EECS 16B

Designing Information Devices and Systems II

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Module 6: Change of Basis

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Where do you live?

- In a 2D plane, we only need to specify three points:
  - Origin (Downtown Library)
  - Location 1 (Cream on Telegraph)
  - Location 2 (Cory Hall)
- Equivalently, we need to specify two vectors
- As long as the vectors are not co-linear, we can specify any other location
- In other words, the vectors must be linearly independent
  - $\overrightarrow{v_1} \neq a\overrightarrow{v_2}$

Generalize to 3D and Beyond

- Any *N* vectors that are linearly independent can form a basis in an *N* dimensional space.

Standard Basis

- $\hat{e}_1, \hat{e}_2, \dots, \hat{e}_N$ are just one representation of mutually orthogonal vectors.
- Note that even the standard basis is not unique as we can easily rotate the standard basis to find a new basis

Sather Gate

- Here we take $\hat{e}_1$ and $\hat{e}_2$ to be North and East
- Which directions are better?
- Both get you to the same location, so they are equivalent.

How to change basis?

- Say you know a location using basis set $\overrightarrow{v_1}, \overrightarrow{v_2}$, ... but you're talking to someone who has no idea where Cory Hall or another other place in Berkeley resides
- What to do? Tell them to pull out a compass and walk along e_1, e_2 instead
- Since the destination is the same, we have:

$$\vec{z} = x_1 \hat{e}_1 + x_2 \hat{e}_2 = y_1 \vec{v}_1 + y_2 \vec{v}_2$$

$\vec{v}_1 \neq k \vec{v}_2$ $\quad$ $\vec{v}_1$ & $\vec{v}_2$ linearly independent

$$x_1 \hat{e}_1 + x_2 \hat{e}_2 + \cdots + x_n \hat{e}_n = I\vec{x}$$

$$I\vec{x} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & & 1 \end{bmatrix} = I\vec{x}$$

$$I\vec{x} = y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n$$

$$I\vec{x} = V\vec{y}$$

$$\boxed{\vec{y} = V^{-1}\vec{x}}$$

Transformation Matrix

$$\vec{z} = A \vec{x}_A = B \vec{x}_B$$

$$A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n]$$

$$B = [\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_n]$$

$$\vec{x}_B = \underbrace{B^{-1} A} \vec{x}_A$$

$$= T_{BA} \vec{x}_A$$

Matrix Product: Standard Basis

- Recall that a matrix-vector multiplication is just a linear combination of the columns of a matrix.
- If we form two matrices, we have:
- But $E = I$, so we have:
- Likewise, since V has independent columns

Non-Standard Basis

- In the previous calculation one basis was the standard basis
- What if we have two non-standard basis vectors A and B

  – Why can we always invert A?
    - The columns of A are linearly independent !

Standard Basis to New Basis

- We found that:

- If B is the standard basis, then $B = I$ and we have

Linear Transformations

- Let's see how a linear transformation in one basis can be written in another basis
- Say L_v is a linear transformation (mapping) from $\vec{x}$ to $\vec{y}$ both represented in basis A
- In other words, for all x we have
- To represent L_v as a matrix, we note that $\vec{x}$ can be written as

$$\mathcal{L}\{\vec{v}\} \to \vec{z}$$

$$\vec{v} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + \cdots$$

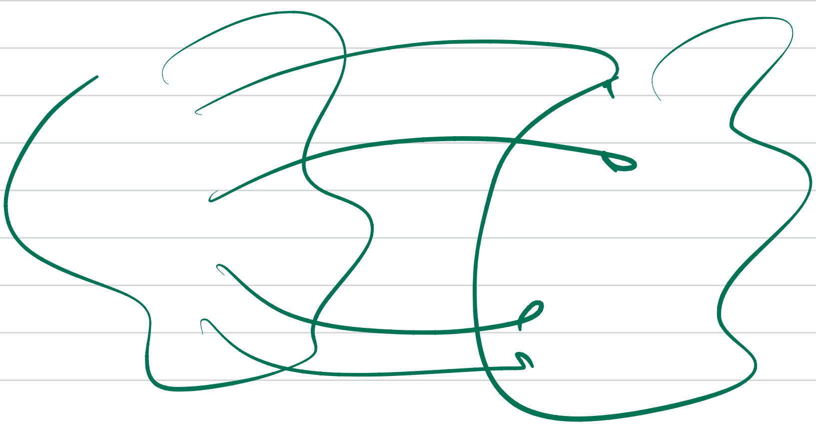
$$\mathcal{L}\{\vec{v}\} = x_1 \underbrace{\mathcal{L}\{\vec{v}_1\}}_{\vec{a}_1} + x_2 \underbrace{\mathcal{L}\{\vec{v}_2\}}_{\vec{a}_2} + \cdots$$

$$= x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + \cdots$$

$$= \underline{A}\vec{x}$$

$\mathcal{L}_{\vec{r}}\{\} = V$

$\mathcal{L}_{\vec{y}}\{\} = Y$



$\mathcal{L}$ same mapping

$\vec{z} = V\vec{r}$

$\vec{r}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix} \neq \alpha \vec{r}_2$