EECS 16B

Designing Information Devices and Systems II

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Module 6: Change of Basis

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Where do you live?

- In a 2D plane, we only need to specify three points:
  - Origin (Downtown Library)
  - Location 1 (Cream on Telegraph)
  - Location 2 (Cory Hall)
- Equivalently, we need to specify two vectors
- As long as the vectors are not co-linear, we can specify any other location
- In other words, the vectors must be linearly independent
  - $\overrightarrow{v_1} \neq a\overrightarrow{v_2}$

Generalize to 3D and Beyond

- Any *N* vectors that are linearly independent can form a basis in an *N* dimensional space.

Standard Basis

- $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_N$ are just one representation of mutually orthogonal vectors.
- Note that even the standard basis is not unique as we can easily rotate the standard basis to find a new basis

Sather Gate

- Here we take $\hat{e}_1$ and $\hat{e}_2$ to be North and East
- Which directions are better?
- Both get you to the same location, so they are equivalent.

How to change basis?

- Say you know a location using basis set $\overrightarrow{v_1}, \overrightarrow{v_2}$, … but you're talking to someone who has no idea where Cory Hall or another other place in Berkeley resides
- What to do? Tell them to pull out a compass and walk along e_1, e_2 instead
- Since the destination is the same, we have:

$$-x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + x_3\overrightarrow{v_3} + \cdots = y_1\ \hat{e}_1 + y_2\hat{e}_2 + y_3\hat{e}_2 + \cdots$$

Matrix Product: Standard Basis

- Recall that a matrix-vector multiplication is just a linear combination of the columns of a matrix.
- If we form two matrices, we have:
  - $\mathrm{V}\,\vec{x} = E\,\vec{y}$
- But $E = I$, so we have:
  - $\vec{y} = \mathrm{V}\,\vec{x}$
- Likewise, since V has independent columns
  - $\vec{x} = \mathrm{V}^{-1}\,\vec{y}$

Non-Standard Basis

- In the previous calculation one basis was the standard basis
- What if we have two non-standard basis vectors A and B
  - $\vec{p} = \mathrm{A}\,\vec{\mathrm{a}}$ and $\vec{p} = \mathrm{B}\,\vec{b}$
  - $\mathrm{A}\,\vec{a} = B\,\vec{b}$
  - $\vec{a} = \mathrm{A}^{-1}\mathrm{B}\,\vec{b} = T_{AB}\vec{b}$
  - Why can we always invert A?
    - The columns of A are linearly independent !

Standard Basis to New Basis

- We found that:
  - $\vec{a} = A^{-1}B\,\vec{b}$
- If B is the standard basis, then $B = I$ and we have
  - $\vec{a} = A^{-1}I\,\vec{b} = A^{-1}\,\vec{b}$

Linear Transformations

- Let's see how a linear transformation in one basis can be written in another basis
- Say L_v is a linear transformation (mapping) from $\vec{x}$ to $\vec{y}$ both represented in basis A
- In other words, for all x we have
  - $\vec{y} = L_v(\vec{x})$
- To represent L_v as a matrix, we note that $\vec{x}$ can be written as
- $\vec{x} = x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + x_3\overrightarrow{a_3} + \cdots$
- $\vec{y} = L_v(\vec{x}) = x_1 L_v(\overrightarrow{a_1}) + x_2 L_v(\overrightarrow{a_2}) + \cdots = A\vec{x}$

Transformation Matrix

- In other words, the matrix columns are the result of applying the transformation to $\overrightarrow{a_1}$, $\overrightarrow{a_2}$ …
- We form a matrix of columns of $L_A = L_v(\overrightarrow{a_k})$

Transformation in Another Basis

- Now we can always use another basis B, and define L_B in the same way
- Since both transformations applied to the same input must map to the same point, we can say
  - $\vec{x}$ is a vector specified in the standard basis
  - $\vec{y} = L_A\vec{x}$ is also in the standard basis
  - $\vec{z} = L_B\vec{y}$ is in the basis B, but it represents the same point since it's the same transformation $\vec{z} = T_{BA}\vec{y}$
  - $\vec{z} = T_{BA}\vec{y} = L_B T_{BA}\vec{x}$

Manipulating…

- Moving vectors from basis *A* to *B:*
  - $T_{BA}\vec{y} = L_B T_{BA}\vec{x}$
- Since the matrix is invertible:
  - $\vec{y} = T_{BA}^{-1} L_B T_{BA}\vec{x} = L_A\vec{x}$
- The mappings are therefore related as follows:
  - $T_{BA}^{-1} L_B T_{BA} = L_A$
  - $L_B = T_{BA} L_A T_{BA}^{-1}$

Diagonalizing Basis

- Is there a natural basis ? Earlier we argued that all basis vectors were equivalent but with respect to a particular linear mapping, what's the best basis ?
- If a particular basis turns a transformation into a diagonal matrix, we say that the basis is the simplest basis for a transformation *T*
- Recall the definition of an eigenvector:
  - $T\vec{v} = \lambda\vec{v}$
  - where the eigenvector is $\vec{v}$ and eigenvalue is λ

Eigenvector Basis

- If the set of eigenvectors forms a basis (linearly independent), we know that the transformation is invertible, and we can use the eigenvectors as a basis.
- Let's perform a transformation using the eigenfunction basis vectors:
  - $T\,\vec{u} = T(\mathrm{u}_1\vec{\mathrm{v}}_1 + \mathrm{u}_2\vec{\mathrm{v}}_2 + \mathrm{u}_3\vec{\mathrm{v}}_3 + \ldots)$
  - $T\,\vec{u} = \mathrm{u}_1 T\vec{\mathrm{v}}_1 + \mathrm{u}_2 T\vec{\mathrm{v}}_2 + \mathrm{u}_3 T\vec{\mathrm{v}}_3 + \ldots$
  - $T\,\vec{u} = \mathrm{u}_1\lambda_1\vec{\mathrm{v}}_1 + \mathrm{u}_2\lambda_2\vec{\mathrm{v}}_2 + \mathrm{u}_3\lambda_3\vec{\mathrm{v}}_3 + \ldots$
- We see the result is very simple as each vector is mapped to itself times a constant

Matrix Form of Eigenvector Basis

- Let's rewrite:
  - $T\,\vec{u} = \mathrm{u}_1\lambda_1\vec{\mathrm{v}}_1 + \mathrm{u}_2\lambda_2\vec{\mathrm{v}}_2 + \mathrm{u}_3\lambda_3\vec{\mathrm{v}}_3 + \ldots$
- As:
  - $T\,\vec{u} = [\vec{\mathrm{v}}_1\ \vec{\mathrm{v}}_2\ \vec{\mathrm{v}}_3\ \ldots]\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}\vec{\mathrm{u}} = \mathrm{Q}\,\Lambda\,\vec{\mathrm{u}}$
- Here we form the eigenvector matrix Q, diagonal matrix Λ

Eigenvector Basis (cont)

- So far, we have found that
  - $T\,\vec{u} = \mathrm{Q}\,\Lambda\,\vec{\mathrm{u}}$
- Now let's write $\vec{\mathrm{u}}$ in terms of the standard basis:
  - $\vec{\mathrm{u}} = \mathrm{Q}^{-1}\vec{\mathrm{x}}$
  - $T\,\mathrm{Q}^{-1}\vec{\mathrm{x}} = \mathrm{Q}\,\Lambda\,\mathrm{Q}^{-1}\vec{\mathrm{x}}$
- The transformation $T = \mathrm{Q}\,\Lambda\,\mathrm{Q}^{-1}$
- Let's rewrite the output $\vec{z}$ in the eigenvector basis $\overrightarrow{\mathrm{z}'}$:
  - $\vec{z} = T\,\vec{u} = \mathrm{Q}\,\Lambda\,\vec{\mathrm{u}}$
  - $\mathrm{Q}^{-1}\vec{\mathrm{z}} = \Lambda\,\vec{\mathrm{u}} \rightarrow \qquad \overrightarrow{\mathrm{z}'} = \Lambda\,\vec{\mathrm{u}}$

Deconvolving a Linear Mapping

- How is this useful? Suppose we want to solve
  - $T\,\vec{x} = \vec{b}$
  - $\mathrm{Q}\,\Lambda\,\mathrm{Q}^{-1}\,\vec{x} = \vec{b}$
- Recall that $\vec{u}$ is the vector $\vec{u}$ in the basis of eigenvectors of T
  - $\vec{u} = Q^{-1}\vec{x}$
  - $\mathrm{Q}\,\Lambda\,\vec{u} = \vec{b}$
  - $\Lambda\,\vec{u} = Q^{-1}\vec{b}$
- Like $Q^{-1}\vec{b}$ is simply $\vec{b}$ transformed into the eigenspace
- $\Lambda\,\vec{u} = \overrightarrow{b_u}$

The World's Easiest Matrix To Invert

- If we know the eigenvectors of a linear transform, it can save us a ton of work.
- Instead of solving a difficult $n \times n$ problem we solve just n equations:

$$-\Lambda \vec{u} = \overrightarrow{b_u}$$

- Λ is a diagonal matrix !
- Clearly the eigenvector basis is the most "natural" for solving problems involving a linear transform T

Use the best "map"

- Summarizing what we have learned, we can say that if we change our perspective (basis) and view things in terms of the eigenspace, then a linear transformation is very simple and only involves multiplying by a diagonal matrix
- Can we apply this to circuit and other dynamic systems described by linear differential equations … ?

Power to the Matrix

- Consider the n'th power of a matrix:
  - $A^k = (\mathrm{Q}\,\Lambda\,\mathrm{Q}^{-1})^k = \mathrm{Q}\,\Lambda\,\mathrm{Q}^{-1}\,\mathrm{Q}\,\Lambda\,\mathrm{Q}^{-1}\,\mathrm{Q}\,\Lambda\,\mathrm{Q}^{-1} \ldots$
  - $A^k = \mathrm{Q}\,\Lambda^k\,\mathrm{Q}^{-1}$
- Note that raising a diagonal matrix is trival !

Preview: Frequency Domain Basis

- We have found that complex exponentials pretty much solve any homogeneous constant coefficient differential equations
- These equations arise from time-invariant circuits and in general time-invariant systems
  - The component values don't change with time
- Another way to see this is to say that the complex exponential is an *eigenfunction* of a linear dynamic system !
- Does it form a basis ?
  - Yes, in fact it forms a continuous orthogonal basis

Preview

- The Fourier Transform and Laplace Transform (EE 120) take you from the time domain to the frequency domain, in other words they are a change of basis operation
- If we solve problems in the frequency domain, these complicated differential equations collapse into simple "diagonal" systems that we can solve …
  - "Convolution" turns into multiplication