeta \pm \sqrt{\zeta^2 - 1}$$

$$\zeta = \frac{1}{2Q}$$

General Solution: Constants

- If the roots are distinct, the form of the general solution is as follows. We can find constants A and B from initial conditions.
- If both roots are real, we have two decaying exponentials:

$$v_C(t) = V_{dd} + A\exp(s_1 t) + B\exp(s_2 t)$$

$$v_C(0) = V_{dd} + A + B = 0$$

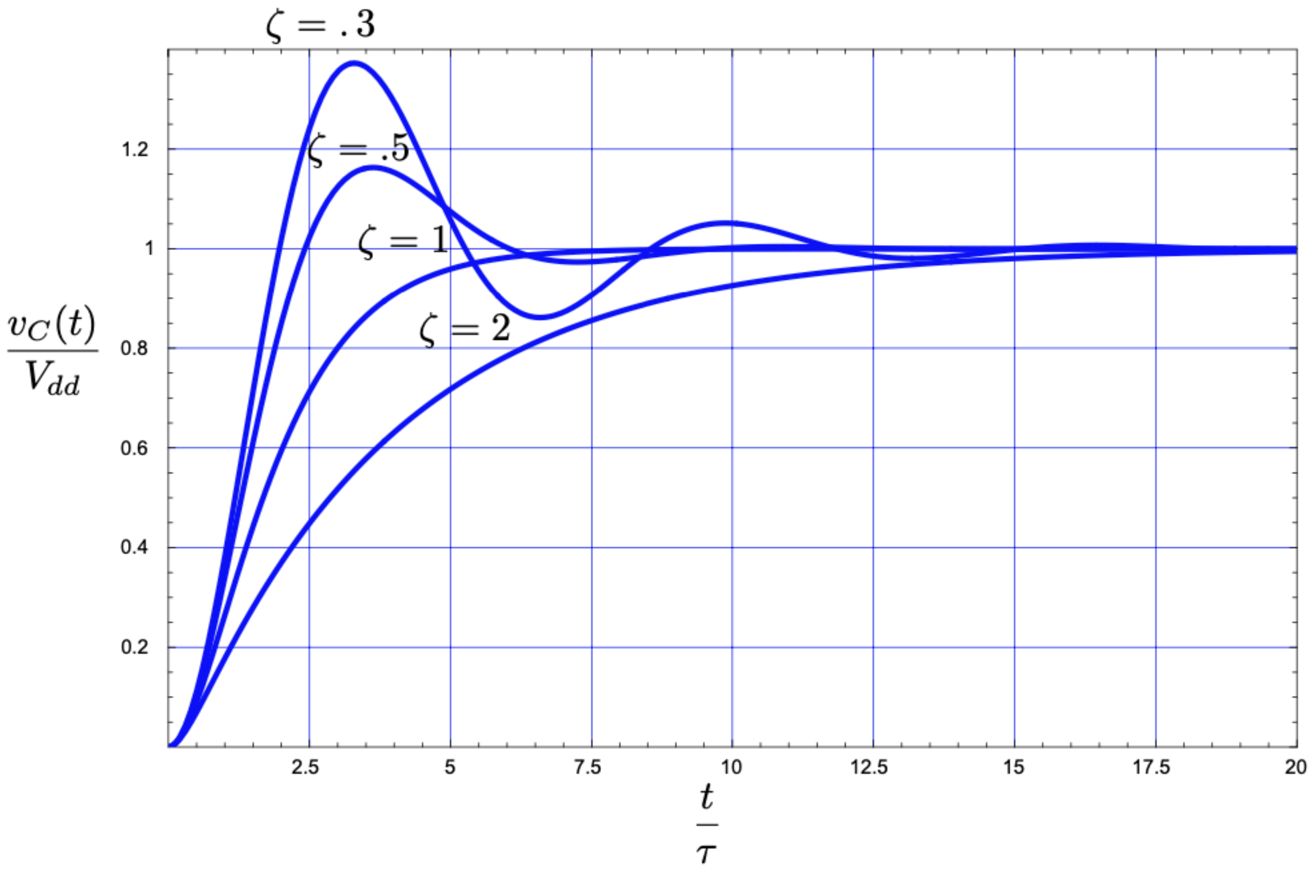
$$i(0) = C\frac{dv_C(t)}{dt}|_{t=0} = 0$$

Damped vs. Oscillatory

- We have a parameter zeta that determines the nature of the solution. We can categorize the solution into three types:
  - Overdamped solutions are decaying exponentials.
  - Underdamped solutions also decay exponentially, but with a twist. They may overshoot and oscillate before fizzling out
  - What's the physical reason ?

$$\begin{array}{ll} \zeta < 1 & \text{Underdamped} \\ \zeta = 1 & \text{Critically Damped} \\ \zeta > 1 & \text{Overdamped} \end{array}$$

Under vs Over Damped



$\zeta < 1$ Underdamped
$\zeta = 1$ Critically Damped
$\zeta > 1$ Overdamped

- Overdamped solutions don't oscillate.

Distince Roots: Details

- A and B satisfy the initial and final conditions:

$$s = \frac{1}{\tau}(-\zeta \pm \sqrt{\zeta^2 - 1}) = \begin{cases} s_1 \\ s_2 \end{cases} < 0$$

$$As_1 + Bs_2 = 0$$

$$A + B = -V_{dd}$$

$$A = \frac{-V_{dd}}{1-\sigma}$$

$$B = \frac{\sigma V_{dd}}{1-\sigma}$$

$$\sigma = \frac{s_1}{s_2}.$$

$$v_C(t) = V_{dd}\left(1 - \frac{1}{1-\sigma}\left(e^{s_1 t} - \sigma e^{s_2 t}\right)\right)$$

Critically Damped

- If the roots are identical, we can obtain the second solution through a limiting process:

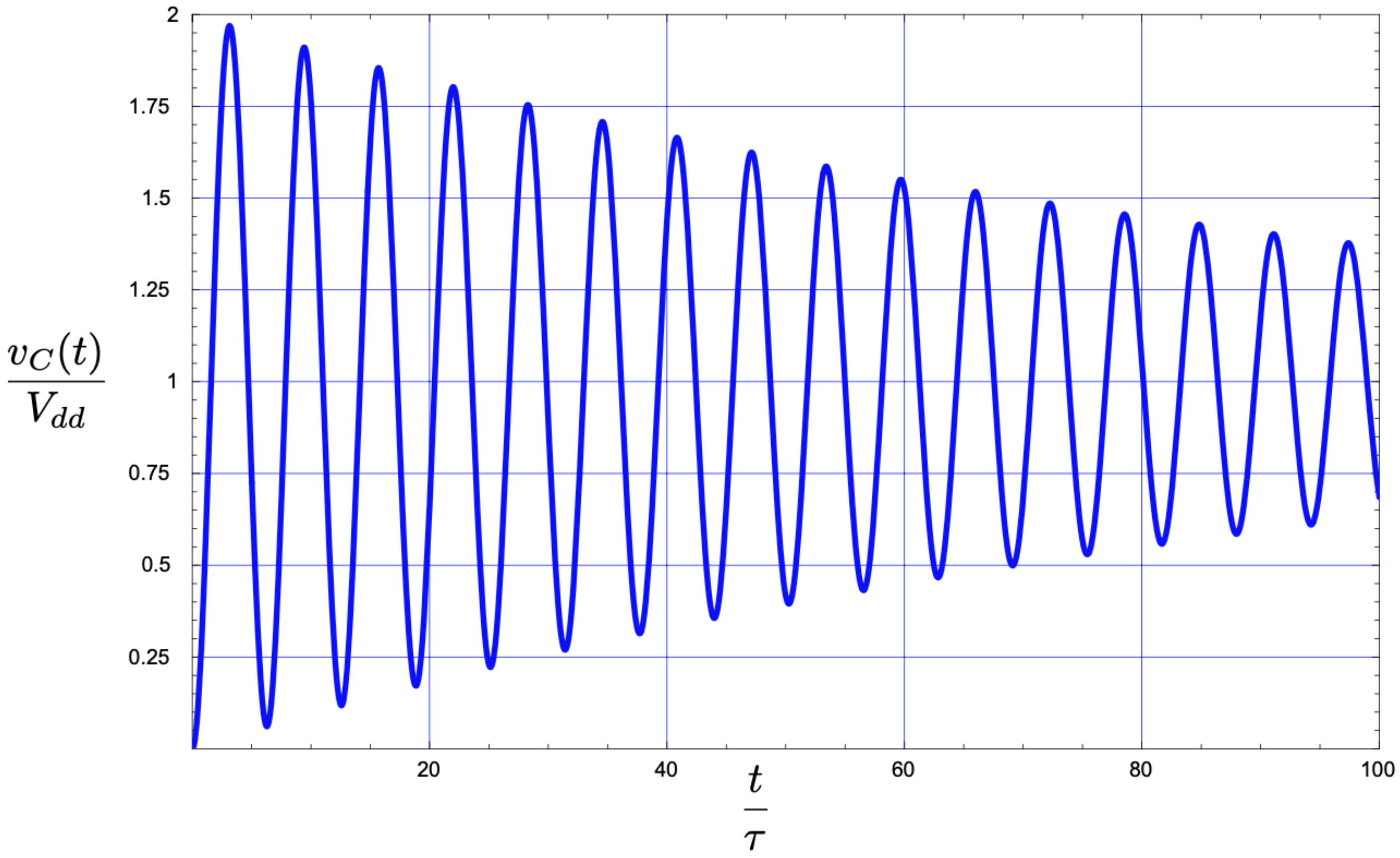
$$s = \frac{1}{\tau}(-\zeta \pm \sqrt{\zeta^2 - 1}) = -\frac{1}{\tau}$$

$$\lim_{\zeta \to 1} v_C(t) = V_{dd}\left(1 - e^{-t/\tau} - \frac{t}{\tau}e^{-t/\tau}\right)$$

Underdamped

- If the roots of the equation are underdamped, they are complex and lead to oscillatory behavior

$$s\tau = -\zeta \pm j\sqrt{1-\zeta^2} = a \pm jb$$



Underdamped Solution Procedure

- We find that A and B are complex conjugates and so we can combine the terms as follows:

$$A^* = \frac{-V_{dd}}{1-\sigma^*} = \frac{V_{dd}}{\sigma^{-1}-1} = \frac{\sigma V_{dd}}{1-\sigma} = B$$

$$|A| = \frac{V_{dd}}{|1+\sigma|}$$

$$v_C(t) = V_{dd} + e^{at/\tau}\left(Ae^{jbt\tau} + A^* e^{-jbt\tau}\right)$$

$$\phi = \angle \frac{V_{dd}}{1+\sigma}$$

$$v_C(t) = V_{dd} + e^{at/\tau} 2|A| \cos(\omega t + \phi)$$

How much energy is stored in the "tank"?

- An "LCR" circuit is often referred to as a "tank"
- Let's assume the tank is lossless. Then the energy stored in the inductor and capacitor is given by:

$$w_L = \tfrac{1}{2} L i^2(t) = \tfrac{1}{2} L I_M^2 \cos^2 \omega_0 t$$

$$w_C = \tfrac{1}{2} C v_C^2(t) = \tfrac{1}{2} C \left(\frac{1}{C} \int i(\tau) d\tau \right)^2$$

$$w_C = \tfrac{1}{2} \frac{I_M^2}{\omega_0^2 C} \sin^2 \omega_0 t$$

Total Tank Energy

- If we sum the energy stored in the inductor and capacitor at any given time, we find that the sum is constant.
- Since the tank is lossless, this is logical and a statement of the conservation of energy.
- We observe that the maximum energy of the inductor or capacitor occurs when other is storing zero energy:

$$w_s = w_L + w_C = \tfrac{1}{2} I_M^2 \left(L \cos^2 \omega_0 t + \frac{1}{\omega_0^2 C} \sin^2 \omega_0 t \right) = \tfrac{1}{2} I_M^2 L$$

$$w_{L,\max} = w_s = \tfrac{1}{2} I_M^2 L \qquad w_{C,\max} = w_s = \tfrac{1}{2} V_M^2 C$$

Lossy Case

- Now let's introduce loss. The energy dissipated by the resistor per cycle is given by:

$$w_d = P \cdot T = \tfrac{1}{2} I_M^2 R \cdot \frac{2\pi}{\omega_0}$$

- Comparing the energy lost to the energy stored in the inductor, we have:

$$\frac{w_s}{w_d} = \frac{\frac{1}{2} L I_M^2}{\frac{1}{2} I_M^2 R \frac{2\pi}{\omega_0}} = \frac{\omega_0 L}{R} \frac{1}{2\pi} = \frac{Q}{2\pi} \qquad Q = 2\pi \frac{w_s}{w_d}$$

Parallel LCR

- Using the concept of "duality", we expect the equations to take on the exam same form as before: