EECS 16B

Designing Information Devices and Systems II

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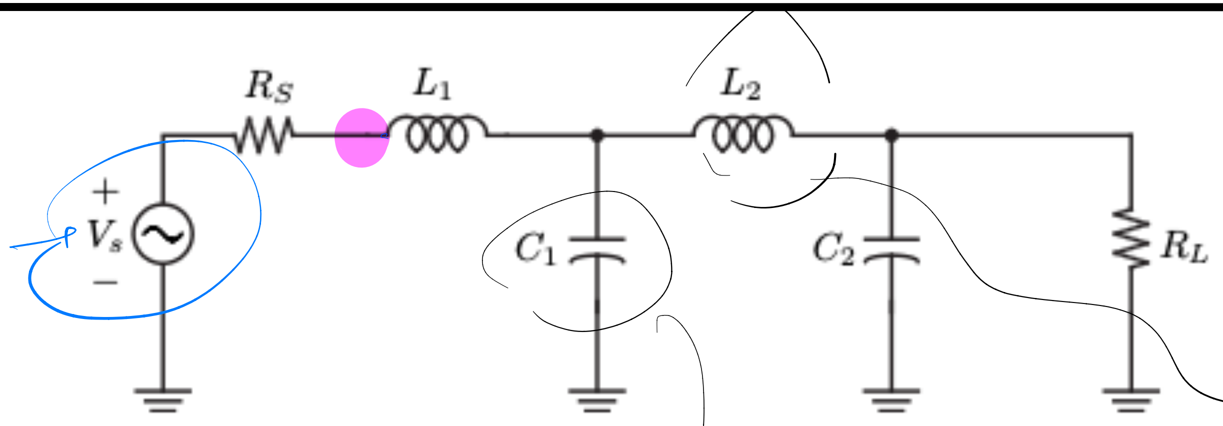
Module 6: Sinusoidal Steady-State Solution to Vector Differential Equations

EECS 16B

Summary

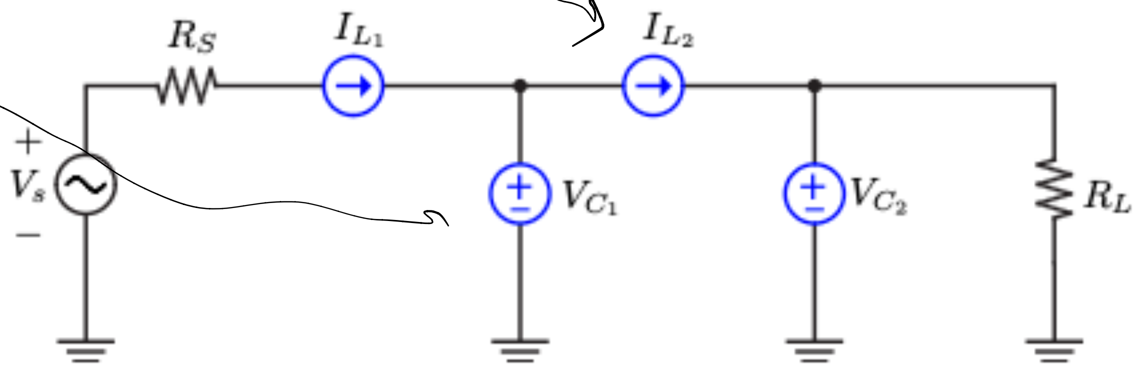
- Solution of VDE in eigenspace with sources
- Steady-state solution of VDE
- Concept of Impedance
- AC Circuits
- Examples

Example Circuit



$$\vec{x} = (V_{C_1}\, V_{C_2}\, I_{L_1}\, I_{L_2})^T$$

$$\frac{d}{dt}\vec{x} = A\vec{x} + B\vec{b}_s$$



$$A = \begin{pmatrix} 0 & 0 & 1/C_1 & -1/C_1 \\ 0 & -1/R_L C_2 & 0 & 1/C_2 \\ -1 & 0 & -R_S/L_1 & 0 \\ 1/L_2 & -1/L_2 & 0 & 0 \end{pmatrix}$$

$B = (0\ 0\ 1\ 0)^T V_s.$

$$\frac{d\vec{x}}{dt} = A\vec{x} + B\vec{b}_s \leftarrow \text{ind. sources}$$

state vector

Numerical Example

L1 = 7410^ − 9; L2 = 114.210^ − 9; C1 = 45.61

Let's see the eigenvalues for this matrix:

Eigenvalues[A]//MatrixForm

4 distict eigenvalues

$$\begin{pmatrix} -6.75676 \times 10^8 + 0.i \\ -1.64354 \times 10^8 + 5.88916 \times 10^8 i \\ -1.64354 \times 10^8 - 5.88916 \times 10^8 i \\ -3.47197 \times 10^8 + 0.i \end{pmatrix}$$

real

λ_k

$\lambda_{k+1} = \lambda_k^*$

complex conjugates

real

Stability

$Re(\lambda_k) < 0$

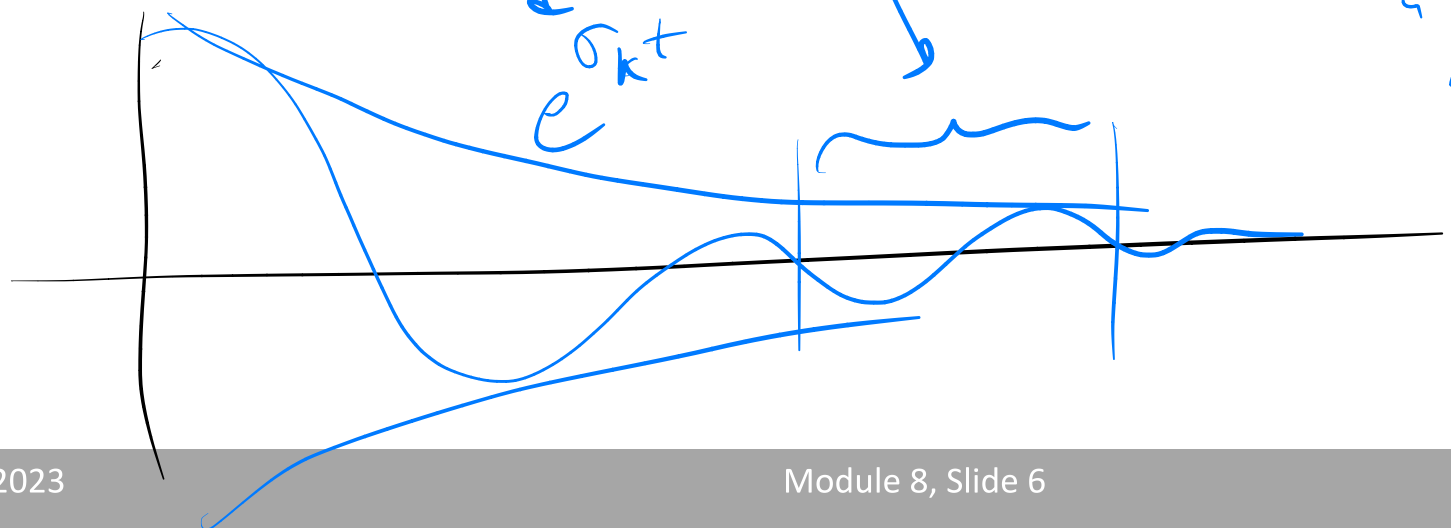
$$\boxed{q_k(t) = q_{k0} e^{\lambda_k t}}$$

λ_k : complex

$$\lambda_k = \sigma_k + j\omega_k$$

$$q_k(t) = q_{k0} e^{\sigma_k t} e^{j\omega_k t}$$

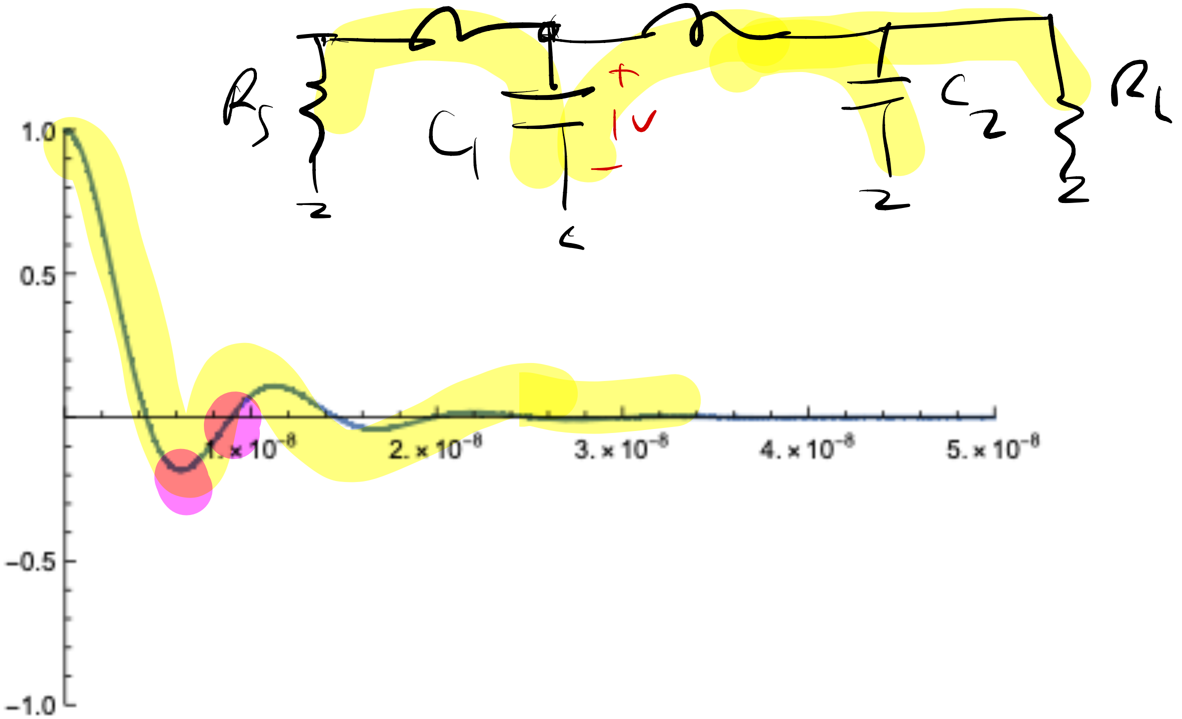
$$= q_{k0} e^{\sigma_k t} (\cos \omega_k t + j \sin \omega_k t)$$



"natural freq"
ω_k

decay rate : σ_k

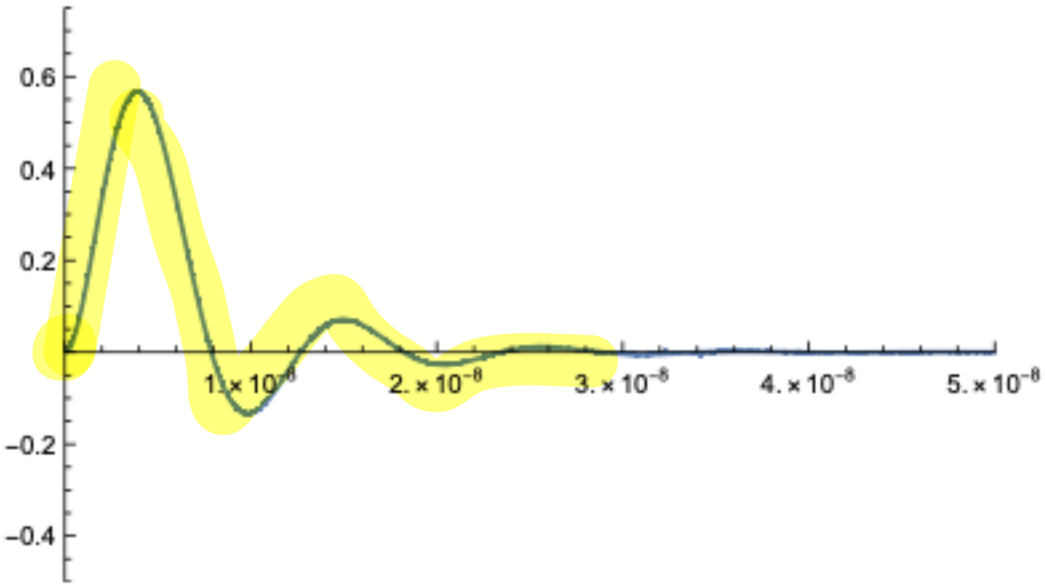
Homogeneous Solution



State 1

$x_1 = v_{C_1}$

$\vec{x}_0 = (1, 0, 0, 0)^T$



State 2

$x_2 = v_{C_2}$

Forced Response

- Since we've reduced the system to simple constant coefficient linear differential equations, we know the forced response !
- Note that the sources are also transformed into the eigenspace through the matrix Q^{-1}

$$q_k^h(t) = q_k^h(0)\, e^{\lambda_k t}$$

$$\boxed{\vec{q}(t) = \vec{q}(0)\, e^{\Lambda t}}$$

Euler Const

Matrix

Notation : $e^{\Lambda} = \begin{bmatrix} e^{\lambda_1} & & & 0 \\ & e^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n} \end{bmatrix}$

Λ : diagonal matrix

$$\vec{x}(t) = Q\, \vec{q}(t)$$

eigenvector matrix

$$\vec{q}(t) = \vec{q}(0)\, e^{\Lambda t}$$

$$\vec{q}(0) = Q^{-1} \vec{X}(0)$$

transformation matrix

initial state

$$\vec{X}(t) = Q\, \vec{q}(t) = \underbrace{Q\, \vec{q}(0)}_{?}\, e^{\Lambda t}$$

Steady-State DC Response

$$\cancel{\frac{d\vec{x}}{dt}} = A\vec{x} + \vec{b}_s$$

$$\lim_{t \to \infty} e^{\lambda t} = \vec{0}$$

All eigenvalues have negative real part

$$0 = A\vec{x} + \vec{b}_s$$

$$\vec{x} = -A^{-1}\vec{b}_s$$

$$= -A^{-1}B\vec{b}_s$$

Review: General Solution to 1st–order ODE

- With constant coefficients, we derived

$$\frac{d\vec{x}}{dt} = \underline{A\vec{x}} + B\vec{b}_s$$

vector of sources

$$\frac{d\vec{x}}{dt} = \underbrace{A\vec{x}}_{Q\Lambda Q^{-1}} + B\vec{b}_s$$

$$\frac{d\vec{x}}{dt} = Q\Lambda Q^{-1}\vec{x} + B\vec{b}_s$$

$$\frac{d(Q^{-1}\vec{x})}{dt} = \Lambda \underbrace{Q^{-1}\vec{x}}_{} + \underbrace{Q^{-1}B\vec{b}_s}_{}$$

$$\boxed{\frac{d\vec{q}}{dt} = \Lambda\vec{q} + \tilde{\vec{b}}_s}$$

Apply Solution to VDE

$$\frac{dq_k}{dt} = \lambda_k q_k + \tilde{b}_{sk}$$

$$e^{-\lambda_k t}\left(\frac{dq_k}{dt} - \lambda_k q_k\right) = \tilde{b}_{sk}\, e^{-\lambda_k t}$$

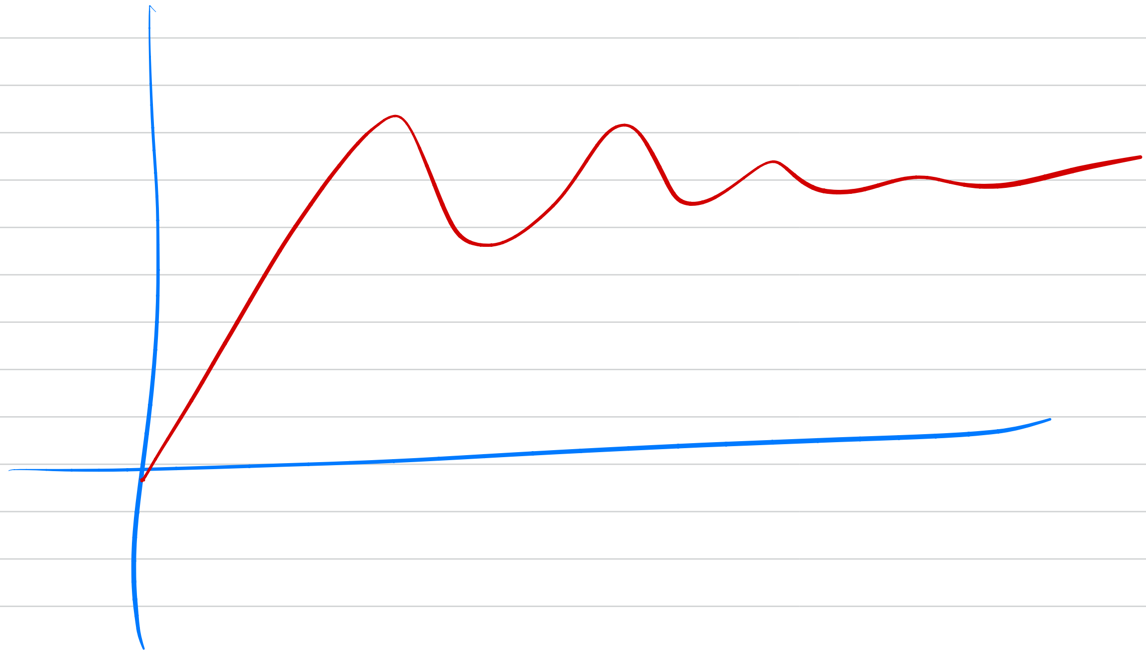
$$\left(e^{-\lambda_k t}\frac{dq_k}{dt} - \underbrace{\lambda_k e^{-\lambda_k t}}\, q_k\right)$$

$$\int \left(q_k(t)\, e^{-\lambda_k t}\right)' ds = \int_{-\infty}^{t} \tilde{b}_{sk}\, e^{-\lambda_k s}\, ds + K_k$$

$$\int \left(q_k(t)\, e^{-\lambda_k t} \right)' ds = \int_{-\infty}^{t} \tilde{b}_{sk}\, e^{-\lambda_k s}\, ds + K_k$$

$$q_k(t) = \underbrace{e^{\lambda_k t} \int_{-\infty}^{t} \hat{\tilde{b}}_{sk}(s)\, e^{-\lambda_k s}\, ds}_{\text{forced solution for zero initial condition}} + \underbrace{e^{\lambda_k t}\, q_k(0)}_{\text{solution to zero input}}$$

$$\vec{q}(t) = e^{-\Lambda t}\, \vec{q}(0) + e^{\Lambda t} \int_{-\infty}^{t} Q^{-1} \vec{b}_s\, e^{-\Lambda s}\, ds$$



$$\vec{q}(t) = e^{\Lambda t}\,\underset{}{\overset{\text{Con}}{\vec{q}(0)}} + e^{\Lambda t}\int_{-\infty}^{t} Q^{-1}\vec{b}_s\, e^{-\Lambda s}\, ds$$

$$\vec{x}(t) = Q\,\vec{q}(t) = \underbrace{Q\, e^{\Lambda t}\, Q^{-1}}\,\vec{x}(0) +$$

$$\underbrace{Q\, e^{\Lambda t}\int_{-\infty}^{t} Q^{-1}\vec{b}_s\, e^{-\Lambda s}\, ds}$$

$$Q\, e^{\Lambda t}\, Q^{-1}\int_{-\infty}^{t}\vec{b}_s\, e^{-\Lambda s}\, ds$$

$$\vec{x}(t) = \underbrace{Q e^{\Lambda t} Q^{-1}}_{e^{At}} \vec{x}(0) + \underbrace{Q e^{\Lambda t} Q^{-1}}\int_{-\infty}^{t} \tilde{b}_s e^{-\Lambda s}\, ds$$

$$\frac{dx}{dt} = ax + b$$

$$\vec{x}(t) = e^{At}\vec{x}(0) + e^{At}\int_{-\infty}^{t} \tilde{b}_s e^{-\Lambda s}\, ds$$

Elegant Notation

- Let's define the matrix exponential
- While for now you can accept this as merely a convenient notation, it turns out to be a more general idea ! (https://youtu.be/O85OWBJ2ayo)

$$e^{At} \triangleq \underbrace{Q e^{\Lambda t} Q^{-1}}$$

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

Forced Response in Steady-State

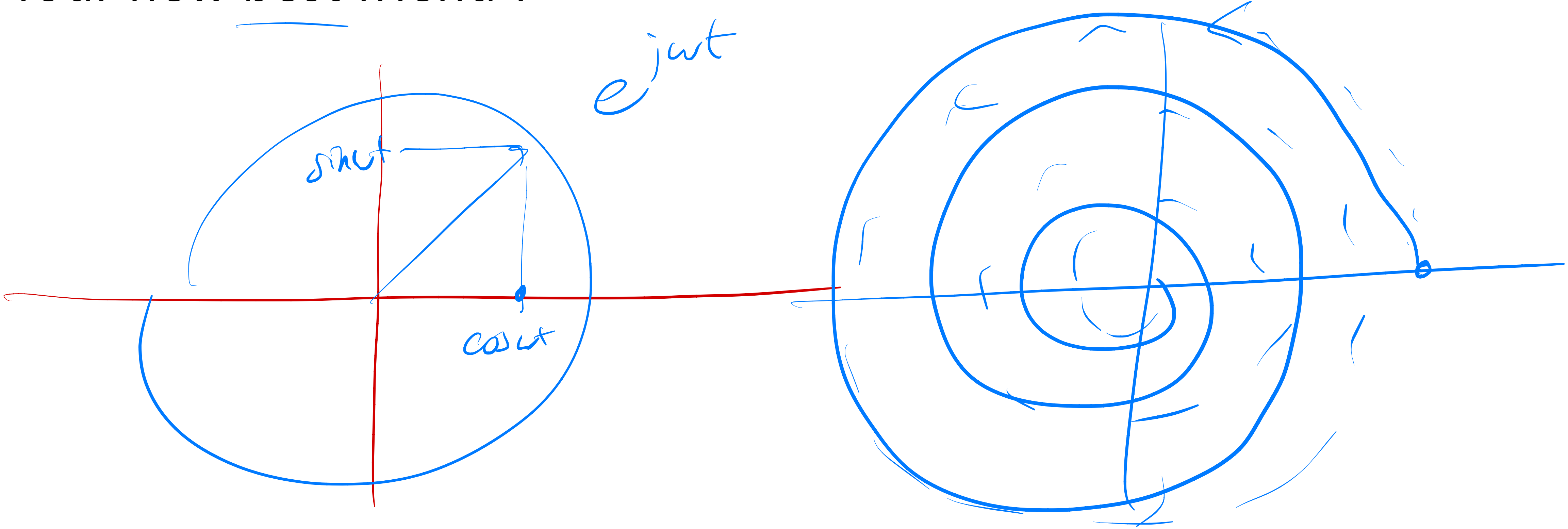
- Assume source is sinusoidal (very common in practice)
  - RF signals essentially sinusoidal on short time scales
  - Musical notes are superpositions of tones (frequencies)
  - Any general waveform can be decomposed into sinusoids ! (Fourier Series and Transform … EE120)

Complex Exponential (again)

- Sine and Cosine are merely shadows of a circle
- Your new best friend !

$e^{-|\sigma|t}\,e^{j\omega t}$

$e^{j\omega t}$



Steady-State Continued

$$q_n(t) = \underbrace{e^{\lambda_n t} q_n(0)}_{0 \; t=\infty} + e^{\lambda_n t}\int_{-\infty}^{t} e^{-\lambda_n t} \underbrace{\tilde{b}_{sn}(s)}\, ds$$

$$\hat{b}_{sn} = Q^{-1} b_s = Q^{-1}\hat{b}_s e^{j\omega t}$$

steady state

$$q_n^f(t) = e^{\lambda_n t}\int_{-\infty}^{t} e^{-\lambda_n s} Q^{-1}\hat{b}_s e^{j\omega s}\, ds$$

$$= e^{\lambda_n t} Q^{-1}\hat{b}_s \int_{-\infty}^{t} e^{(j\omega-\lambda_n)s}\, ds$$

by stability $Re(\lambda_n) < 0$

$$= e^{\lambda_n t} Q^{-1}\hat{b}_s \left\{ \left(\frac{e^{(j\omega-\lambda_n)s}}{j\omega-\lambda_n} \right|_{-\infty}^{t} = \frac{e^{(j\omega-\lambda_n)t}}{j\omega-\lambda_n} - \underbrace{\cancel{\frac{e^{j\omega t} e^{+\lambda_n t}}{j\omega-\lambda_n}}}_{t\to\infty} \right\}$$

$$\tilde{b}_{sk} = b_1 e^{j(\omega t + \phi_1)} + b_2 e^{j(\omega t + \phi_2)} + b_3 e^{j(\omega t + \phi_3)}$$

$$= b_1 e^{j\omega t} e^{j\phi_1} + b_2 e^{j\omega t} e^{j\phi_2} + b_3 e^{j\omega t} e^{j\phi_3}$$

$$= e^{j\omega t} \left(b_1 e^{j\phi_1} + b_2 e^{j\phi_2} + \cdots \right)$$

$$\left(b_1 \cos\phi_1 + b_1 j \sin\phi_1 + b_2 \cos\phi_2 + j b_2 \sin\phi_2 + \cdots \right)$$

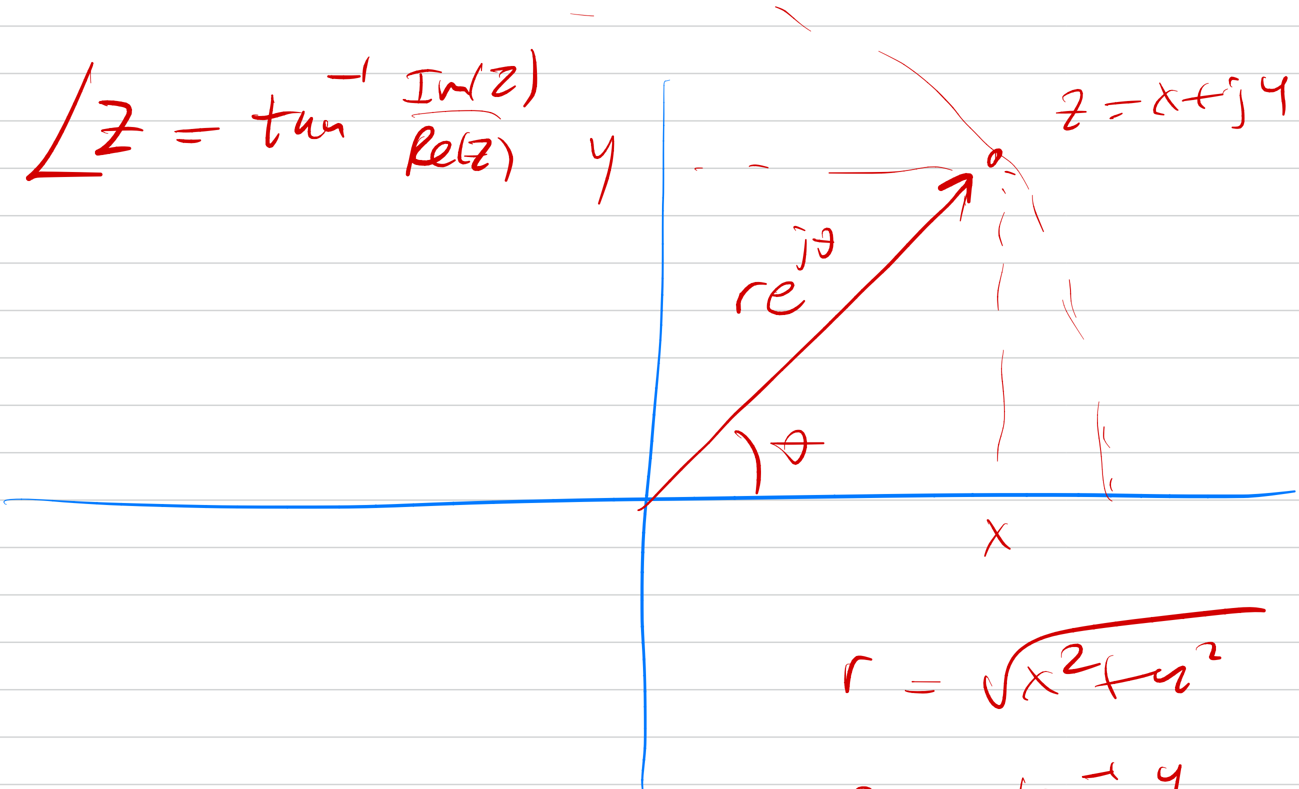
$$= x + jy = r e^{j\theta}$$

$$= \hat{b}_{sk} e^{j\omega t}$$

rotating around circle

$$\hat{b}_{sk} = |\hat{b}_{sk}| e^{j \angle \hat{b}_{sk}}$$

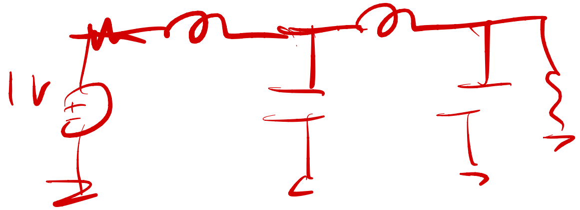
$$\angle Z = \tan^{-1} \frac{Im(z)}{Re(z)}$$

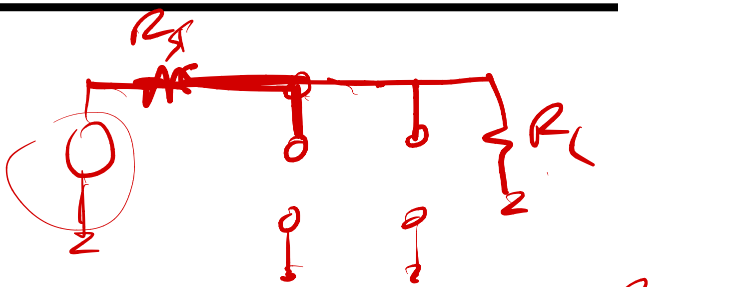


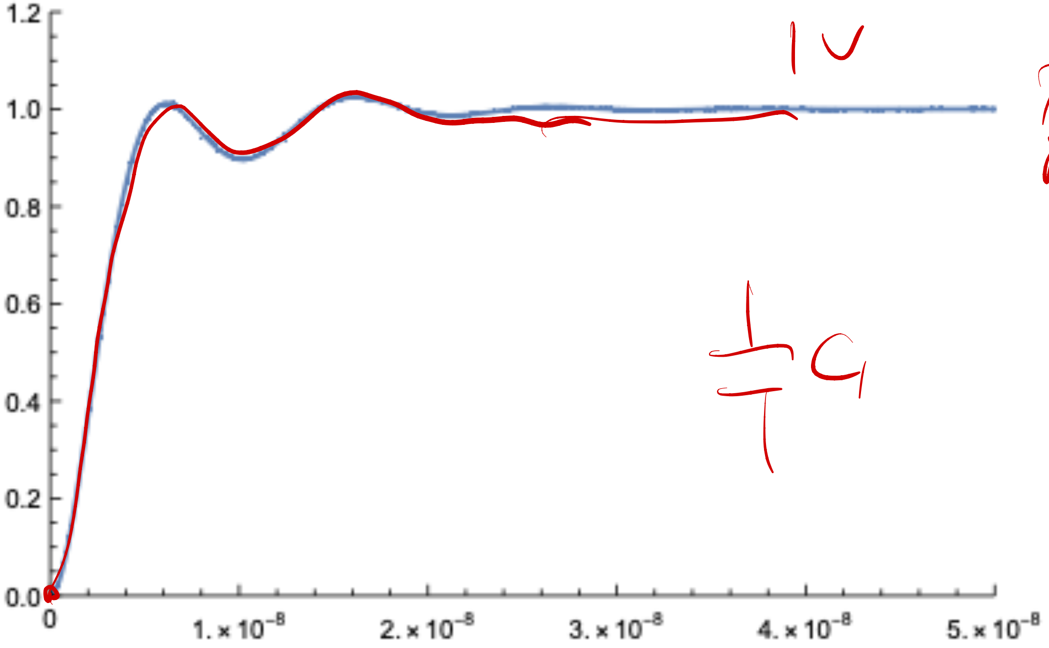
$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1} \frac{y}{x}$$

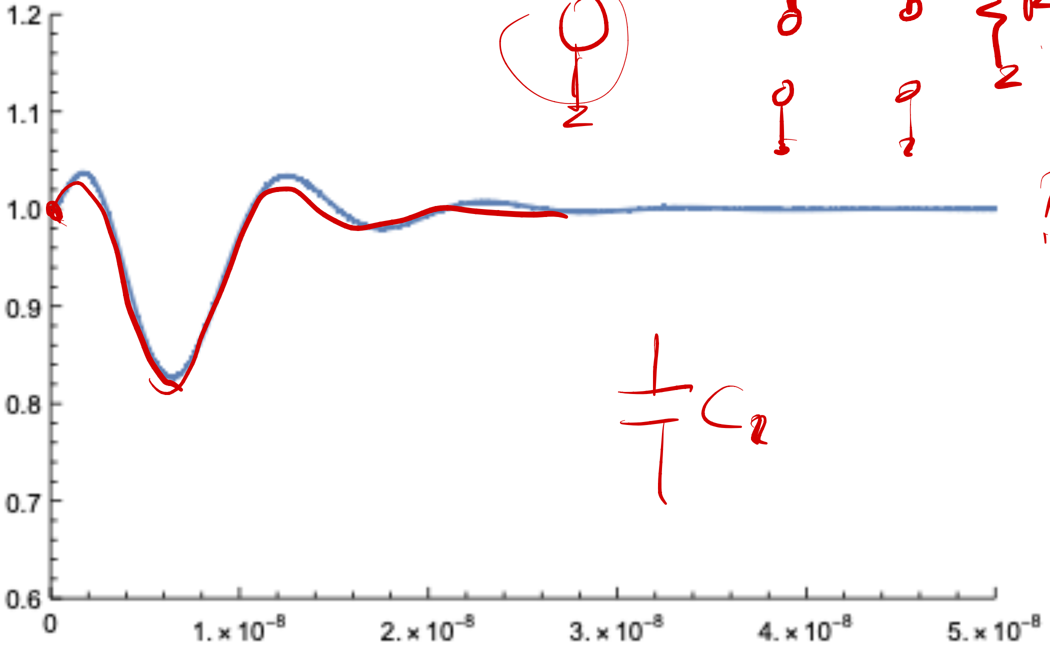
Example: Forced Solution







State 1



State 2

? $\frac{R_L}{R_L+R_S}$

? $\frac{R_L}{R_S+R_L}$

$\vec{x}(0) = (0, 1, 0, 0)^T$

Steady-State

$$q_k(t) = \frac{\hat{b}_k}{\underbrace{j\omega - \lambda_k}} e^{j\omega t} = g_k e^{j\omega t}$$

some complex #

$$\vec{x} = Q\,\vec{q} \quad \neq$$

$$x_k(t) = \underbrace{\left(\sum_k \hat{h}_k\right)} e^{j\omega t}$$

$$x_k(t) = H_k\, e^{j\omega t}$$