EECS 16B

Designing Information Devices and Systems II

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Module 6: Change of Basis

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Where do you live?

- In a 2D plane, we only need to specify three points:
  - Origin (Downtown Library)
  - Location 1 (Cream on Telegraph)
  - Location 2 (Cory Hall)
- Equivalently, we need to specify two vectors
- As long as the vectors are not co-linear, we can specify any other location
- In other words, the vectors must be linearly independent
  - $\overrightarrow{v_1} \neq a\overrightarrow{v_2}$

Generalize to 3D and Beyond

- Any N vectors that are linearly independent can form a basis in an N dimensional space.

Standard Basis

- $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_N$ are just one representation of mutually orthogonal vectors.
- Note that even the standard basis is not unique as we can easily rotate the standard basis to find a new basis

Sather Gate

- Here we take $\hat{e}_1$ and $\hat{e}_2$ to be North and East
- Which directions are better?
- Both get you to the same location, so they are equivalent.

How to change basis?

- Say you know a location using basis set $\overrightarrow{v_1}, \overrightarrow{v_2}$, … but you're talking to someone who has no idea where Cory Hall or another other place in Berkeley resides
- What to do? Tell them to pull out a compass and walk along e_1, e_2 instead
- Since the destination is the same, we have:

$$\vec{z} = \underline{x_1 \hat{e}_1 + x_2 \hat{e}_2} = y_1 \vec{v}_1 + y_2 \vec{v}_2$$

$$\vec{v}_1 \neq k \vec{v}_2 \qquad \vec{v}_1 \;\&\; \vec{v}_2 \text{ linearly independent}$$

$$x_1 \hat{e}_1 + x_2 \hat{e}_2 + \cdots + x_n \hat{e}_n = I \vec{x}$$

$$I \vec{x} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & & \end{bmatrix} = I \vec{x}$$

$$I \vec{x} = y_1 \vec{v}_1 + y_2 \vec{v}_2 + \cdots + y_n \vec{v}_n$$

$$I \vec{x} = V \vec{y} \qquad \boxed{\vec{y} = V^{-1} \vec{x}}$$

Transformation Matrix

$$\vec{z} = A\,\vec{x}_A = B\,\vec{x}_B$$

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$$

$$B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$$

$$\vec{x}_B = \underbrace{B^{-1} A}\,\vec{x}_A$$

$$= T_{BA}\,\vec{x}_A$$

Matrix Product: Standard Basis

- Recall that a matrix-vector multiplication is just a linear combination of the columns of a matrix.
- If we form two matrices, we have:
- But $E = I$, so we have:
- Likewise, since V has independent columns

Non-Standard Basis

- In the previous calculation one basis was the standard basis
- What if we have two non-standard basis vectors A and B

  – Why can we always invert A?
    - The columns of A are linearly independent !

Standard Basis to New Basis

- We found that:

- If B is the standard basis, then $B = I$ and we have

Linear Transformations

- Let's see how a linear transformation in one basis can be written in another basis
- Say L_v is a linear transformation (mapping) from $\vec{x}$ to $\vec{y}$ both represented in basis A
- In other words, for all x we have
- To represent L_v as a matrix, we note that $\vec{x}$ can be written as

$$\mathcal{L}\{\vec{v}\} \longrightarrow \vec{z}$$

$$\vec{v} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + \cdots$$

$$\mathcal{L}\{\vec{v}\} = x_1\underbrace{\mathcal{L}\{\vec{v}_1\}}_{\vec{a}_1} + x_2\underbrace{\mathcal{L}\{\vec{v}_2\}}_{\vec{a}_2} + \cdots$$

$$= x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + \cdots$$

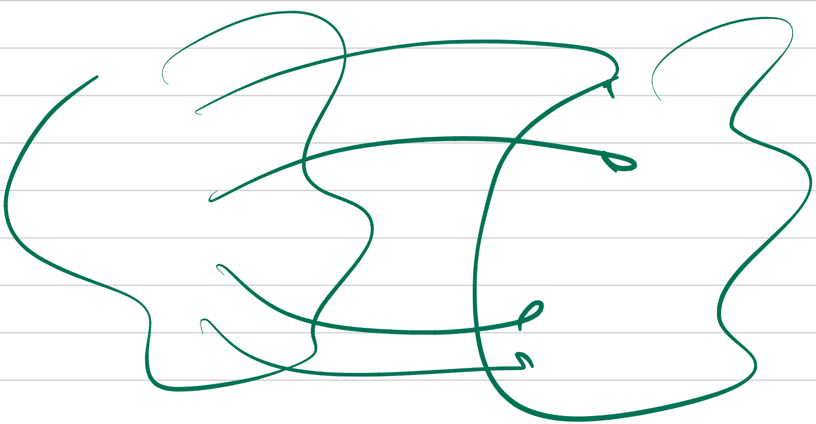
$$= \underline{A}\,\vec{x}$$

$\mathcal{L}_{\vec{r}}\{\} = V$

$\mathcal{L}$ same mapping

$\mathcal{L}_{\vec{y}}\{\} = Y$

$\vec{z} = V\vec{r}$



$\vec{r}_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$

$\begin{pmatrix}0\\\alpha\\0\\\vdots\\0\end{pmatrix} \neq \alpha\vec{r}_2$

Transformation Matrix

- In other words, the matrix columns are the result of applying the transformation to $\overrightarrow{a_1}$, $\overrightarrow{a_2}$ …
- We form a matrix of columns of $L_A = L_v(\overrightarrow{a_k})$

Transformation in Another Basis

- Now we can always use another basis B, and define L_B in the same way
- Since both transformations applied to the same input must map to the same point, we can say

$$\vec{y} = T\vec{x}$$

$$\vec{y}_B = T_B \vec{x}_B$$

T in standard basis

T & T_B should be related

Review

$$\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n$$

Standard basis $\vec{x} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + \cdots$

$$B = [\vec{b}_1 ; \vec{b}_2 ; \cdots ; \vec{b}_n]$$

$$\vec{x}_B = B^{-1} \vec{x}$$

Manipulating…

- Moving vectors from basis ~~*A*~~ I to *B:*

$$\vec{y} = T\vec{x} \qquad B^{-1}\vec{y} = B^{-1}T\vec{x}$$

$$\vec{y}_B = B^{-1}T\vec{x}$$

- Since the matrix is invertible:

$$\vec{X}_B = B^{-1}\vec{x} \qquad \vec{x} = (B\vec{x}_B)$$

- The mappings are therefore related as follows:

$$\vec{y}_B = \underbrace{B^{-1}TB}\,\vec{X}_B$$

T in standard basis

$$T_B = B^{-1}TB$$

T in basis B

T & T_B are "similar matrices"

$$\boxed{T_B = B^{-1} T B}$$

$$B T_B B^{-1} = T B B^{-1}$$

$$\boxed{B T_B B^{-1} = T}$$

Diagonalizing Basis

- Is there a natural basis ? Earlier we argued that all basis vectors were equivalent but with respect to a particular linear mapping, what's the best basis ?
- If a particular basis turns a transformation into a diagonal matrix, we say that the basis is the simplest basis for a transformation T
- Recall the definition of an eigenvector:
  - $T\vec{v} = \lambda\vec{v}$ By definition
  - where the eigenvector is $\vec{v}$ and eigenvalue is λ

$$B^{-1} T B = D \quad \text{diagonal}$$

what basis B turns our transformation T into a diagonal matrix?

Eigenvector Basis

- If the set of eigenvectors forms a basis (linearly independent), we know that the transformation is invertible, and we can use the eigenvectors as a basis: $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$
- Let's perform a transformation using the eigenfunction basis vectors:

$$T\vec{u} = T(u_1\vec{v}_1 + u_2\vec{v}_2 + \cdots)$$

$$= u_1 T\vec{v}_1 + u_2 T\vec{v}_2 + \cdots$$

$$= u_1\lambda_1\vec{v}_1 + u_2\lambda_2\vec{v}_2 + \cdots$$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

- We see the result is very simple as each vector is mapped to itself times a constant

Matrix Form of Eigenvector Basis

- Let's rewrite:
  - $T\,\vec{u} = \mathrm{u}_1\lambda_1\vec{\mathrm{v}}_1 + \mathrm{u}_2\lambda_2\vec{\mathrm{v}}_2 + \mathrm{u}_3\lambda_3\vec{\mathrm{v}}_3 + \ldots$
- As:

$$Q = [\vec{v}_1 ; \vec{v}_2 ; \cdots]$$

Q^{-1} transformation matrix from standard basis to eigen-basis

$$T\vec{u} = Q \Lambda \vec{u}$$

$$= Q \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \vec{u}$$

- Here we form the eigenvector matrix Q, diagonal matrix Λ

$$\Lambda \vec{u} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \lambda_1 u_1 \\ \lambda_2 u_2 \\ \vdots \\ \lambda_n u_n \end{pmatrix}$$

$$Q \Lambda \vec{u} = Q \begin{pmatrix} \lambda_1 u_1 \\ \vdots \\ \lambda_n u_n \end{pmatrix}$$

$$= \vec{r}_1 \lambda_1 u_1 + \vec{r}_2 \lambda_2 u_2 + \cdots$$

Eigenvector Basis (cont)

- So far, we have found that

$$T\vec{u} = Q\Lambda(\vec{u})$$

- Now let's write $\vec{u}$ in terms of the standard basis:

$$\vec{u} = Q^{-1}\vec{x}$$

- The transformation: $T\vec{u} = TQ^{-1}\vec{x} = \underbrace{Q\Lambda Q^{-1}}\vec{x}$

In the new basis T is Λ

Deconvolving a Linear Mapping

- How is this useful? Suppose we want to solve

$$T\vec{x} = \vec{b} \qquad Q\Lambda \underbrace{Q^{-1}\vec{x}} = \vec{b}$$

- Recall that $\vec{u}$ is the vector $\vec{u}$ in the basis of eigenvectors of T

$$\vec{u} = Q^{-1}\vec{x} \qquad Q\Lambda\vec{u} = \vec{b}$$

- Like $Q^{-1}\vec{b}$ is simply $\vec{b}$ transformed into the eigenspace

$$\underbrace{Q^{-1}Q}_{I}\Lambda\vec{u} = Q^{-1}\vec{b} \qquad \vec{\tilde{b}} \text{ is } \vec{b} \text{ in eigen-basis}$$

$$\boxed{\Lambda\vec{u} = \vec{\tilde{b}}}$$

$$\begin{tikzcd} \vec{x} \arrow[r, bend left, "Q^{-1}"] & \vec{u} \arrow[l, bend left, "Q"] \end{tikzcd}$$

The World's Easiest Matrix To Invert

- If we know the eigenvectors of a linear transform, it can save us a ton of work.
- Instead of solving a difficult $n \times n$ problem we solve just n equations:

$$\Lambda^{-1} = \begin{pmatrix} 1/\lambda_1 & & 0 \\ & \ddots & \\ 0 & & 1/\lambda_n \end{pmatrix}$$

- Λ is a diagonal matrix !
- Clearly the eigenvector basis is the most "natural" for solving problems involving a linear transform T

Use the best “map”

- Summarizing what we have learned, we can say that if we change our perspective (basis) and view things in terms of the eigenspace, then a linear transformation is very simple and only involves multiplying by a diagonal matrix
- Can we apply this to circuit and other dynamic systems described by linear differential equations … ?

Power to the Matrix

- Consider the n'th power of a matrix:

$$A^n = A \cdots A = (Q \Lambda \underbrace{Q^{-1})(Q}_{I} \Lambda Q^{-1}) \cdots (Q \Lambda Q^{-1})$$

$$= Q \Lambda I \Lambda Q^{-1} \cdots$$

$$= Q \Lambda^2 \underbrace{Q^{-1}(Q}_{I} \Lambda Q^{-1}) \cdots \underbrace{(Q}_{I} \Lambda Q^{-1})$$

- Note that raising a diagonal matrix is trivial ! $= Q \Lambda^n Q^{-1}$

Preview: Frequency Domain Basis

- We have found that complex exponentials pretty much solve any homogeneous constant coefficient differential equations
- These equations arise from time-invariant circuits and in general time-invariant systems
  - The component values don't change with time
- Another way to see this is to say that the complex exponential is an *eigenfunction* of a linear dynamic system !
- Does it form a basis ?
  - Yes, in fact it forms a continuous orthogonal basis

Preview (cont)

- The Fourier Transform and Laplace Transform (EE 120) take you from the time domain to the frequency domain, in other words they are a change of basis operation
- If we solve problems in the frequency domain, these complicated differential equations collapse into simple "diagonal" systems that we can solve …
  - "Convolution" turns into multiplication