EECS 16B

Designing Information Devices and Systems II

Prof. Ali Niknejad and Prof. Kannan Ramchandran

Department of Electrical Engineering and Computer Sciences, UC Berkeley,
niknejad@berkeley.edu

Module 6: Change of Basis

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Where do you live?

- In a 2D plane, we only need to specify three points:
  - Origin (Downtown Library)
  - Location 1 (Cream on Telegraph)
  - Location 2 (Cory Hall)
- Equivalently, we need to specify two vectors
- As long as the vectors are not co-linear, we can specify any other location
- In other words, the vectors must be linearly independent
  - $\overrightarrow{v_1} \neq a\overrightarrow{v_2}$

Generalize to 3D and Beyond

- Any N vectors that are linearly independent can form a basis in an N dimensional space.

Standard Basis

- $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_N$ are just one representation of mutually orthogonal vectors.
- Note that even the standard basis is not unique as we can easily rotate the standard basis to find a new basis

Sather Gate

- Here we take $\hat{e}_1$ and $\hat{e}_2$ to be North and East
- Which directions are better?
- Both get you to the same location, so they are equivalent.

How to change basis?

- Say you know a location using basis set $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots$ but you're talking to someone who has no idea where Cory Hall or another other place in Berkeley resides
- What to do? Tell them to pull out a compass and walk along e_1, e_2 instead
- Since the destination is the same, we have:

Matrix Product: Standard Basis

- Recall that a matrix-vector multiplication is just a linear combination of the columns of a matrix.
- If we form two matrices, we have:
- But $E = I$, so we have:
- Likewise, since V has independent columns

Non-Standard Basis

- In the previous calculation one basis was the standard basis
- What if we have two non-standard basis vectors A and B

  – Why can we always invert A?
    - The columns of A are linearly independent !

Standard Basis to New Basis

- We found that:

- If B is the standard basis, then $B = I$ and we have

Linear Transformations

- Let's see how a linear transformation in one basis can be written in another basis
- Say L_v is a linear transformation (mapping) from $\vec{x}$ to $\vec{y}$ both represented in basis A
- In other words, for all x we have
- To represent L_v as a matrix, we note that $\vec{x}$ can be written as

Transformation Matrix

- In other words, the matrix columns are the result of applying the transformation to $\overrightarrow{a_1}$, $\overrightarrow{a_2}$ …
- We form a matrix of columns of $L_A = L_v(\overrightarrow{a_k})$

Transformation in Another Basis

- Now we can always use another basis B, and define L_B in the same way
- Since both transformations applied to the same input must map to the same point, we can say

Manipulating…

- Moving vectors from basis A to B:

- Since the matrix is invertible:

- The mappings are therefore related as follows:

Diagonalizing Basis

- Is there a natural basis ? Earlier we argued that all basis vectors were equivalent but with respect to a particular linear mapping, what's the best basis ?
- If a particular basis turns a transformation into a diagonal matrix, we say that the basis is the simplest basis for a transformation T
- Recall the definition of an eigenvector:
  - $T\vec{v} = \lambda\vec{v}$
  - where the eigenvector is $\vec{v}$ and eigenvalue is λ

Eigenvector Basis

- If the set of eigenvectors forms a basis (linearly independent), we know that the transformation is invertible, and we can use the eigenvectors as a basis.
- Let's perform a transformation using the eigenfunction basis vectors:
- We see the result is very simple as each vector is mapped to itself times a constant

Matrix Form of Eigenvector Basis

- Let's rewrite:
  - $T\,\vec{u} = \mathrm{u}_1\lambda_1\vec{\mathrm{v}}_1 + \mathrm{u}_2\lambda_2\vec{\mathrm{v}}_2 + \mathrm{u}_3\lambda_3\vec{\mathrm{v}}_3 + \ldots$
- As:

- Here we form the eigenvector matrix Q, diagonal matrix Λ

Eigenvector Basis (cont)

- So far, we have found that
- Now let's write $\vec{u}$ in terms of the standard basis:
- The transformation:

Deconvolving a Linear Mapping

- How is this useful? Suppose we want to solve

- Recall that $\vec{u}$ is the vector $\vec{u}$ in the basis of eigenvectors of T

- Like $Q^{-1}\vec{b}$ is simply $\vec{b}$ transformed into the eigenspace

The World's Easiest Matrix To Invert

- If we know the eigenvectors of a linear transform, it can save us a ton of work.
- Instead of solving a difficult $n \times n$ problem we solve just n equations:

- Λ is a diagonal matrix !
- Clearly the eigenvector basis is the most "natural" for solving problems involving a linear transform T

Use the best "map"

- Summarizing what we have learned, we can say that if we change our perspective (basis) and view things in terms of the eigenspace, then a linear transformation is very simple and only involves multiplying by a diagonal matrix
- Can we apply this to circuit and other dynamic systems described by linear differential equations … ?

Power to the Matrix

- Consider the n'th power of a matrix:

- Note that raising a diagonal matrix is trivial !

Preview: Frequency Domain Basis

- We have found that complex exponentials pretty much solve any homogeneous constant coefficient differential equations
- These equations arise from time-invariant circuits and in general time-invariant systems
  - The component values don't change with time
- Another way to see this is to say that the complex exponential is an *eigenfunction* of a linear dynamic system !
- Does it form a basis ?
  - Yes, in fact it forms a continuous orthogonal basis

Preview (cont)

- The Fourier Transform and Laplace Transform (EE 120) take you from the time domain to the frequency domain, in other words they are a change of basis operation
- If we solve problems in the frequency domain, these complicated differential equations collapse into simple "diagonal" systems that we can solve …
  - "Convolution" turns into multiplication