EECS 16B

Designing Information Devices and Systems II

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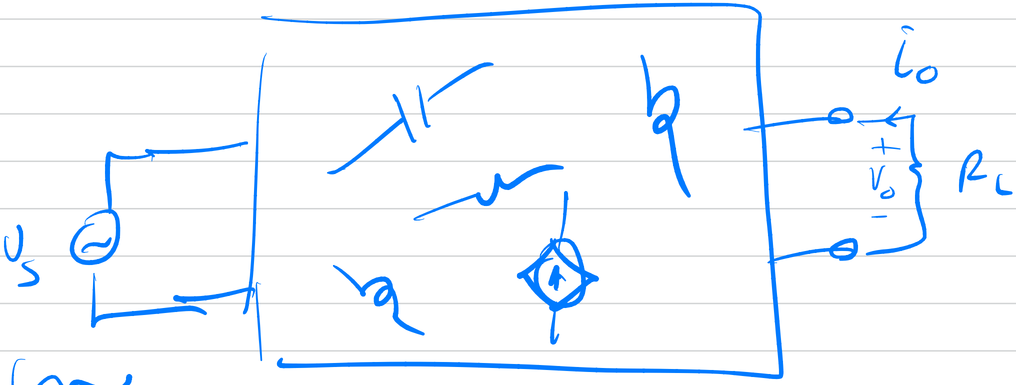
Module 9: AC Analysis

EECS 16B



input

Any generic circuit

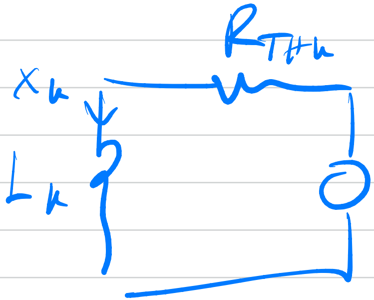
$$\Rightarrow \frac{d\vec{x}}{dt} = A\vec{x} + B\vec{v}_s$$

$\vec{x}$: state

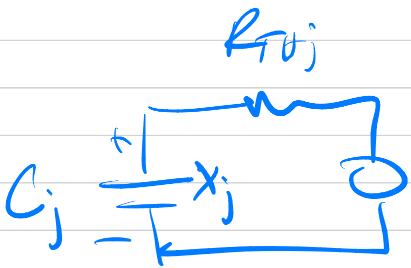
$$\left\{ \begin{array}{l} \text{voltages on caps} \\ \text{currents thru ind} \end{array} \right\}$$

L, R, C

dependent source



$$v_{TH_k} = \sum_j x_j \Rightarrow L_k \frac{dx_k}{dt} = \sum_j x_j \alpha_j$$



$$v_{TH_j} = \sum_j x_j \Rightarrow C_j \frac{dx_j}{dt} = \sum_j x_j \beta_j$$

⋮

$$\frac{d\vec{x}}{dt} = A\vec{x} + B\vec{b}_s$$

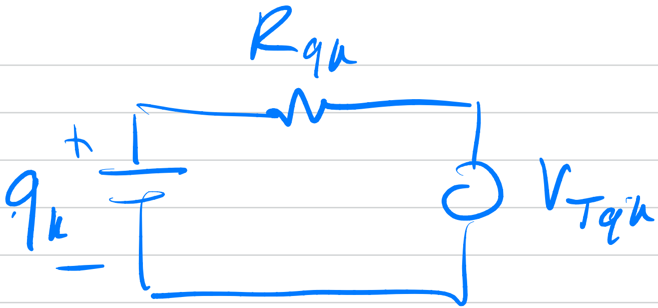
$$\frac{d\vec{x}}{dt} = Q\Lambda Q^{-1}\vec{x} + B\vec{b}_s$$

$$\boxed{\frac{d\vec{q}}{dt} = \Lambda\vec{q} + Q^{-1}\vec{b}_s}$$

$\vec{x}$: state space

$\vec{q}$: state space in the eigenbasis

"q_k" modes of the circuit



eq. circuit of each row

NOT NEEDED

Just a reminder that we've problem before.

$$q_k(t) = e^{\lambda_k t} q(0) + e^{\lambda_k t} \int_{-\infty}^{t} e^{-\lambda_k s} \underbrace{\tilde{b}_{sk}(s)}\, ds$$

general solution

$$b_s(t) = e^{j\omega t}$$

$\tilde{b}_s(t)$ = source in the eign basis

$\Rightarrow$ sum of any number of

$\alpha_k e^{j\omega t} \Rightarrow \hat{b}\, e^{j\omega t}$

$$= b_1 e^{j\omega t} + b_2 e^{j(\omega t + \phi_2)}$$

$$= b_1 e^{j\omega t} + b_2 e^{j\phi_2} e^{j\omega t}$$

$$= e^{j\omega t}\left(b_1 + b_2 e^{j\phi_2}\right) = \hat{b}\, e^{j\omega t}$$

Summary

- Steady-state solution of VDE
- Concept of Impedance
- AC Circuits
- Examples

Review: Forced Solution

$$q_k^f(t) = e^{\lambda_k t}\, \hat{b}_k \frac{e^{j(\omega - \lambda_k)t}}{j\omega - \lambda_k}$$

$$= e^{j\omega t}\left(\frac{\hat{b}_k}{j\omega - \lambda_k} \right) = \hat{h}_k(j\omega)\, e^{j\omega t}$$

constant, not function of time

$$x_k^f(t) = \sum_j \alpha_j^k q_k(t) = h_k(j\omega)\, e^{j\omega t}$$

$$= \underbrace{\underbrace{\left(\frac{\alpha_1^k}{j\omega - \lambda_1} + \frac{\alpha_2^k}{j\omega - \lambda_2} + \cdots \right)}_{h_k(j\omega)}}_{= \text{Amp \& Phase Shift}} \underbrace{e^{j\omega t}}_{\text{function of time}}$$

$H(j\omega)$

$$H(j\omega) = \left(\frac{\alpha_1^k}{j\omega - \lambda_1} + \frac{\alpha_2^k}{j\omega - \lambda_2} + \cdots \right)$$

$$= \frac{N(j\omega)}{D(j\omega)} = \frac{(j\omega - z_1)\cdots(j\omega - z_2)}{(j\omega - \lambda_1)(j\omega - \lambda_2)\cdots(j\omega - \lambda_n)}$$

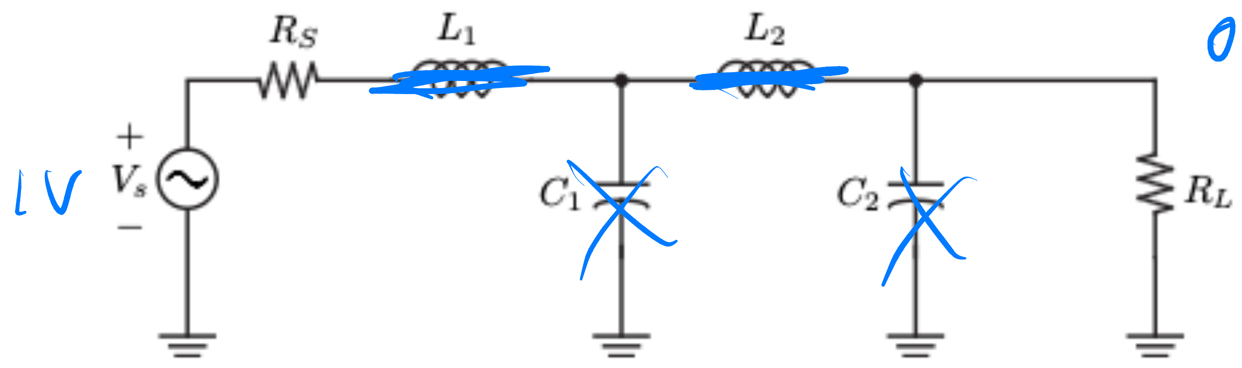
) Transfer Function z_i λ_i

zeros

poles

Forced Sinusoidal Solution

Example Circuit



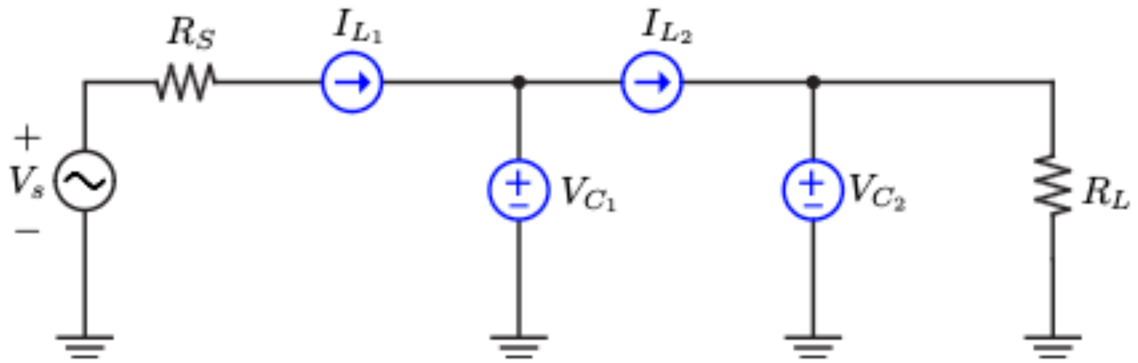
0.5V $R_L = R_S$

$$\vec{x} = (V_{C_1}\ V_{C_2}\ I_{L_1}\ I_{L_2})^T$$

$$\frac{d}{dt}\vec{x} = A\vec{x} + B\vec{b_s}$$

$$A = \begin{pmatrix} 0 & 0 & 1/C_1 & -1/C_1 \\ 0 & -1/R_L C_2 & 0 & 1/C_2 \\ -1/L_1 & 0 & -R_S/L_1 & 0 \\ 1/L_2 & -1/L_2 & 0 & 0 \end{pmatrix}$$

$B = (0\ 0\ 1/L_1\ 0)^T V_s.$



Numerical Example

```
In[120]:= EV = Eigenvalues[A] ; EV // MatrixForm
```

Out[120]//MatrixForm=

$$\begin{array}{l} \begin{pmatrix} -1.9034\times 10^{8} + 7.2085\times 10^{8}\, i \\ -1.9034\times 10^{8} - 7.2085\times 10^{8}\, i \\ -4.8545\times 10^{8} + 2.82637\times 10^{8}\, i \\ -4.8545\times 10^{8} - \underline{2.82637\times 10^{8}}\, i \end{pmatrix} \end{array} \begin{array}{l} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 = \sigma_4 + j\omega_4 \end{array}$$

Note that the eigenvalues have an imaginary part close to the cut-off frequency of the filter (100 MHz):

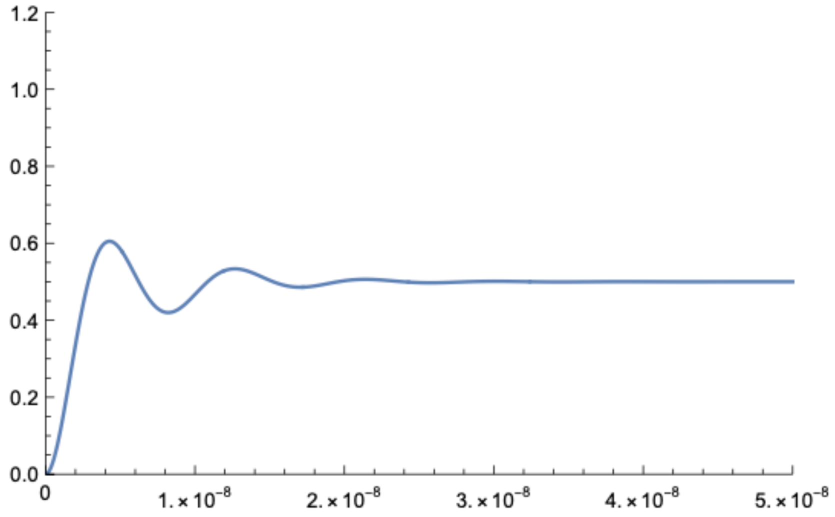
```
In[121]:= Im[EV] / (2 Pi 100 × 10 ^ 6) // N
Out[121]= {1.14727, -1.14727, 0.449831, -0.449831}
```

115 MHz

45 MHz

$$bs = \{\{0\}, \{0\}, \{\underline{1/L1}\}, \{0\}\};$$

$Plot[Re[xforce[t][[2]]], \{t, 0, 5010^\wedge - 9\}, PlotRange->\{\{0, 5010^\wedge - 9\}, \{0, 1.2\}\}]$



Forced Response

- Since we've reduced the system to simple constant coefficient linear differential equations, we know the forced response !
- Note that the sources are also transformed into the eigenspace through the matrix Q^{-1}

```
xforce[t_] = Q.MatrixExp[Dlam t].Qi.xi + Q.MatrixExp[Dlam t].Qi .Integrate[Q.MatrixExp[-Dlam sigma ].Qi . bs , {sigma, 0, t}];
```

homogeneous solution

$$Q \quad e^{\Lambda t} \quad Q^{-1} \int Q\, e^{-\Lambda \sigma}\, Q^{-1}\, \vec{b_s}\, d\sigma$$

Steady-State DC Response

```
In[140]:= Limit[Q.MatrixExp[Dlam t].Qi .Integrate[Q.MatrixExp[-Dlam sigma].Qi . bs , {sigma, 0, t}], t → Infinity]
Out[140]= {{0.5}, {0.5 + 0. i}, {0.01 + 0. i}, {0.01}}
```

The final values are also easily derived from the basic vector equation by setting all derivatives to zero:

```
In[141]:= -Inverse[A] . bs
Out[141]= {{0.5}, {0.5}, {0.01}, {0.01}}
```

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}_s = \vec{0}$$

DC steady state:

$$\vec{x}_{ss} = -A^{-1}\vec{b}_s$$

L → short
C → open } "obvious"

Sinusoidal Steady-State

```
w1 = 2 Pi 100×10^6;
```

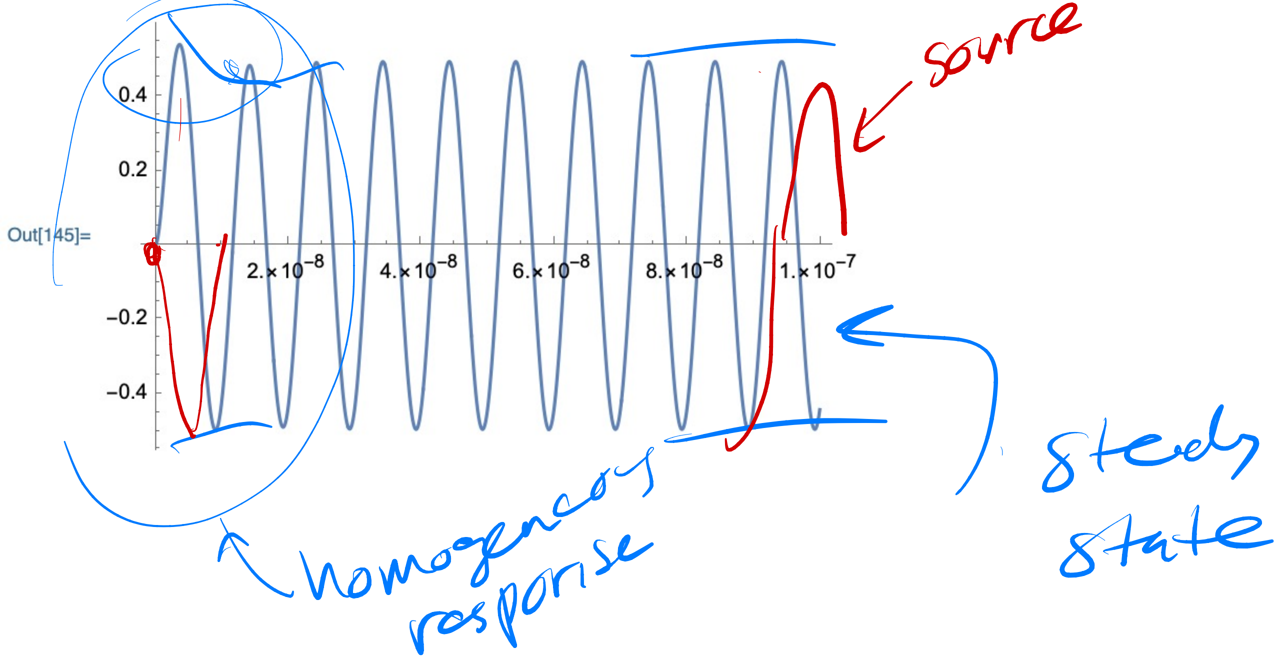
100MHz

```
In[143]:= xac1[t_] = Q.MatrixExp[Dlam t].Qi.xi + Q.MatrixExp[Dlam t].Qi .Integrate[Q.MatrixExp[-Dlam sigma ].Qi . bs Cos[w1 sigma] , {sigma, 0, t}];

In[145]:= Plot[Re[xac1[t][[2]]], {t, 0, 100×10^-9}]
```

sinusoidal source

Out[145]=

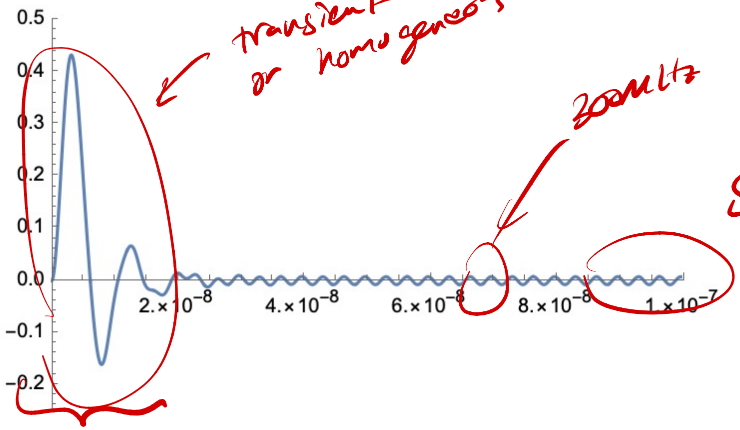


Higher Frequency Response

```
xac2[t_] = Q.MatrixExp[Dlam t].Qi.xi + Q.MatrixExp[Dlam t].Qi .Integrate[Q.MatrixExp[-Dlam sigma ].Qi . bs Cos[3 w1 sigma] , {sigma, 0, t}];
```

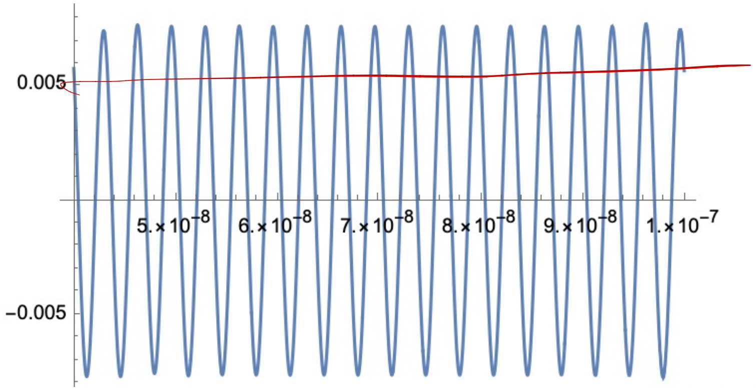
$3 \times 100\text{MHz}$

```
Plot[Re[xac2[t][[2]]], {t, 0, 100×10^-9}, PlotRange → {{0, 100×10^-9}, {-.25, .5}}]
```



Steady state resp

```
Plot[Re[xac2[t][[2]]], {t, 40×10^-9, 100×10^-9}]
```



5mV

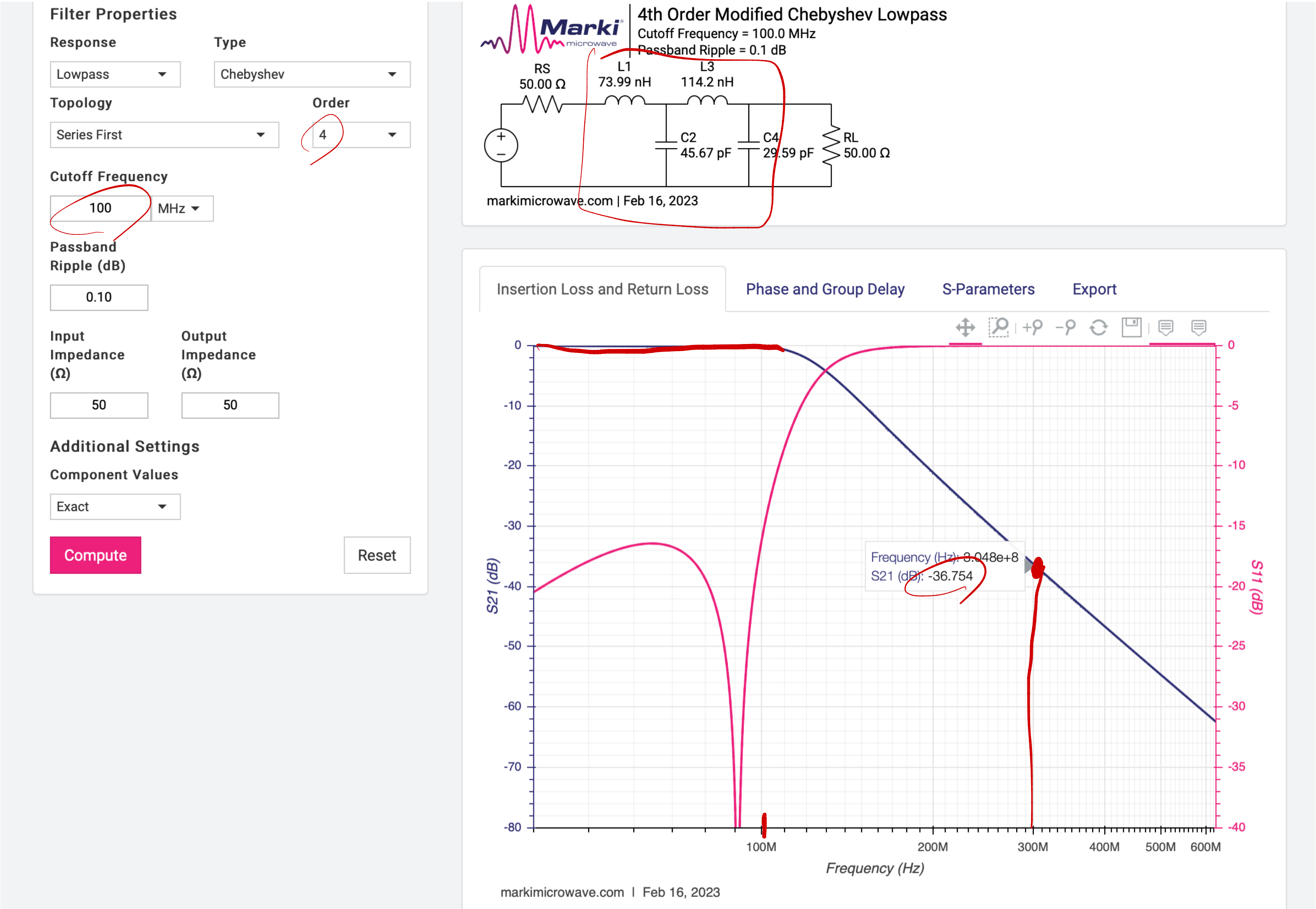
$$\frac{|H(3\omega_0)|}{|H(\omega_0)|} \simeq \frac{5mV}{0.5V}$$

100MHz

Steady-State Sinusoidal Response

- Weighted integral perspective

Filter Design



AC Analysis

- Can we get here directly without setting up all the VDE and solving it ?

Goal: Directly Calculate

Sinusoidal Steady-State Resp

"Phasors"

- A complex number by another name

$\hat{I}, \hat{V}$: a complex number

$$v(t) = V_0 \cos(\omega t + \phi) \Rightarrow \hat{V} \Leftarrow V_0 e^{j(\omega t + \phi)}$$

$$\Leftarrow \underbrace{V_0 e^{j\phi}}_{\hat{V}} e^{j\omega t}$$

Concept of Impedance

$$Z_C = \frac{1}{j\omega C} \qquad Z_L = j\omega L$$

Reactance

If Z is purely imaginary

Z is reactive

$$Z = R + jX$$

resistance

reactance

$$Y = G + jB = Z^{-1}$$

conductance

susceptance

Ohm's Law

$$V = I \cdot R \qquad I = G V$$

$$\hat{V} = \hat{I} Z \qquad \hat{I} = Y \hat{V}$$

sinusoidal (steady state) equiv.

Impedance

Admittance

Capacitors in AC Circuits

$$i_c = e^{j\omega t}\hat{I}_c$$

$$v_c = e^{j\omega t}\hat{V}_c$$

$$\left.\begin{array}{l}\hat{I}_c \\ \hat{V}_c\end{array}\right\} \text{constants}$$

(complex #)

$$i_c = C\frac{dv_c}{dt}$$

$$e^{j\omega t}\hat{I}_c = C\frac{d}{dt}\left(e^{j\omega t}\hat{V}_c\right) = C\hat{V}_c\, j\omega e^{j\omega t}$$

$$\cancel{e^{j\omega t}}\,\underbrace{\hat{I}_C = C \cdot j\omega \cdot \hat{V}_c}\,\cancel{e^{j\omega t}}$$

A capacitor in sinusoidal steady state looks like a "complex resistor"

$$I = G \cdot V$$

$$\boxed{\hat{I}_c = j\omega C\,\hat{V}_c}$$

"complex resistor" $\Rightarrow$ impedance

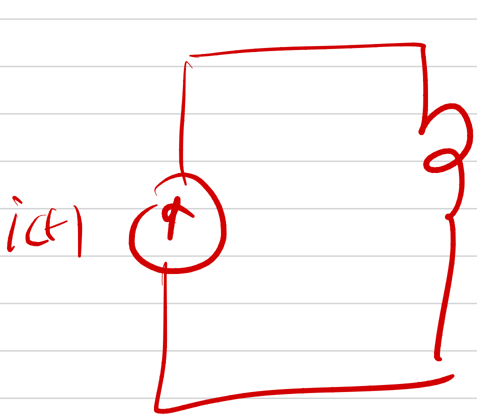
Inductors in AC Circuits

$$v_L = L \frac{di_L}{dt}$$

S.S.S.

$$\boxed{\hat{V}_L = \underbrace{j\omega L}_{\text{complex "resistor"}} \hat{I}_L}$$

$$Z_L = j\omega L \qquad Z_C = \frac{1}{j\omega C}$$


i(t)

Back In Time:

$$\hat{V}_c = \frac{1}{j\omega C}\hat{I}_c$$

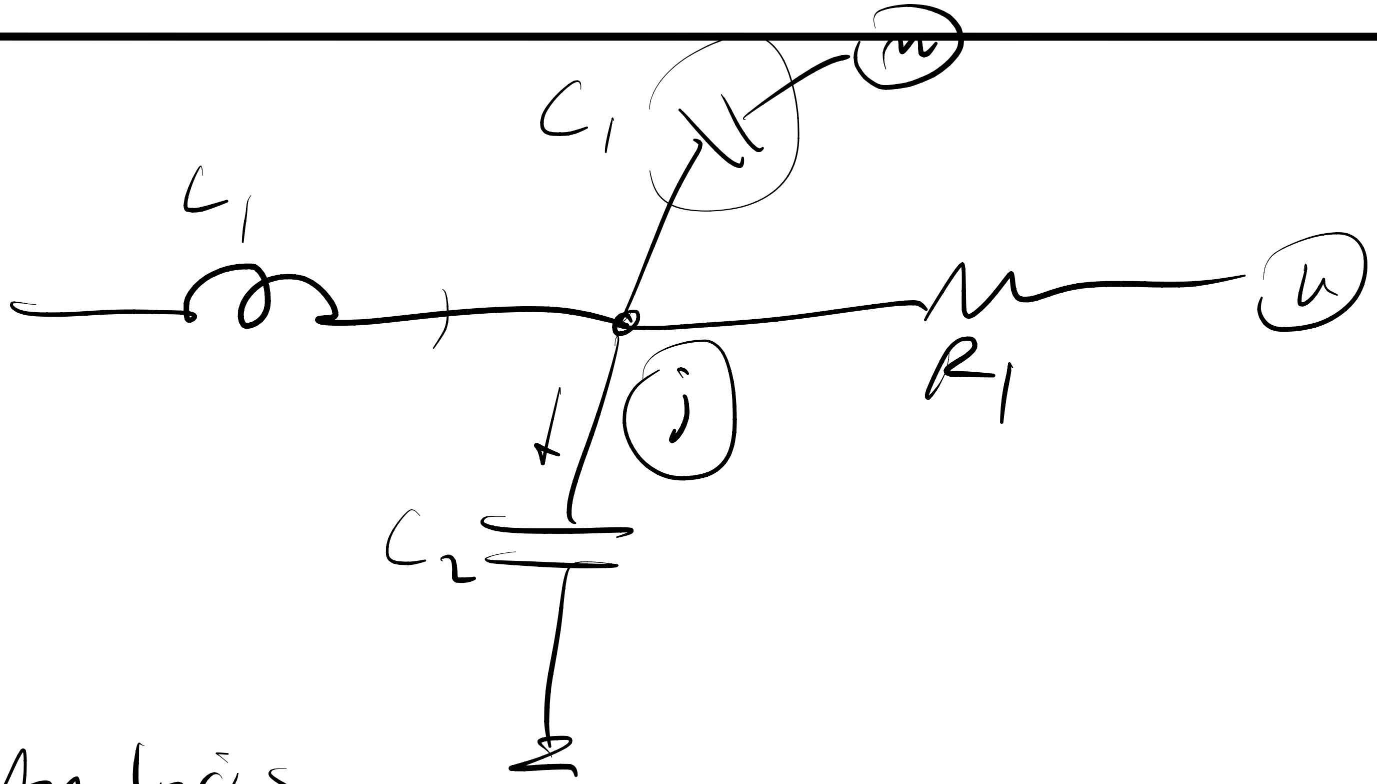
$$v(t) \overset{?}{\Longleftrightarrow} \frac{1}{j\omega C} e^{j\phi_c} e^{j\omega t}$$

$$= -j \frac{1}{\omega C} e^{j\phi_c} e^{j\omega t}$$

$$= e^{-\frac{\pi}{2}j} \frac{1}{\omega C} e^{j\phi_c} e^{j\omega t} = e^{j\phi_c'} \frac{1}{\omega C} e^{j\omega t}$$

$$v(t) = \frac{1}{\omega C}\cos(\omega t + \phi_c')$$

Direct Solution (no VDE required)



Nodal Analysis

$$C_2 \frac{dV_j}{dt} + \frac{V_j - V_n}{R_1} + C_1 \frac{d}{dt}\left(V_j - V_m\right) + i_{L1}$$

NODE k

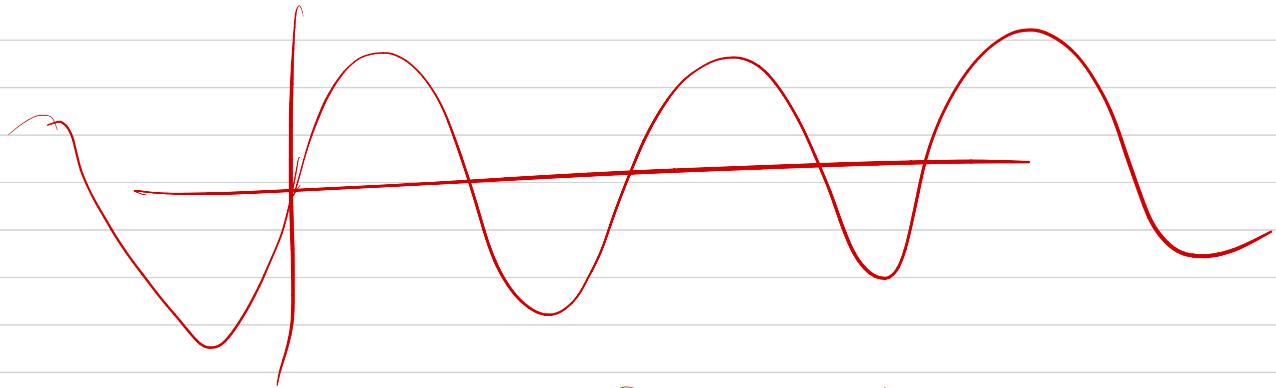
$$0 = -I_s + \sum_j C_{kj} \frac{d(V_k - V_j)}{dt} + \frac{V_k - V_j}{R_{kj}} + i_{Lj}$$

$$0 = -I_s + \sum_j C_{kj} \frac{d(V_k - V_j)}{dt} + \frac{V_k - V_j}{R_{kj}} + \frac{1}{L_j}\int (V_k - V_j)\,dt$$

$$V_k = \hat{V}_k e^{j\omega t}$$

$$0 = \cancel{e^{j\omega t}} \sum_j C_{kj}(j\omega)(\hat{V}_k - \hat{V}_j) + \frac{\hat{V}_k - \hat{V}_j}{R_{kj}} + \frac{1}{L_j}\frac{1}{j\omega}(\hat{V}_k - \hat{V}_j)$$

$$+ \hat{I}_s \cancel{e^{j\omega t}}$$



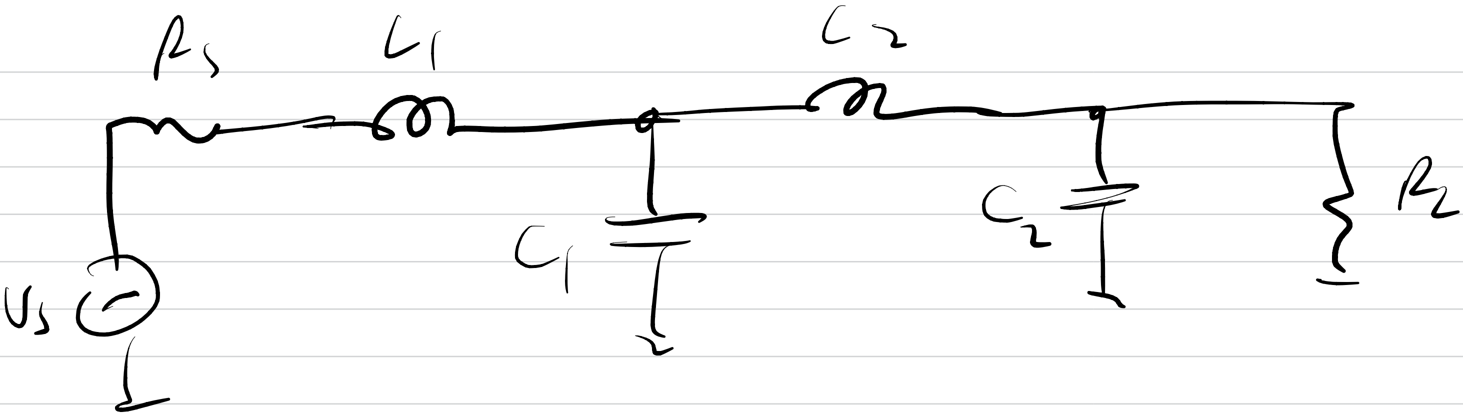
$$v(t) = V_0 \sin(\omega t + \phi)$$

freq domain

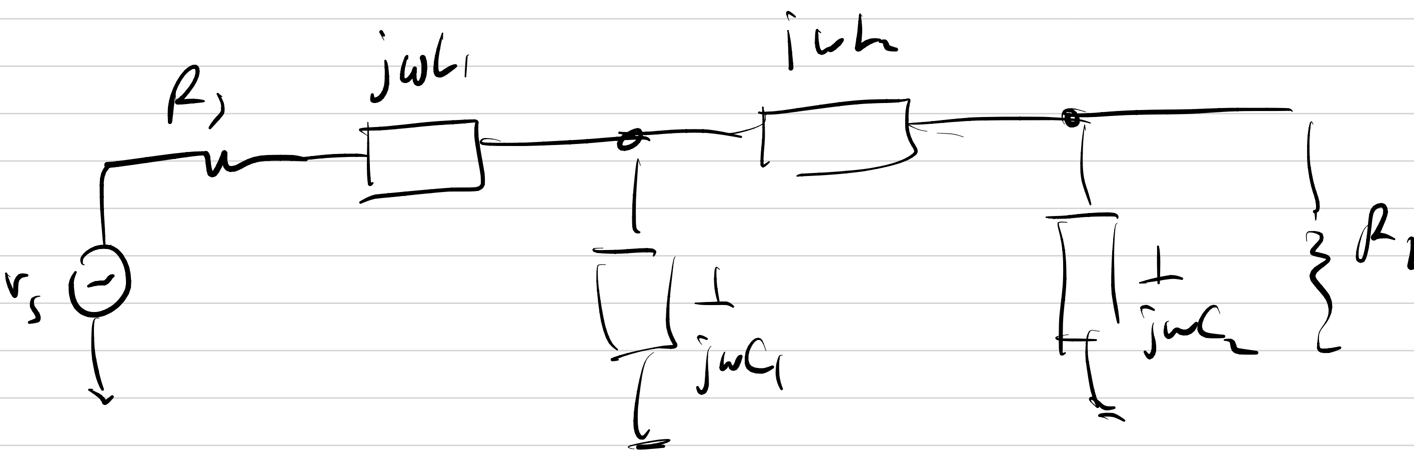
$$\left(\hat{V}_0 e^{j\omega t} \right)$$

Phasor Algebra

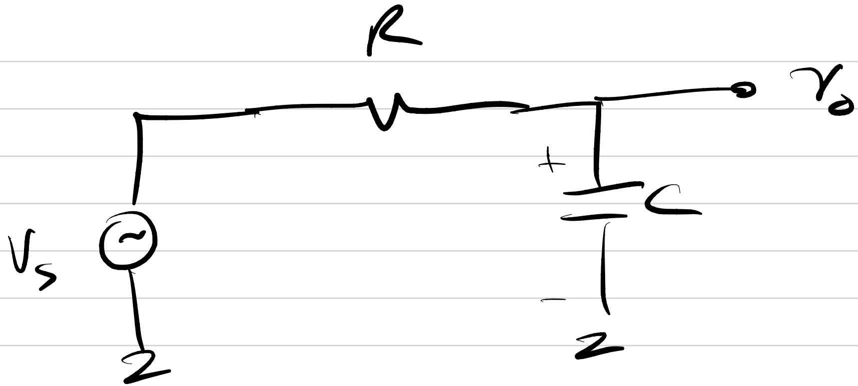
- Same as any complex number



R_s
L_1
L_2
C_1
C_2
R_L
V_s



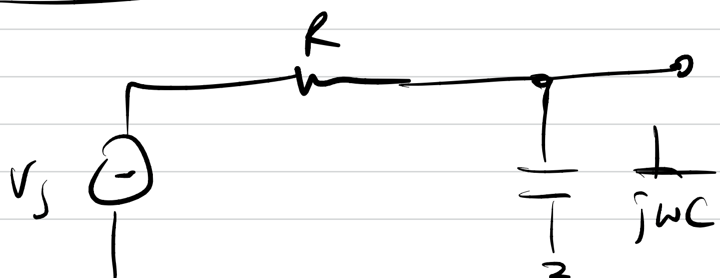
R_s
$j\omega L_1$
$j\omega L_2$
v_s
$\frac{1}{j\omega C_1}$
$\frac{1}{j\omega C_2}$
R_L



input

output

AC Response



$$\frac{v_o}{v_s} = \frac{1/j\omega C}{R + \frac{1}{j\omega C}}$$

$$= \frac{1}{1 + j\omega RC}$$

$$\mathrm{Re}\{e^{j\omega t}\} \longrightarrow \boxed{\text{circuit}} \longrightarrow \mathrm{Re}\{H(j\omega)\, e^{j\omega t}\}$$

$$\frac{\cos\omega t}{2} \longrightarrow \boxed{\phantom{\text{circuit}}} \longrightarrow \mathrm{Re}$$

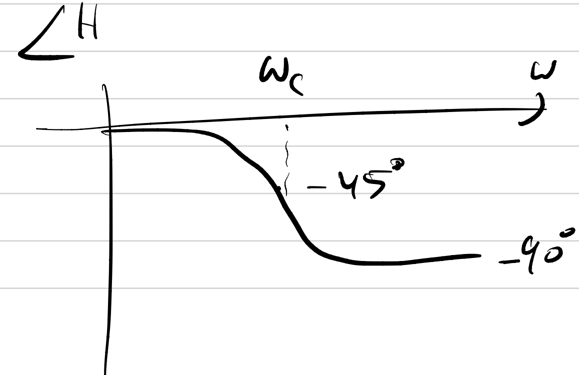
$$\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right) \longrightarrow \boxed{\phantom{\text{circuit}}} \longrightarrow H(j\omega)\, e^{j\omega t} + H(-j\omega)\, e^{-j\omega t}$$

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

$$|H(j\omega)| \sim \left|\frac{1}{j\omega RC}\right| = \frac{1}{\omega RC} \sim \frac{1}{\omega}$$

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 (RC)^2}}$$

$$\omega_c = \frac{1}{RC}$$



Magnitude Reop

$$\angle H(j\omega) = -\tan\left(\frac{\omega RC}{1}\right)$$

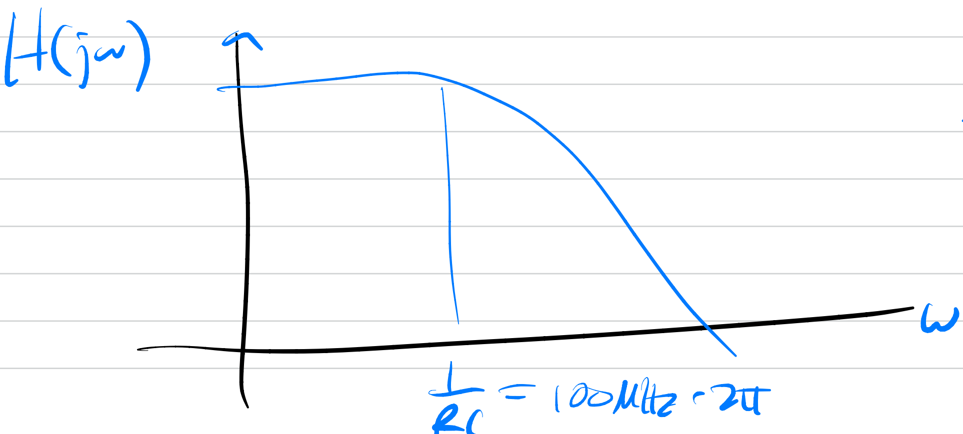
$$\hat{V}_o = H \hat{V}_s$$

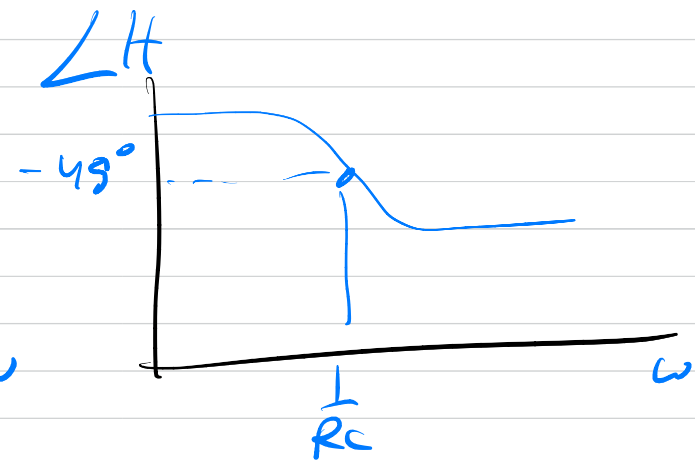
phasor

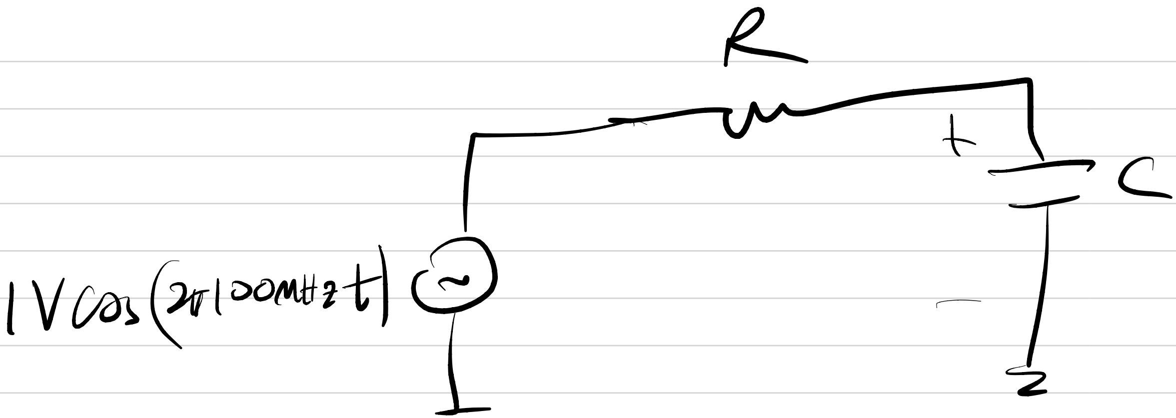
Phasor

$$v_s(t) = V_s \cos \omega t$$

$$v_o(t) = |H(\omega)| V_s \cos(\omega t + \angle H(\omega))$$







$$v_o(t) = 1V \cdot \left| H\left(j \frac{1}{RC}\right) \right| \cos\left(2\pi \cdot 100\,\text{MHz}\, t + \angle H\right)$$

$$= 1V \underbrace{\frac{1}{\sqrt{2}}}_{\text{Atten.}} \cos\left(2\pi\, 100\,\text{MHz}\, t - 45^\circ\right)$$

Phase Shift