EECS 16B

Designing Information Devices and Systems II

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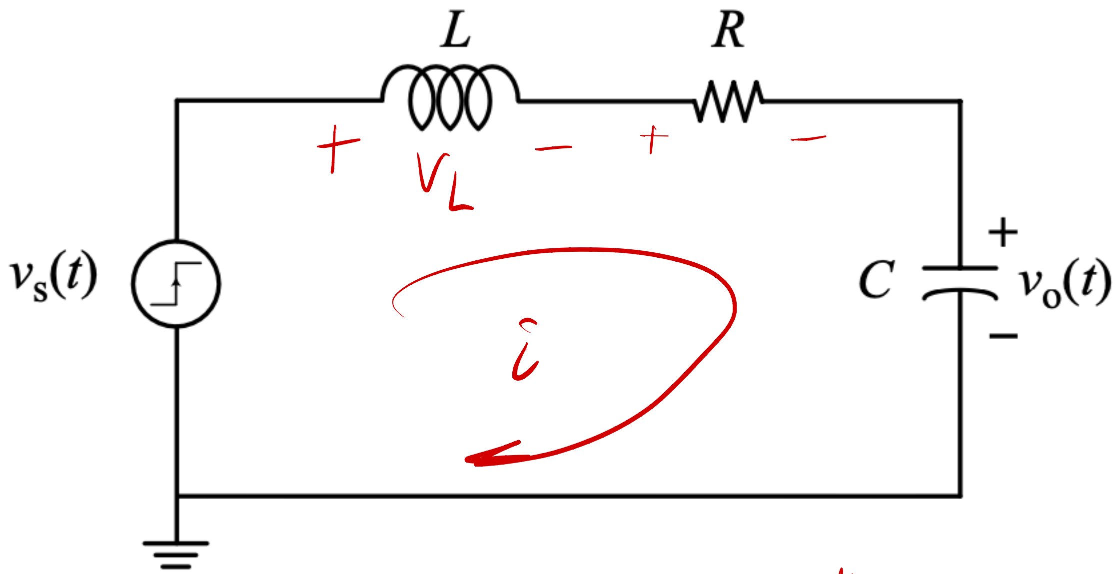
Module 5: RLC Circuits

EECS 16B

Series RLC Circuit



$$V_s(t) = V_L + V_R + V_C = L\frac{di}{dt} + i \cdot R + V_C(0) + \frac{1}{C}\int_0^t i(t)dt$$

- This is a very important circuit and we'll spend some time understanding the behavior of the circuit.

$$i = C\frac{dv_c}{dt} \Rightarrow \frac{di}{dt} = C\frac{d^2v_c}{dt^2}$$

$$v_s = L\frac{di}{dt} + i \cdot R + v_c$$

$$= LC\frac{d^2v_c}{dt^2} + RC\frac{dv_c}{dt} + v_c$$

KVL For Series RLC Circuit

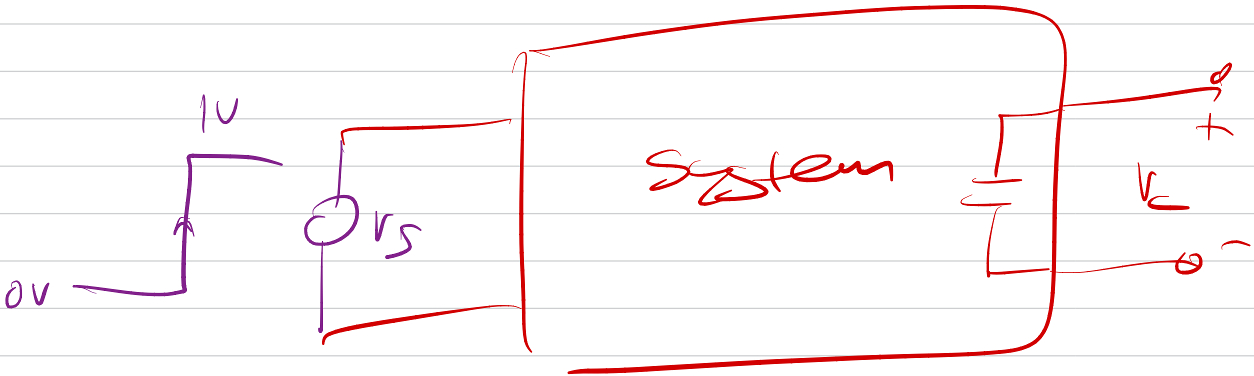
$$\boxed{v_s(t) = v_C(t) + RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2}}$$

$$\left.\begin{aligned} v_0(t) &= v_C(t) = 0\text{V} \\ i(0) &= i_L(0) = 0\text{A} \end{aligned}\right\}$$

initial conditions

- Due to interaction between current in inductor and voltage on capacitor, we end up with a 2nd order differential equation
- We must specify two initial conditions, the voltage (or charge) on the capacitor and the current (or flux) in the inductor

{ Initial State of the System }



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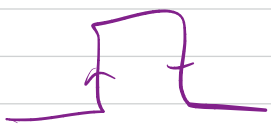


2

3

Solution for Constant Inputs

- We'll solve the situation when we apply a constant input to the circuit at some time.
- Note the final value of the state of the circuit is predictable based on DC steady-state:

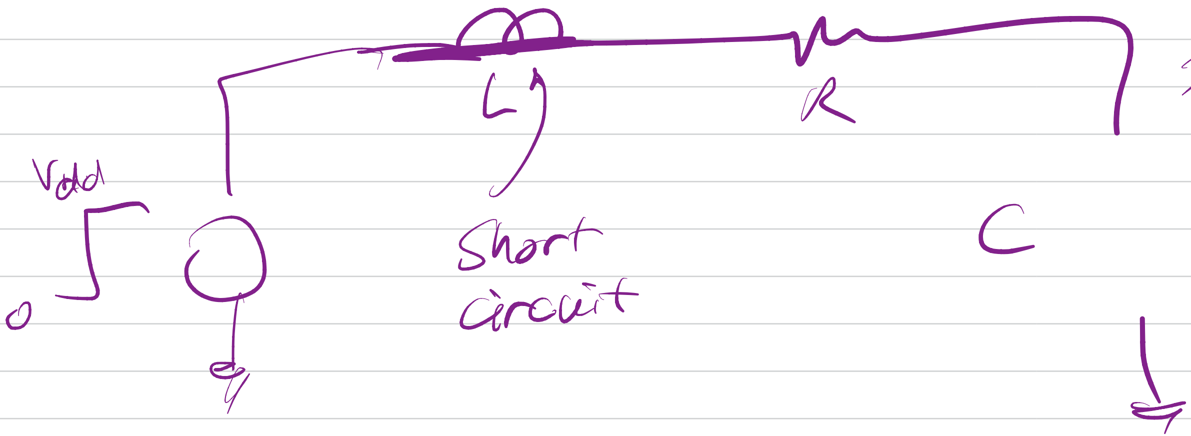
$t = 0$

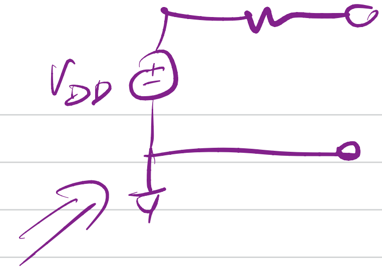
$$\boxed{V_{dd} = v_C(t)} + \cancel{RC\frac{dv_C}{dt}} + \cancel{LC\frac{d^2v_C}{dt^2}} \qquad V_{dd} = v_C(\infty)$$

$v_c(t) = v_{dd}$ $\qquad \frac{dv_c}{dt} = 0 \qquad \frac{d^2v_c}{dt^2} = 0$

solutions

DC Steady State





$V_C \to$ constant

$$\frac{dV_C}{dt} = 0$$

$$\frac{di}{dt} \to 0$$

$$V = L\frac{di}{dt} = 0$$

Steady-State Solution

- Let's simply plug in the steady-state solution and solve for the unknown *transient*solution, which is the solution to the homogeneous differential equation:

$$v_C(t) = V_{dd} + v(t)$$

$$V_{dd} = V_{dd} + v(t) + RC\frac{dv}{dt} + LC\frac{d^2v}{dt^2}$$

$$v(t) = v_p(t)$$

$$v'(t) = v_p(t) + A\, v_h(t)$$

DE

homogeneous soln	inhomogeneous soln
natural soln	forced soln
transient	particular soln
complementary solution	

Homogeneous Solution

- Try an exponential solution as before to satisfy the homogeneous equation:

$$0 = v(t) + RC\frac{dv}{dt} + LC\frac{d^2v}{dt^2}$$

$v(t) = Ae^{st}$

$0 = \cancel{A}\cancel{e^{st}} + RC \cdot \cancel{A}s\,\cancel{e^{st}} + LC\,\cancel{A}s^2\cancel{e^{st}}$

$$\boxed{0 = 1 + RCs + LCs^2 = 1 + (s\tau)2\zeta + (s\tau)^2}$$

$\tau \triangleq \sqrt{LC}$

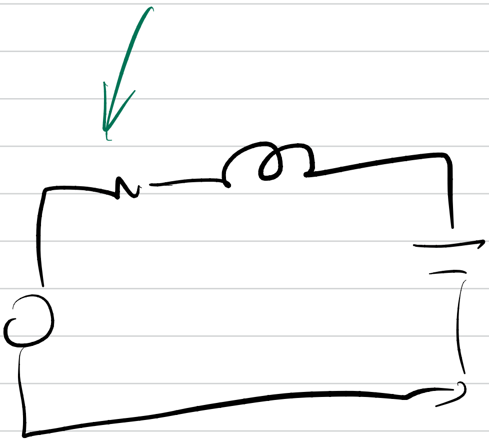
$$\tau = \frac{1}{\omega_0}$$

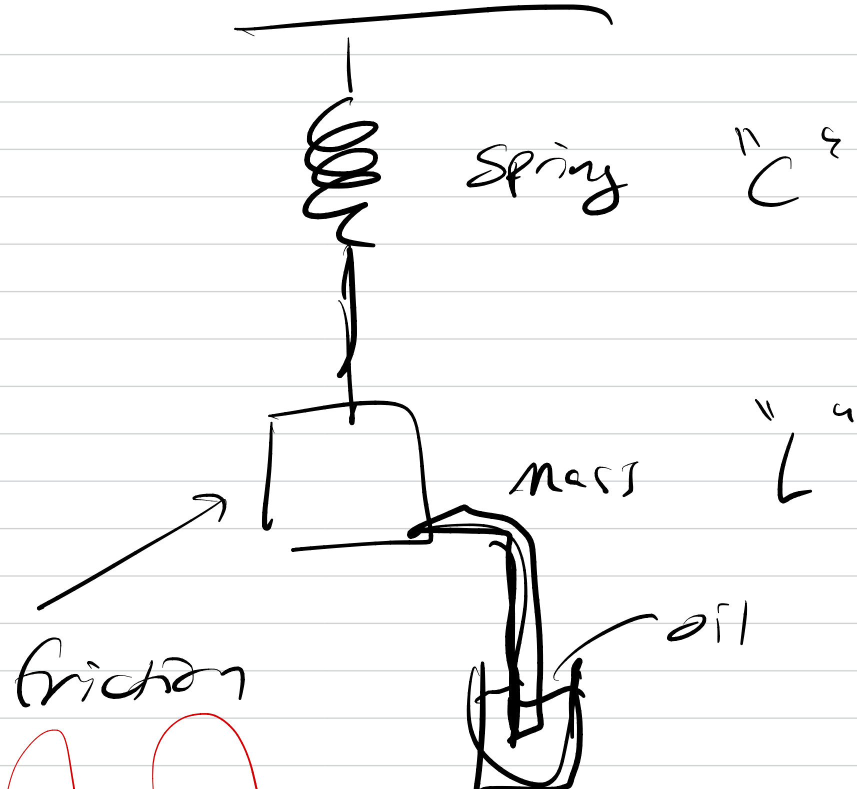
$\omega_0 = \frac{1}{\tau} = \frac{1}{\sqrt{LC}}$

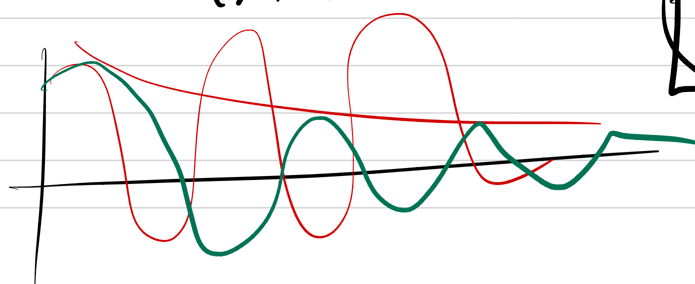
$$\boxed{\zeta = \frac{1}{2Q}}$$

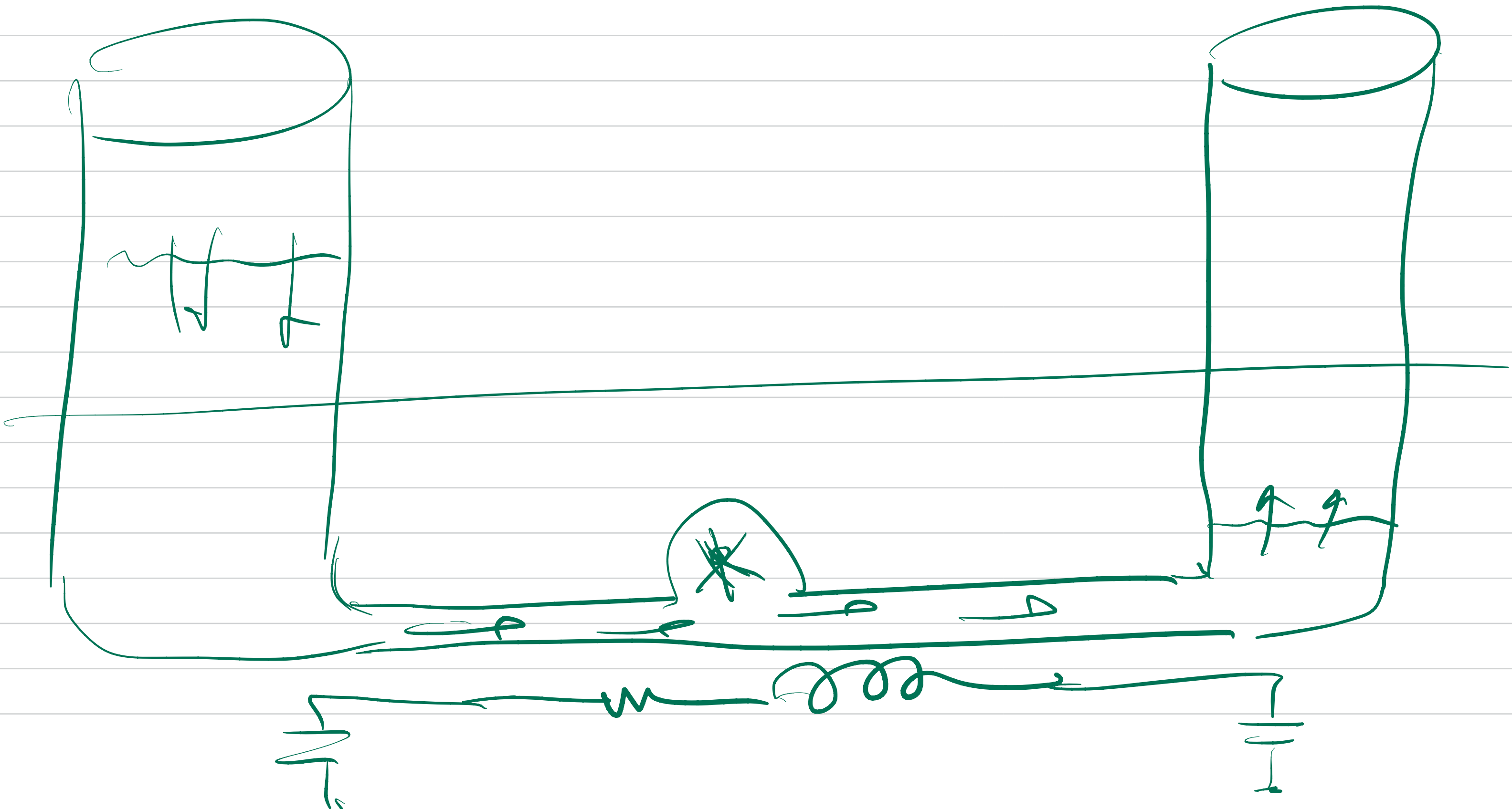
$$\boxed{s\tau = -\zeta \pm \sqrt{\zeta^2 - 1}}$$

$RC = 2\zeta$




Spring
"C"
mass
"L"
friction
oil





$$s\tau = -\zeta \pm \sqrt{\zeta^2 - 1}$$

$$\zeta = \begin{cases} \zeta < 1 & \text{2 COMPLEX ROOTS} \quad \text{UNDER DAMPED} \\ \zeta = 1 & \text{1 ROOT } s = -1/\tau \text{ ? ROOT} \quad \text{CRIT. DAMPED} \\ \zeta > 1 & \{\text{2 DISTINCT ROOTS}\} \quad \text{OVERDAMPED} \\ & \{\text{ROOTS ARE NEGATIVE}\} \end{cases}$$

$$e^{st}$$

$$s = \frac{1}{\tau}\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

$s = a + jb$ $\quad e^{st} = e^{at}\, e^{jbt} \Rightarrow$ next page

$$e^{st} = \underbrace{e^{at}}_{\text{exponential decay}} \underbrace{e^{jbt}}$$

$a < 0$

RC

$$\omega_0^2 = \frac{1}{LC}$$

$$Q \triangleq \frac{\omega_0 L}{R}$$

General Solution: Constants

- If the roots are distinct, the form of the general solution is as follows. We can find constants A and B from initial conditions.
- If both roots are real, we have two decaying exponentials:

$$\underline{v_C(t)} = \underline{V_{dd}} + \underbrace{A \overbrace{\exp(s_1 t)}^{=1} + B \overbrace{\exp(s_2 t)}^{=1}}$$

$v_C' = 0 + s_1 A e^{s_1 t} + s_2 B e^{s_2 t}$

$v'(0) = s_1 A + s_2 B$

homogeneous Soln

particular solution

$$\boxed{(1) \quad v_C(0) = V_{dd} + A + B = 0}$$

$$\boxed{(2) \quad i(0) = C\frac{dv_C(t)}{dt}|_{t=0} = 0 = s_1 A + s_2 B}$$

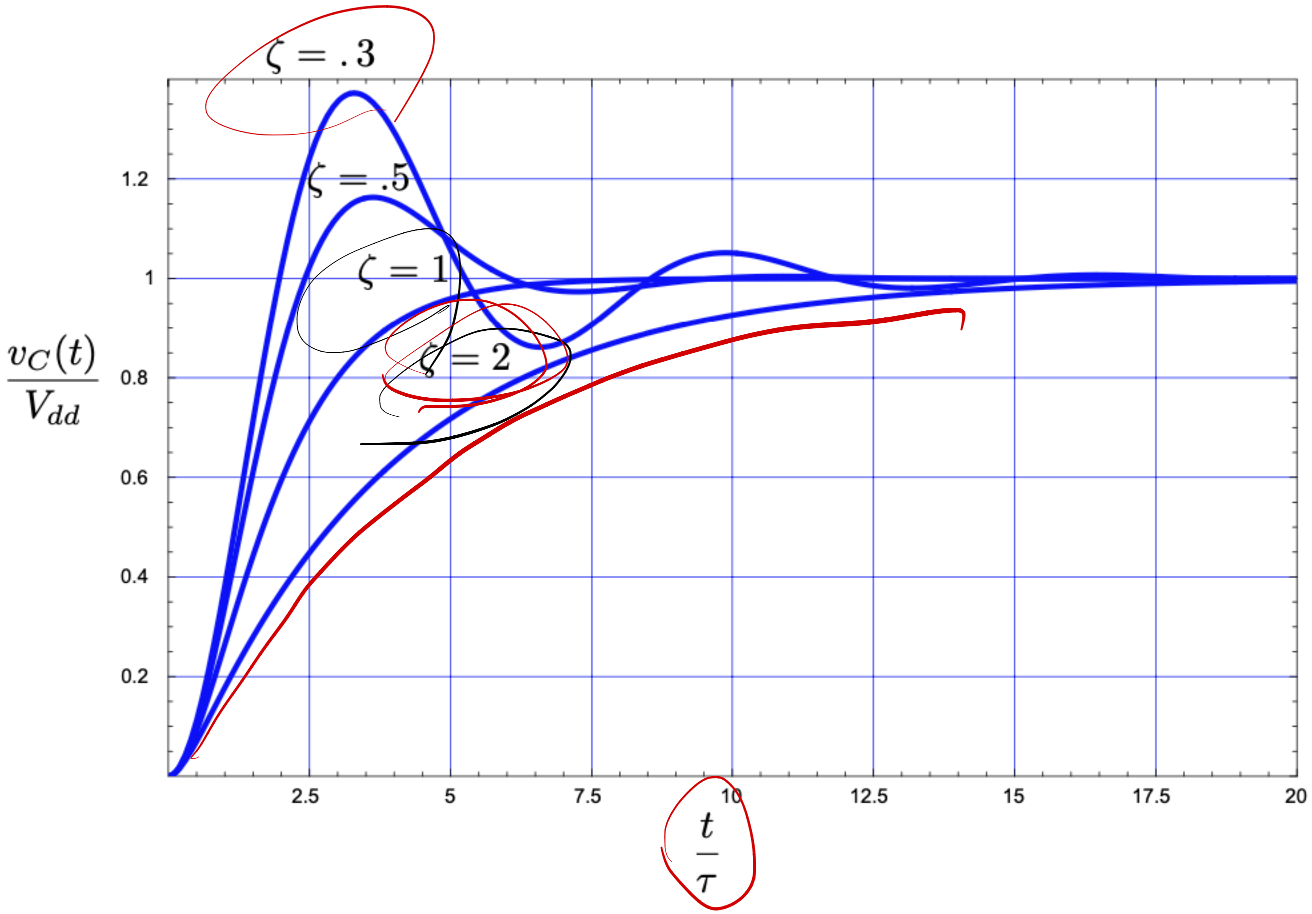
$i(0) = C \left.\frac{dv_c}{dt}\right|_{t=0}$

Damped vs. Oscillatory

- We have a parameter zeta that determines the nature of the solution. We can categorize the solution into three types:
  - Overdamped solutions are decaying exponentials.
  - Underdamped solutions also decay exponentially, but with a twist. They may overshoot and oscillate before fizzling out
  - What's the physical reason ?

$$\begin{array}{ll} \zeta < 1 & \text{Underdamped} \\ \zeta = 1 & \text{Critically Damped} \\ \zeta > 1 & \text{Overdamped} \end{array}$$

Under vs Over Damped



$\sqrt{\xi^2 - 1}$

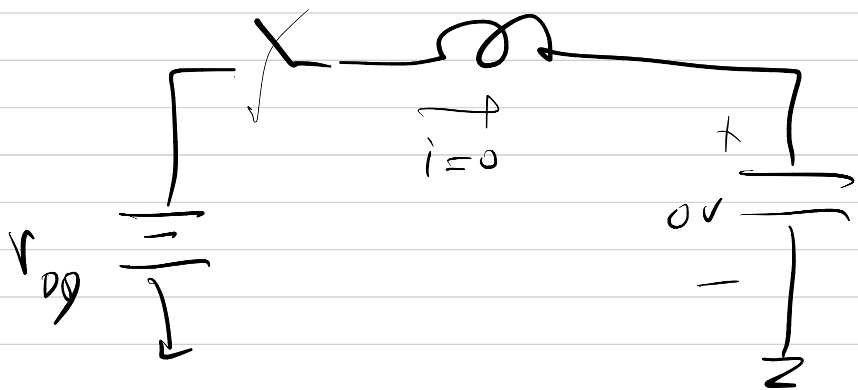
$\zeta < 1$ Underdamped
$\zeta = 1$ Critically Damped
$\zeta > 1$ Overdamped

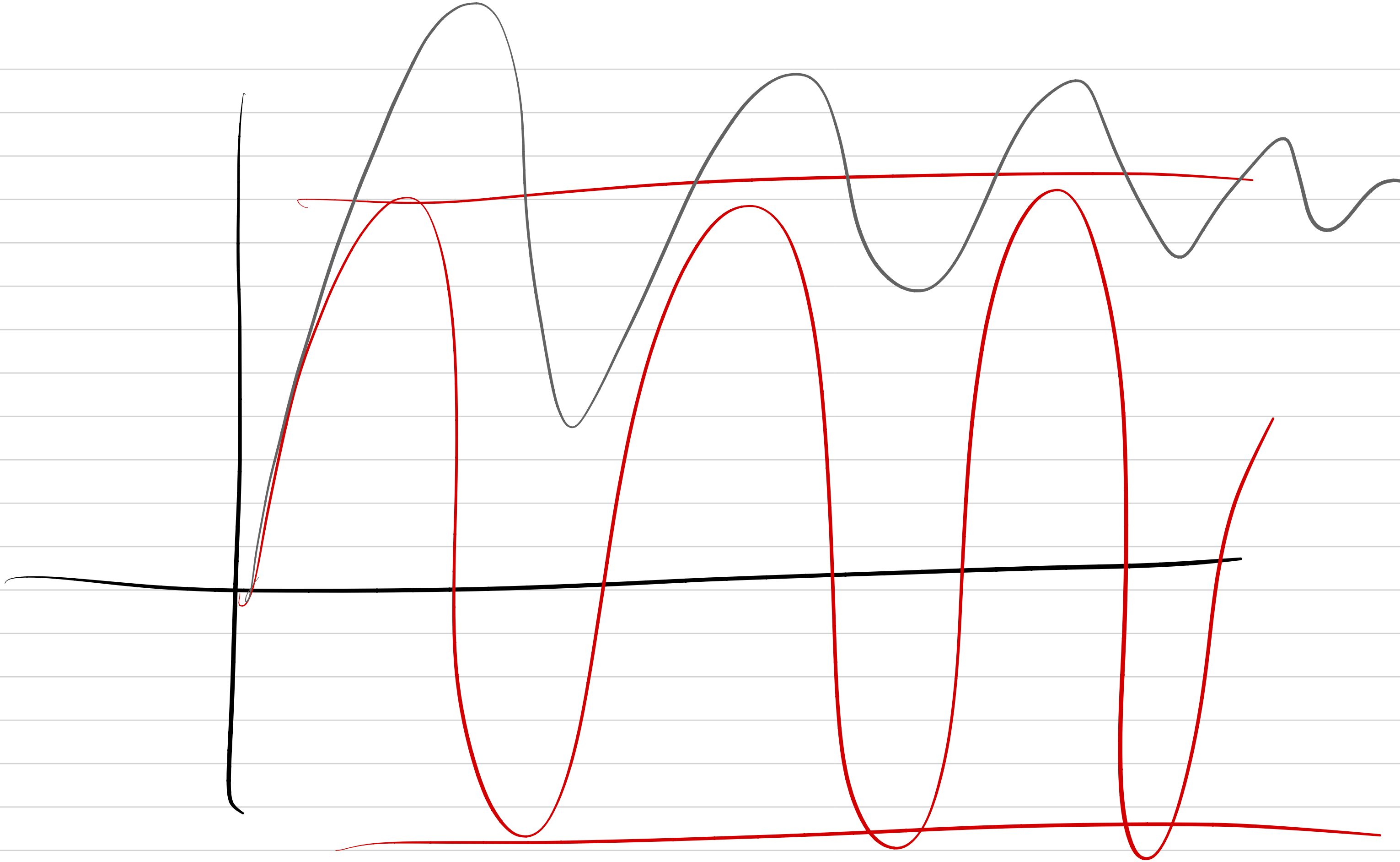
$(s\tau)^2 + (s\tau)2\xi + 1 = 0$

$\tau = \frac{1}{\omega_0}$ $\quad$ $\omega_0^2 = \frac{1}{LC}$

- Overdamped solutions don't oscillate.

$$\xi = \frac{1}{2} R C = 0 \qquad R = 0\,\Omega$$





Distince Roots: Details

- A and B satisfy the initial and final conditions:

$$s = \frac{1}{\tau}(-\zeta \pm \sqrt{\zeta^2 - 1}) = \begin{cases} s_1 \\ s_2 \end{cases} < 0$$

eq 1 $\quad As_1 + Bs_2 = 0$

eq 2 $\quad A + B = -V_{dd}$

$$\boxed{A = \frac{-V_{dd}}{1-\sigma} \qquad B = \frac{\sigma V_{dd}}{1-\sigma}}$$

$$\sigma = \frac{s_1}{s_2}$$

$$A^* = \frac{-V_{dd}}{1-\sigma^*}$$

$$\sigma^* = \left(\frac{\alpha + j\beta}{\alpha - j\beta}\right)^* = \frac{\alpha - j\beta}{\alpha + j\beta}$$

$$\sigma^* = \frac{1}{\sigma}$$

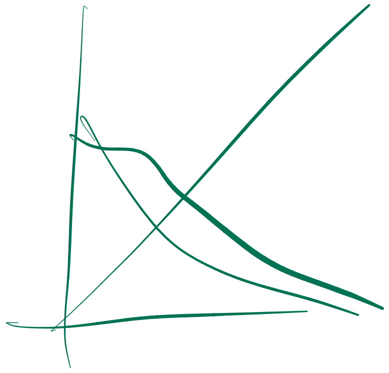
$$v_C(t) = V_{dd}\left(1 - \frac{1}{1-\sigma}\left(e^{s_1 t} - \sigma e^{s_2 t}\right)\right)$$

Critically Damped

- If the roots are identical, we can obtain the second solution through a limiting process:

$$s = \frac{1}{\tau}(-\zeta \pm \sqrt{\zeta^2 - 1}) = -\frac{1}{\tau}$$

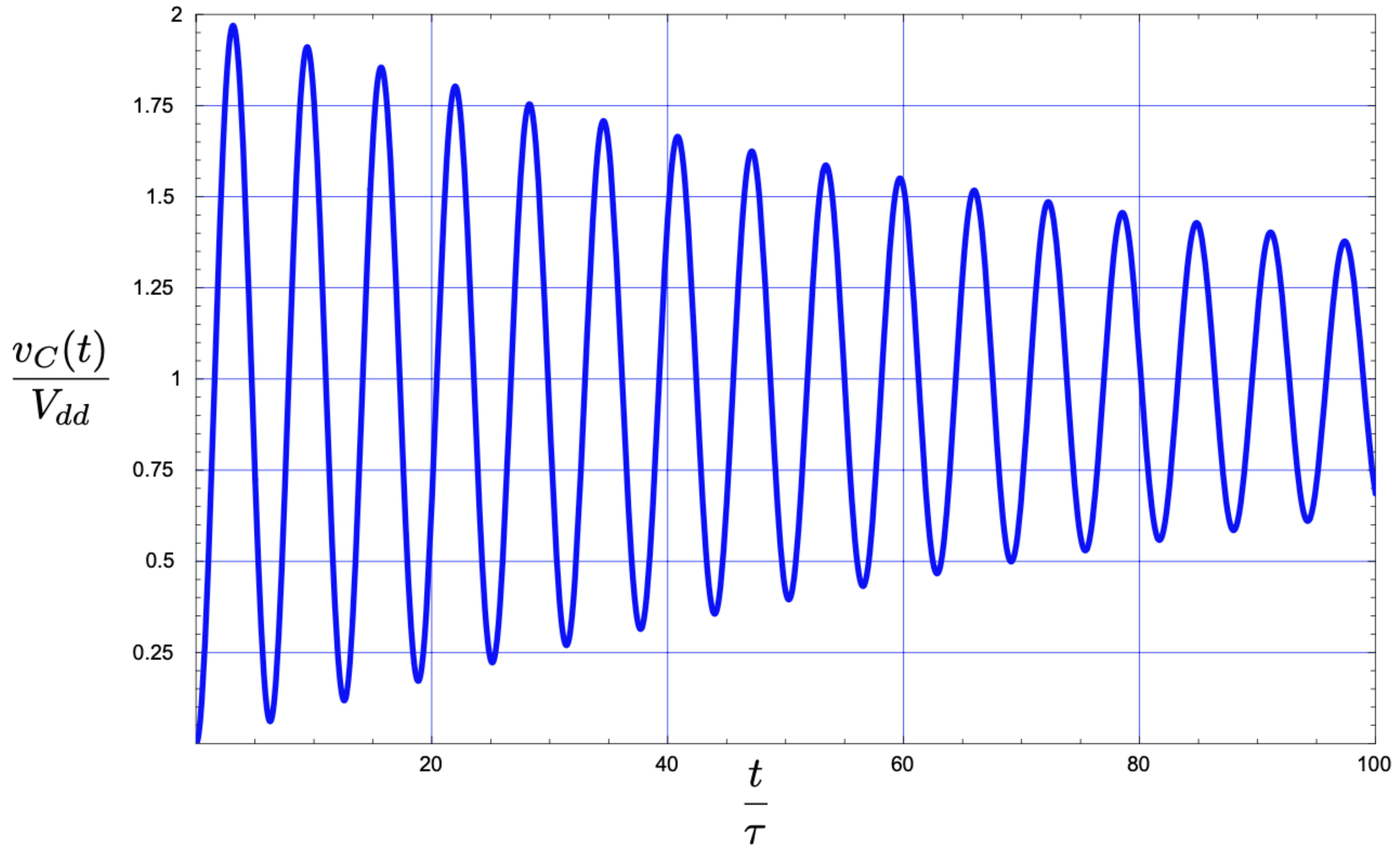
$$\lim_{\zeta \to 1} v_C(t) = V_{dd}\left(1 - e^{-t/\tau} - \frac{t}{\tau}e^{-t/\tau}\right)$$



Underdamped

- If the roots of the equation are underdamped, they are complex and lead to oscillatory behavior

$$s\tau = -\zeta \pm j\sqrt{1-\zeta^2} = a \pm jb$$



$$s_{1,2}\tau = -\zeta \pm j\sqrt{1-\zeta^2}$$

real part

imag.

$$s_1 = -\zeta + j\sqrt{1-\zeta^2}$$

$$s_2 = \overline{s_1}$$

Underdamped Solution Procedure

- We find that A and B are complex conjugates and so we can combine the terms as follows:

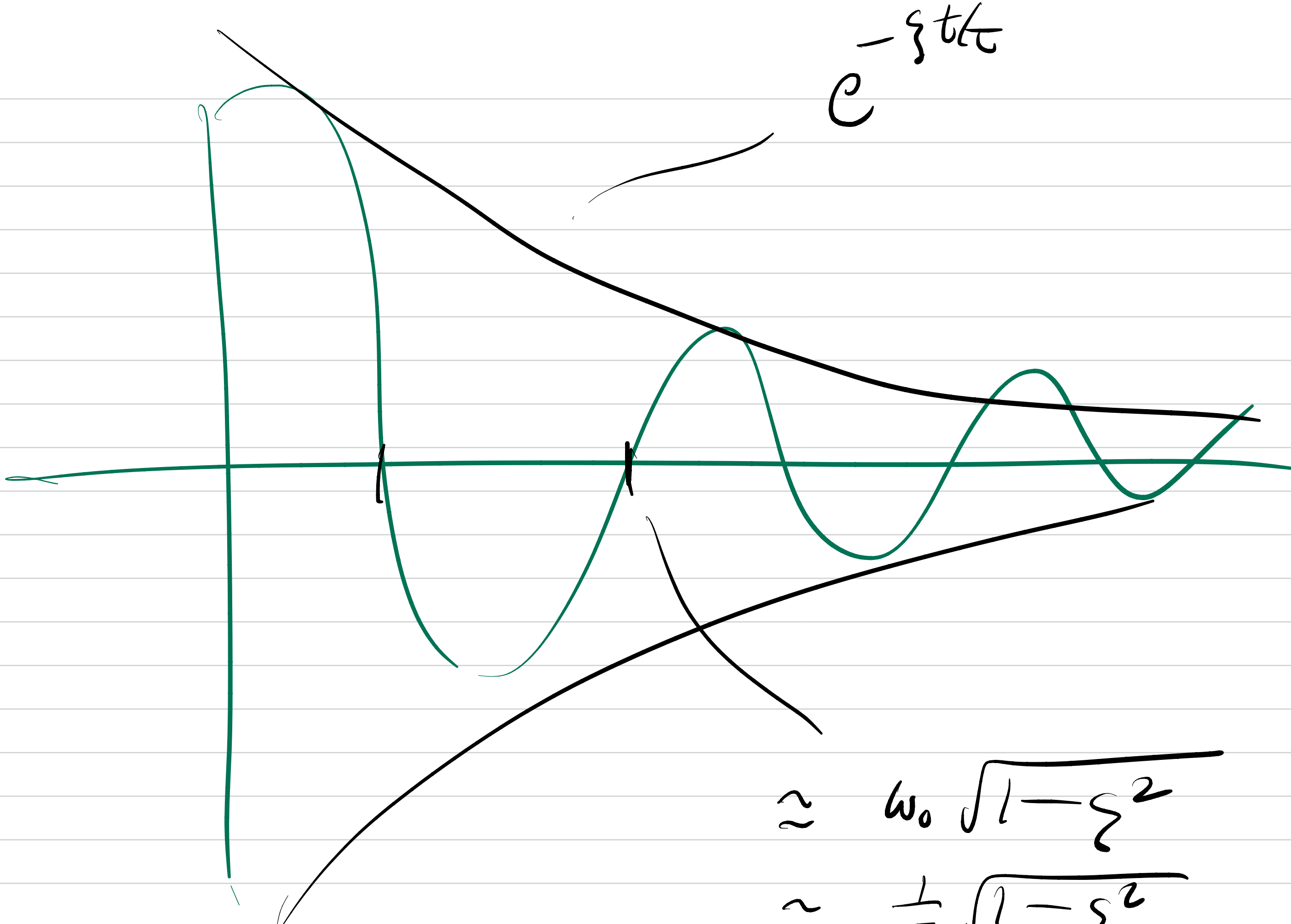
$$A^* = \frac{-V_{dd}}{1-\sigma^*} = \frac{V_{dd}}{\sigma^{-1}-1} = \frac{\sigma V_{dd}}{1-\sigma} = B$$

$$|A| = \frac{V_{dd}}{|1+\sigma|}$$

$$v_C(t) = V_{dd} + e^{at/\tau}\underbrace{\left(Ae^{jbt\tau} + A^*e^{-jbt\tau}\right)}$$

$$\phi = \angle\frac{V_{dd}}{1+\sigma}$$

$$\boxed{v_C(t) = V_{dd} + e^{at/\tau}2|A|\cos(\omega t + \phi)}$$


$e^{-\xi t/\tau}$
$\simeq \omega_0 \sqrt{1-\xi^2}$
$\simeq \frac{1}{\tau}\sqrt{1-\xi^2}$

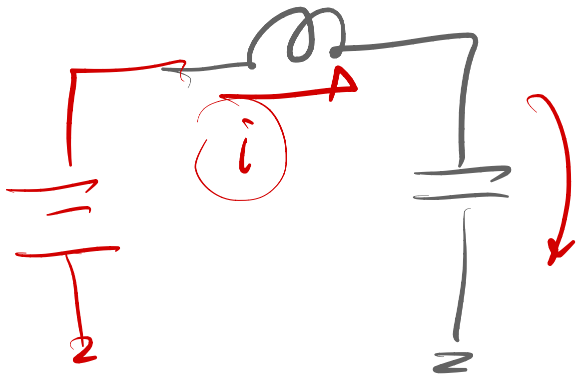
How much energy is stored in the "tank"?

- An "LCR" circuit is often referred to as a "tank"
- Let's assume the tank is lossless. Then the energy stored in the inductor and capacitor is given by:

$R = 0$

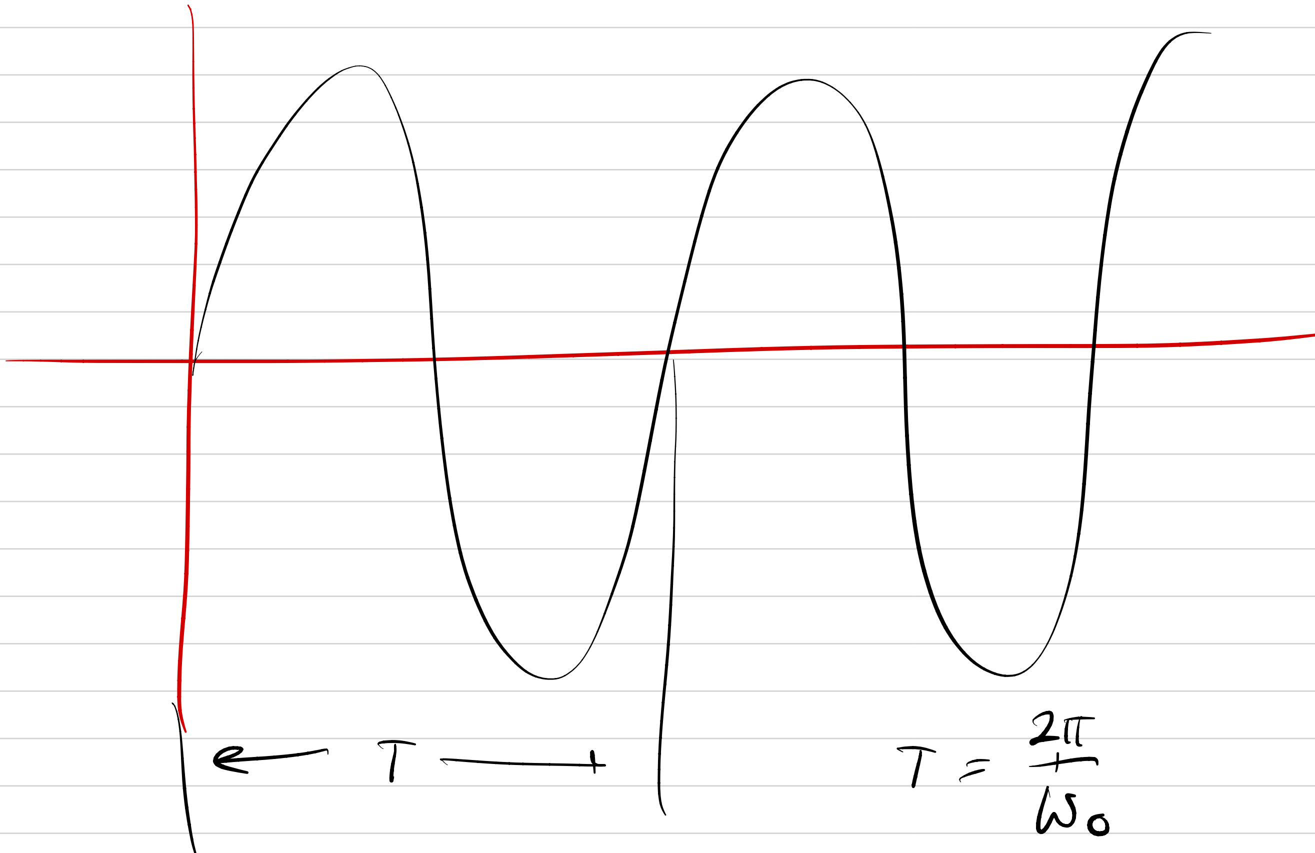
$$w_L = \tfrac{1}{2} L i^2(t) = \tfrac{1}{2} L I_M^2 \cos^2 \omega_0 t$$

$i = I_M \cos \omega_0 t$



$$w_C = \tfrac{1}{2} C v_C^2(t) = \tfrac{1}{2} C \left(\underbrace{\frac{1}{C} \int i(\tau) d\tau} \right)^2$$

$$w_C = \frac{1}{2} \frac{I_M^2}{\omega_0^2 C} \sin^2 \omega_0 t$$



$$T = \frac{2\pi}{\omega_0}$$

Total Tank Energy

- If we sum the energy stored in the inductor and capacitor at any given time, we find that the sum is constant.
- Since the tank is lossless, this is logical and a statement of the conservation of energy.
- We observe that the maximum energy of the inductor or capacitor occurs when other is storing zero energy:

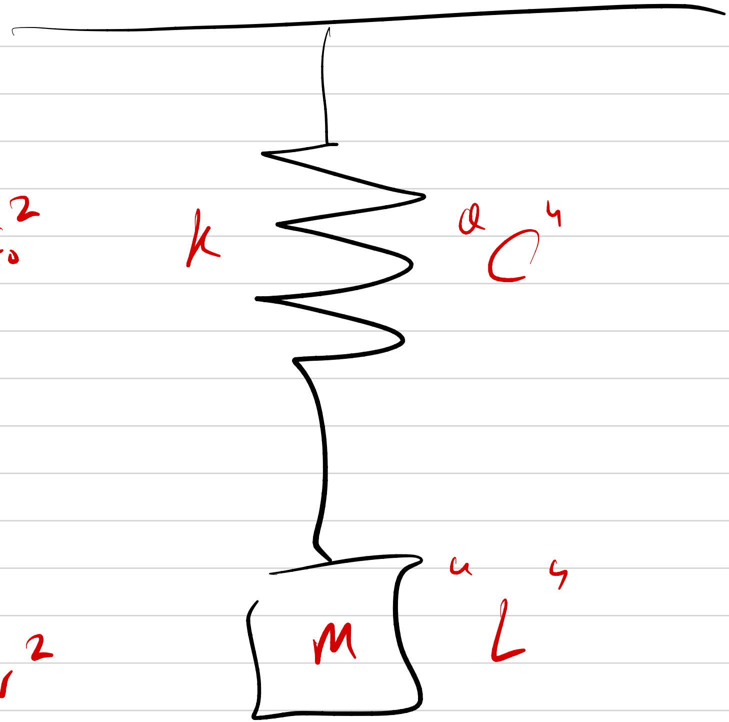
$$\omega_0^2 = \frac{1}{LC}$$

$$w_s = w_L + w_C = \tfrac{1}{2} I_M^2 \left(L \cos^2 \omega_0 t + \boxed{\frac{1}{\omega_0^2 C}} \sin^2 \omega_0 t \right) = \tfrac{1}{2} I_M^2 \underline{L}$$

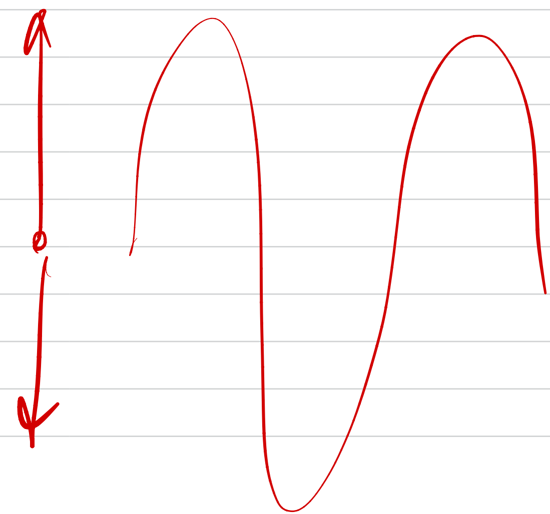
$$L = \frac{1}{\omega_0^2 C}$$

$$w_{L,\max} = w_s = \tfrac{1}{2} I_M^2 L \qquad w_{C,\max} = w_s = \tfrac{1}{2} V_M^2 C$$

$\frac{1}{2} k x_0^2$



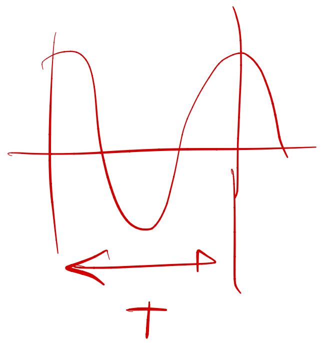
$\frac{1}{2} m v^2$



Lossy Case

- Now let's introduce loss. The energy dissipated by the resistor per cycle is given by:

$$w_d = P \cdot T = \left(\frac{1}{2} I_M^2 R\right) \cdot \overbrace{\left(\frac{2\pi}{\omega_0}\right)}^{T}$$



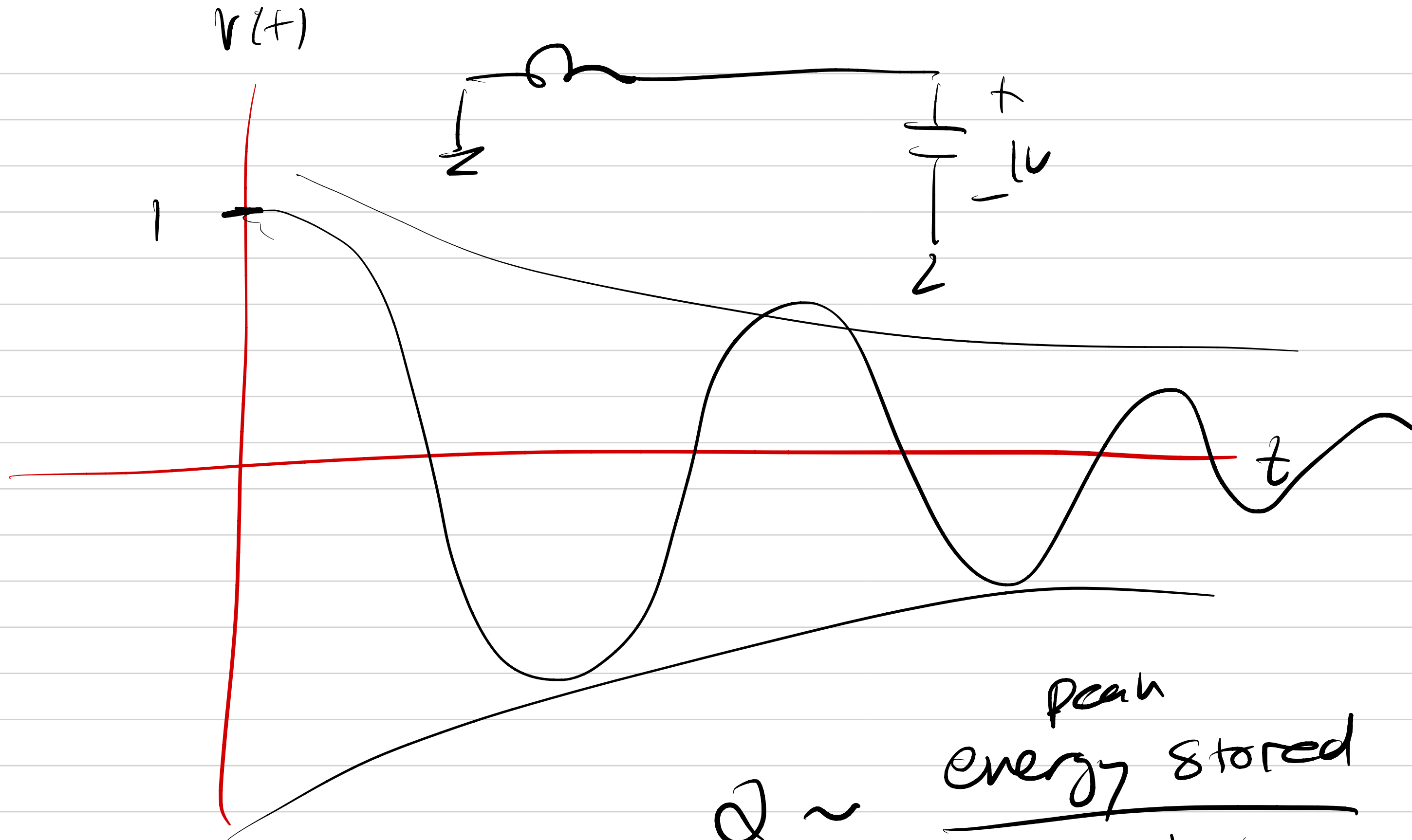
- Comparing the energy lost to the energy stored in the inductor, we have:

$$\frac{w_s}{w_d} = \frac{\cancel{\frac{1}{2}} L \cancel{I_M^2}}{\cancel{\frac{1}{2}} \cancel{I_M^2} R \frac{2\pi}{\omega_0}} = \left(\frac{\omega_0 L}{R}\right) \frac{1}{2\pi} = \frac{Q}{2\pi}$$

$$\boxed{Q = 2\pi \frac{w_s}{w_d} = \frac{1}{2\zeta}}$$

$$\zeta = \frac{1}{2} RC = \frac{1}{2} \frac{R}{L\omega_0^2} = \frac{1}{2} \frac{R}{\omega_0 L} \cdot \left(\frac{1}{\omega_0}\right)$$

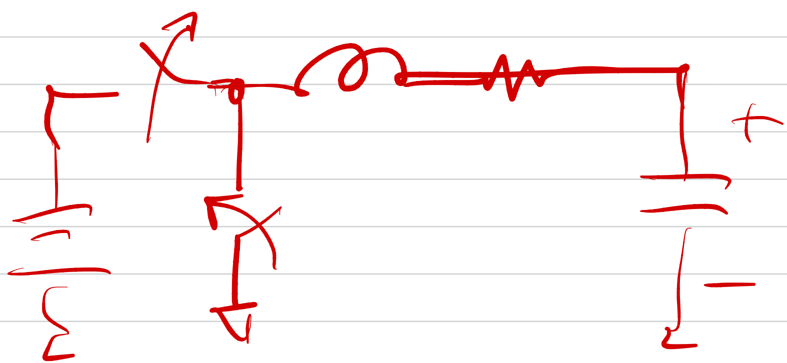
$$\omega_0^2 = \frac{1}{LC} \qquad C = \frac{1}{L\omega_0^2}$$

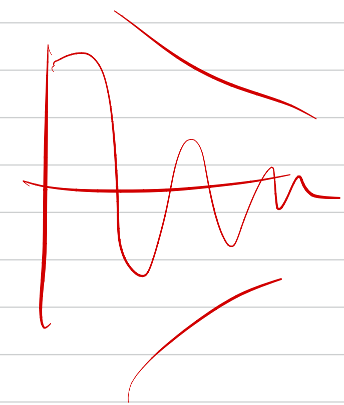


$$Q \sim \frac{\text{peak energy stored}}{\text{energy lost per cycle}}$$

$$P_S \simeq \frac{1}{2} L I_{peak}^2$$

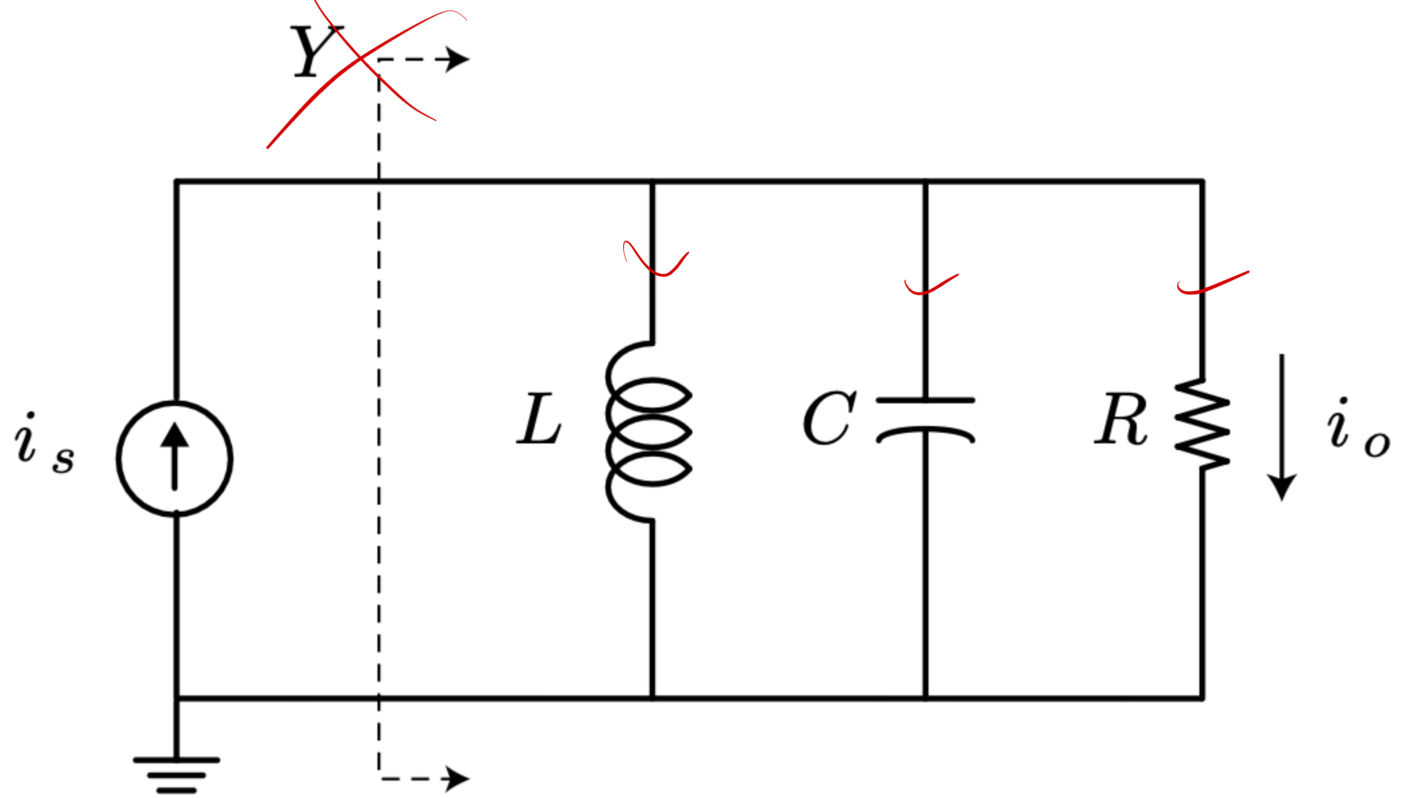
$$P_L = \frac{1}{2} I_{peak}^2 \cdot R \cdot \alpha$$

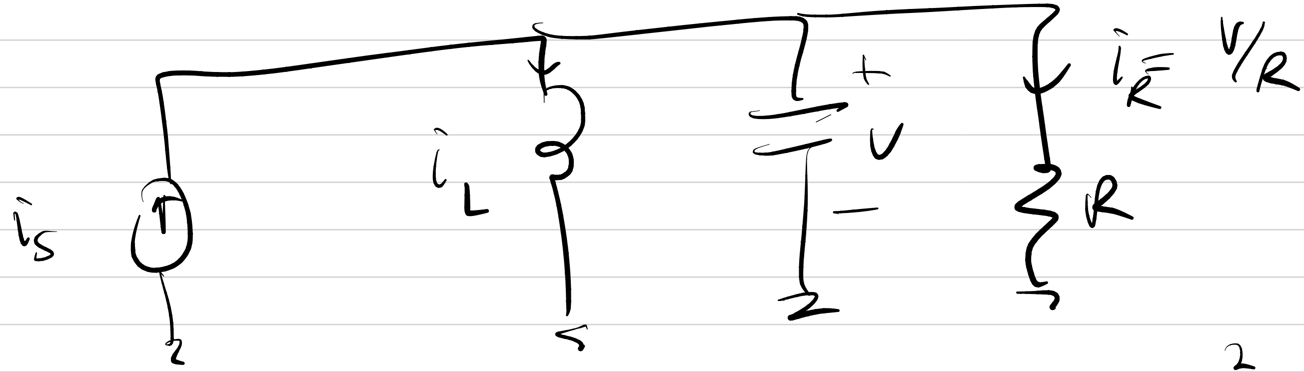




Parallel LCR

- Using the concept of "duality", we expect the equations to take on the exam same form as before:





$$i_s' = i_L' + C\left(\frac{dv}{dt}\right)' + \frac{1}{R} v'$$

$$\omega_0^2 = \frac{1}{LC}$$

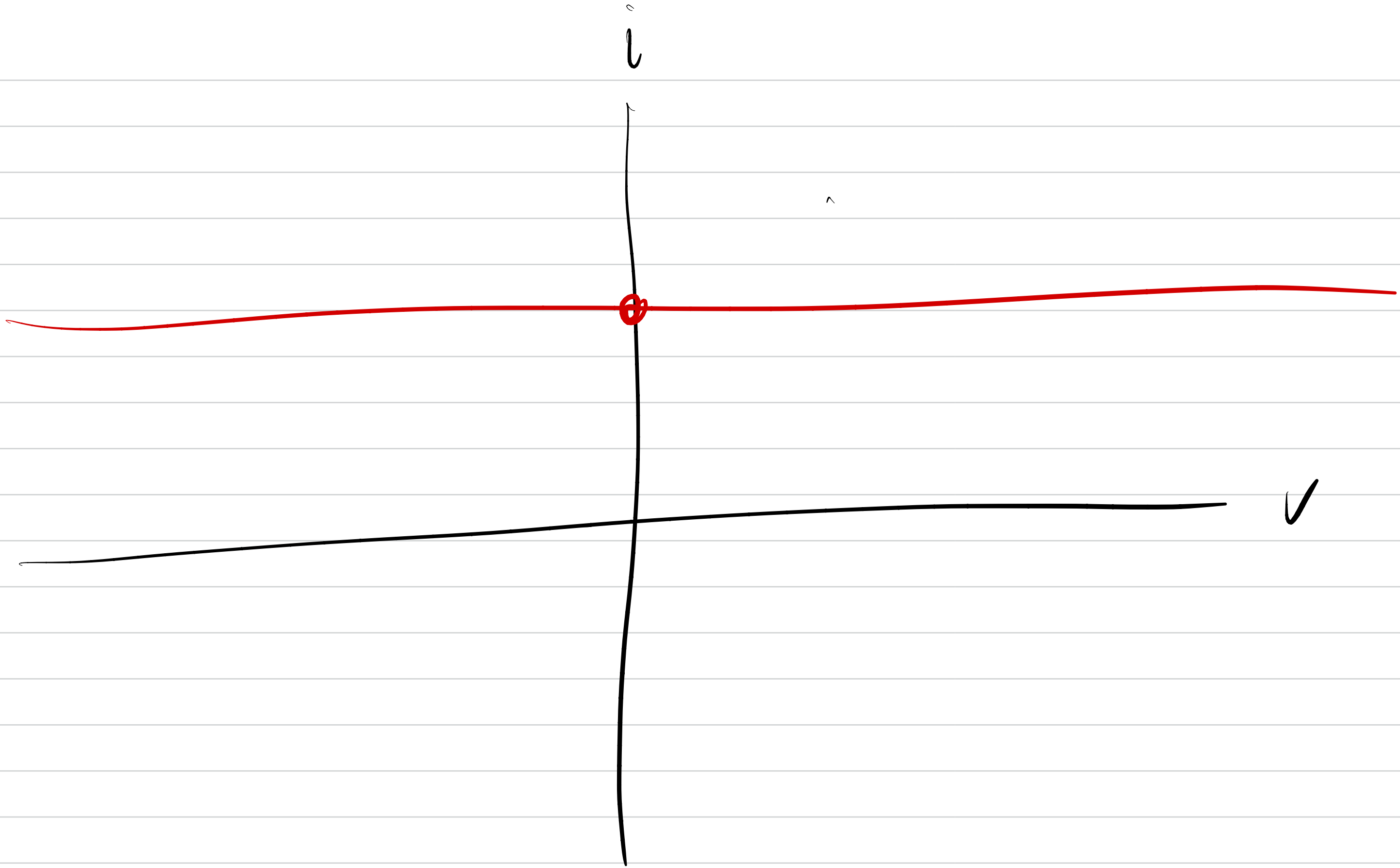
$$v = L \frac{di_L}{dt}$$

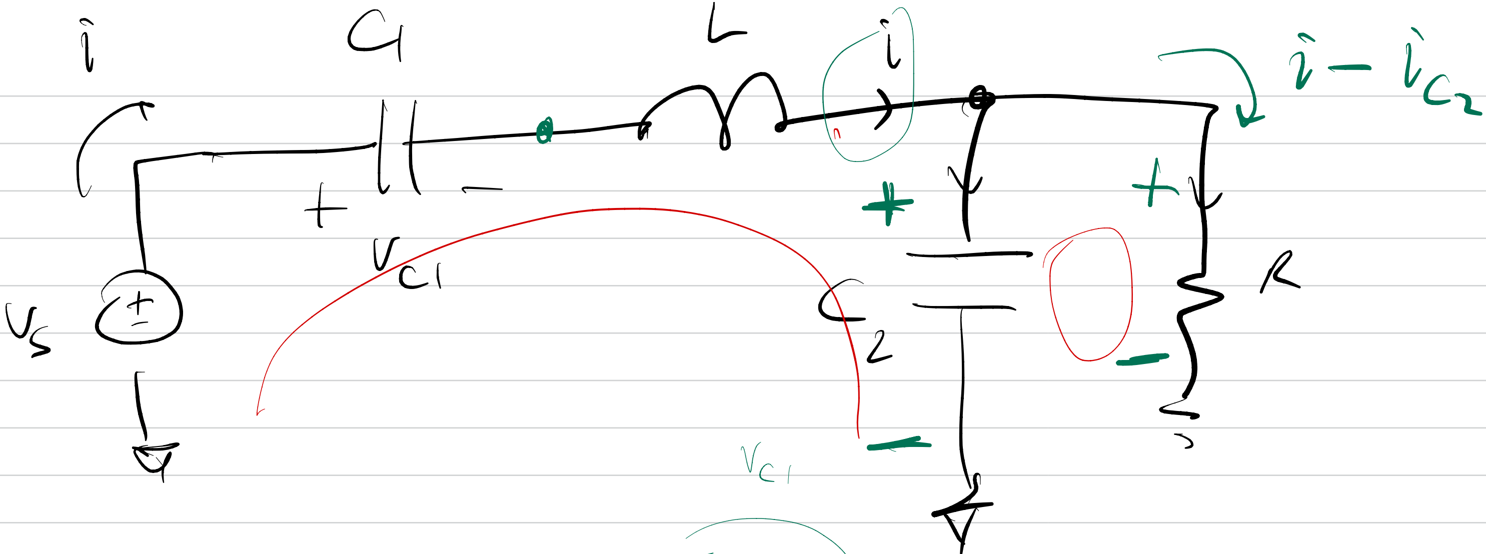
$\zeta \neq$ same

$$\boxed{i_s' = \frac{v}{L} + C \frac{d^2v}{dt^2} + \frac{1}{R} \frac{dv}{dt}}$$

$$Q = \frac{R}{\omega_0 L}$$

$$= \omega_0 RC$$


i
V



$$\begin{cases} V_s = v_{C1} + L\dfrac{di}{dt} + v_{C2} \\ i = C_2 \dfrac{dv_{C2}}{dt} + \dfrac{V_{C2}}{R} \\ i' = C_1 \left(\dfrac{dv_{C1}}{dt}\right)' \end{cases}$$

3 eq

3 unk

$$v_s = v_{C1} + L\frac{di}{dt} + v_{C2}$$

$$i = C_2 \frac{dv_{C2}}{dt} + \frac{v_{C2}}{R}$$

$$i = C_1 \frac{dv_{C1}}{dt}$$

$$\vec{x} = \begin{pmatrix} v_{C1} \\ v_{C2} \\ i \end{pmatrix}$$

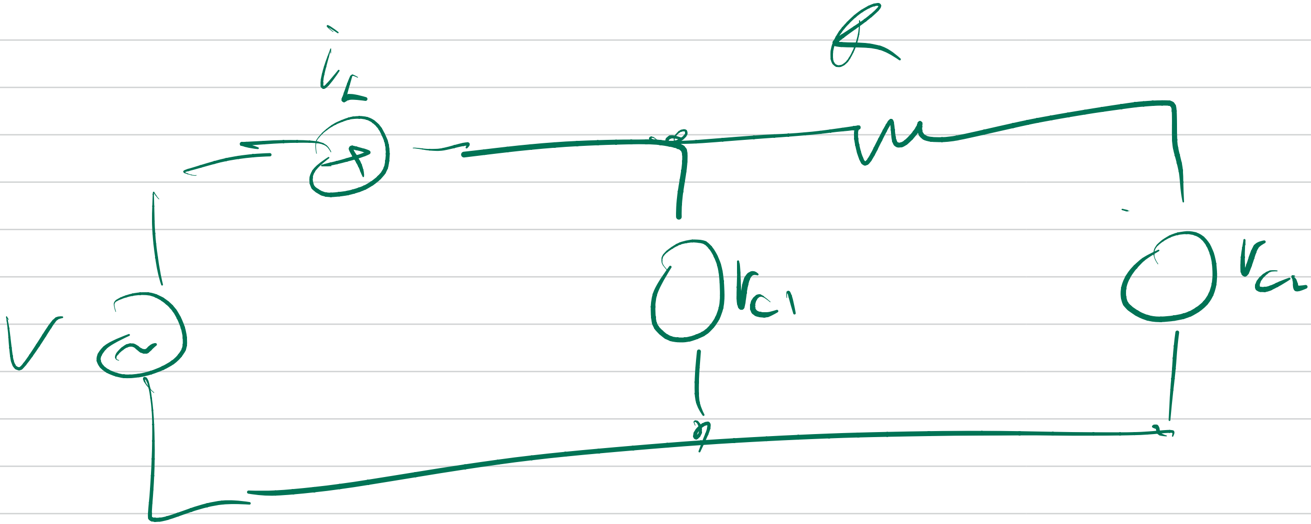
State of System

$$\frac{dv_{C1}}{dt} = \frac{1}{C_1} i$$

$$\frac{dv_{C2}}{dt} = -\frac{1}{C_2} i - \frac{1}{C_2 R} v_{C2}$$

$$\frac{di}{dt} = +\frac{v_s}{L} - \frac{v_{C1}}{L} - \frac{v_{C2}}{L}$$

$$\frac{d}{dt}\vec{x} = \begin{pmatrix} 0 & 0 & \frac{1}{C_1} \\ 0 & -\frac{1}{C_2 R} & +\frac{1}{C_2} \\ -\frac{1}{L} & -\frac{1}{L} & 0 \end{pmatrix} \begin{pmatrix} v_{C1} \\ v_{C2} \\ i \end{pmatrix} + \frac{1}{L}\begin{pmatrix} 0 \\ 0 \\ v_s \end{pmatrix}$$



i_L
R
V
v_{C1}
v_{C2}