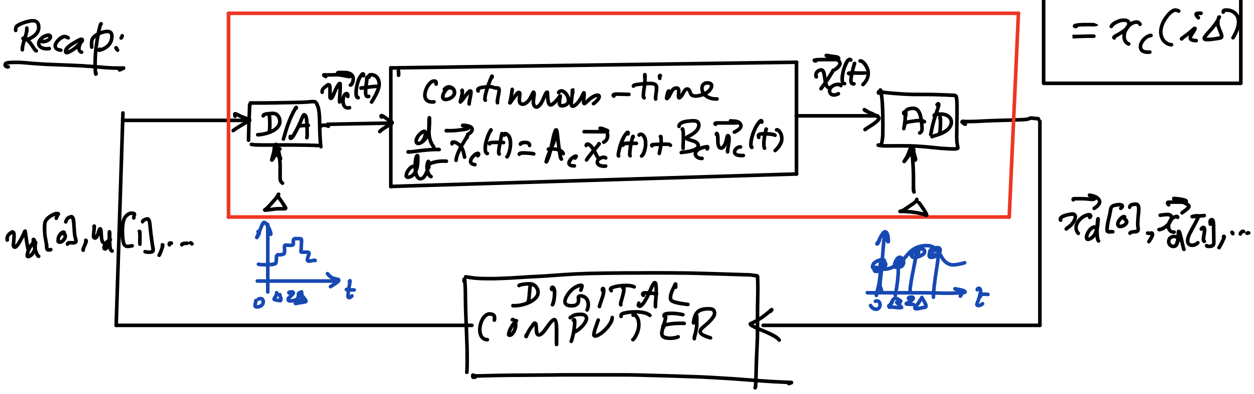
Today: • CT–DT model conversion for control sys.
• System ID

(DISCRETIZATION)

DT diff. eq.: $\vec{x}_d[i+1] = A_d \vec{x}_d[i] + B_d \vec{u}_d[i]$

want $\vec{x}_d[i] = x_c(i\Delta)$

Recap:



$\vec{x}$: vector of state variables (V_c, I_L in RLC circuits, position, velocity in motion systems, flows and pressures in an engine model, etc.)

$\vec{u}$: control input (indep voltage & current sources in RLC circuits, forces & torques in mechanical systems, etc.)

$\vec{x}_d[i+1]$ should be the solution to the

DE: $$\frac{d\vec{x}_c(t)}{dt} = A_c\vec{x}_c(t) + B_c\vec{u}_c(t)$$

@ time $t = (i+1)\Delta$ from

initial cond. $\vec{x}_c(i\Delta) = \vec{x}_d[i]$

@ time $t = i\Delta$ and input $\vec{u}_d[i]$

Scalar case:

C-T: $\dfrac{dx_c(t)}{dt} = \lambda\, x_c(t) + b\, u_c(t)$ —— (1)

D-T: $x_d[i+1] = \lambda_d\, x_d[i] + b_d\, u_d[i]$ —— (2)

Goal: Find λ_d, b_d of D-T sys. as function of λ, b of C-T sys. (and Δ)

$x_d[i] = x_c(i\Delta)$ in (1)

where $u_c(t) = u_d[i]$ for $i\Delta \le t < (i+1)\Delta$

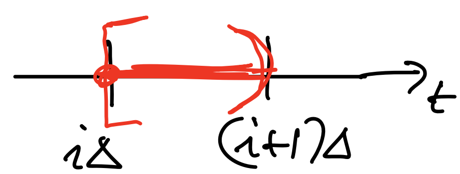
Soln to (1):

$$\boxed{\text{For } i\Delta \le t < (i+1)\Delta: \quad x_c(t) = e^{\lambda(t-t_0)} x_c(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)} d\tau\, b\, u_c(\tau)}$$

Q] What is $u_c(\tau)$ for $i\Delta \le t < (i+1)\Delta$?

$u_c(i\Delta) = u_d[i]$

$t_0 = i\Delta$; $t = (i+1)\Delta$



$u_c(t) = u_d[i]$ throughout this time-interval

$$\underbrace{x_c((i+1)\Delta)}_{x_d[i+1]} = e^{\lambda\Delta}\,\overbrace{x_c(i\Delta)}^{x_d[i]} + b\,u_d[i]\,\underbrace{e^{\lambda t}\int_{i\Delta}^{(i+1)\Delta} e^{-\lambda\tau}\,d\tau}_{= \begin{cases} \left(\frac{1-e^{\lambda\Delta}}{\lambda}\right) ; & \lambda \neq 0 \\ \Delta ; & \lambda = 0 \end{cases}}$$

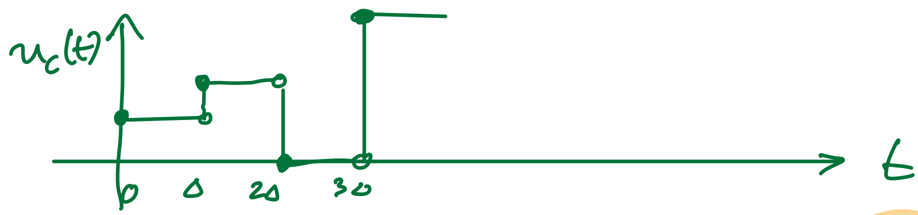
$$\boxed{x_d[i+1] = \underbrace{e^{\lambda\Delta}}_{\lambda_d}\, x_d[i] + \underbrace{b\left(\frac{1-e^{-\lambda\Delta}}{\lambda}\right)}_{b_d \text{ (for } \lambda \neq 0)}\, u_d[i]}$$

Ex. of discretization:

$$\frac{d}{dt} x_c(t) = 3\, x_c(t) + 5\, u_c(t),$$

$A_c = \lambda$ $\qquad$ $B_c = b$

where $u_c(t)$ is p.c. over intervals of length Δ



Discretization:
(Difference Eqn.)

$$x_d[i+1] = e^{3\Delta} x_d[i] + \frac{e^{3\Delta} - 1}{3} \cdot 5\, u_d[i]$$

$A_d = \lambda_d$ $\qquad$ $B_d = b_d$

$$x_d[i+1] = \lambda_d \, x_d[i] + b \, u_d[i] ; \quad i = 0, 1, 2, \ldots$$

$$x_d[1] = \lambda_d x_d[0] + b_d u_d[0]$$

$$x_d[2] = \lambda_d x_d[1] + b_d u_d[1]$$

$$= \lambda_d \left[\lambda_d x_d[0] + b_d u_d[0] \right] + b_d u_d[1]$$

$$= \lambda_d^2 x_d[0] + \lambda_d b_d u_d[0] + b_d u_d[1]$$

$$x_d[3] = \lambda_d^3 x_d[0] + \lambda_d^2 b_d u_d[0] + \lambda_d b_d u_d[1] + b_d u_d[2]$$

$$\vdots$$

$$\boxed{x_d[k] = \lambda_d^k x_d[0] + \sum_{i=0}^{k-1} \lambda^{k-1-i} b_d u_d[i]}$$

$$\boxed{\vec{x}_d[k] = A_d^k \vec{x}_d[0] + \sum_{i=0}^{k-1} A_d^{k-1-i} B_d \vec{u}_d[i]}$$

General vector case: $k = 1, 2, \ldots$

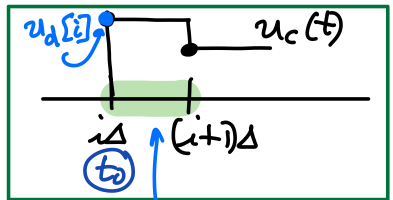
Vector Diff. Eq..

$$(*) \quad \boxed{\frac{d}{dt}\vec{x}_c(t) = A_c \vec{x}_c(t) + B_c \underbrace{\vec{u}_c(t)}_{\vec{u}_d[i]}}$$

True for all t. Let's use $t \in [i\Delta, (i+1)\Delta)$

$t_0 = i\Delta$

$t = (i+1)\Delta$



t is in this range

Recall lecture on diagonalization & change-of-basis

$$\vec{y}_c = V^{-1}\vec{x}_c \Rightarrow \vec{x}_c = V\vec{y}_c$$

$$\frac{d}{dt}\vec{y}_c(t) = V^{-1}\frac{d}{dt}\vec{x}_c(t) = V^{-1}A_c\vec{x}_c(t) + \underbrace{V^{-1}B_c\vec{u}_d[i]}_{\triangleq \vec{z}[i]}$$

↑ from (*)

$\vec{x}_c$: "standard basis"

$\vec{y}_c$: "eigenbasis of A_c"

columns of V
$$= \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$
eigenvectors of A_c

$$\frac{d\vec{y}_c(t)}{dt} = \underbrace{V^{-1}A_cV}\vec{y}_c + \vec{z}[i]$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\boxed{\frac{d}{dt} \underbrace{y_{c,k}(t)}_{k^{th} \text{ entry of } \vec{y}_c(t)} = \lambda_k y_{c,k}(t) + \underbrace{z_{i,k}}_{k^{th} \text{ entry of } \vec{z}[i]}}$$

for $k = 1, 2, \dots n$

DECOUPLED SCALAR EQS. !!!

From scalar case:

$$y_{d,k}[i+1] = e^{\lambda_k \Delta} y_{d,k}[i] + \underbrace{\frac{e^{\lambda_k\Delta}-1}{\lambda_k}}_{\text{If } \lambda_k \neq 0, \text{ else replace by } \Delta} z_{i,k}$$

for $k = 1, 2, \dots n$

Stacking up these n scalar eqns:

$$\vec{y_d}[i+1] = \begin{bmatrix} e^{\lambda_1\Delta} & & & \\ & e^{\lambda_2\Delta} & & \\ & & \ddots & \\ & & & e^{\lambda_n\Delta} \end{bmatrix} \vec{y_d}[i] + \begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{e^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix} \vec{z[i]}$$

$$\vec{x_d} = V\vec{y_d}$$

$$\vec{y_d} = V^{-1}\vec{x_d}$$

multiply both sides by V

$$\boxed{\vec{x_d}[i+1] = V \begin{bmatrix} e^{\lambda_1\Delta} & & & \\ & e^{\lambda_2\Delta} & & \\ & & \ddots & \\ & & & e^{\lambda_n\Delta} \end{bmatrix} V^{-1}\vec{x_d}[i] + V \begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{e^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix} \vec{z[i]}}$$

$\vec{z[i]} = V^{-1}B_c \vec{u_d}[i]$

FINAL SOLN. !!!

So far:

$$\frac{dx_c(t)}{dt} = \lambda\, x_c(t) + b\, u_c(t) \qquad \text{(CT-DE)}$$

$$x_d[k+1] = \underbrace{\lambda_d}_{e^{\lambda\Delta}} x_d[k] + \underbrace{b_d}_{\begin{cases} \frac{b(e^{\lambda\Delta}-1)}{\lambda} & \text{if } \lambda \neq 0 \\ b\Delta & \text{if } \lambda = 0 \end{cases}} u_d[k] \qquad \text{(DT-diff. eqn)}$$

These are so-called "model-based" control : system parameters λ, b, λ_d, b_d (more generally, A_c, B_c, A_d, B_d are well-defined and known)

Alternate approach: "Data-centric" approach

Model parameters λ_d, b_d are not known.

How to proceed? Try various inputs & see what happens? called system identification.

SYSTEM ID:

$$x_d[k+1] = \lambda_d x_d[k] + b_d u_d[k] + \underbrace{e[k]}_{\substack{\text{error or disturbance due to}\\ \text{model mismatch, noise, perturbation, etc.}}}$$

need to be estimated

MODEL PARAMETERS NEED TO BE ESTIMATED

Goal: "Learn" λ_d, b_d from data

Unknowns: $\lambda_d, b_d \Rightarrow 2$ unknowns

$\begin{bmatrix} \lambda_d \\ b_d \end{bmatrix}$ vector of unknowns

Start system @ time $k=0 \Longrightarrow x_d[0]$: observation

$$x_d[k+1] = \lambda_d x_d[k] + b_d u_d[k] + e[k]$$

$$x_d[1] = \lambda_d x_d[0] + b_d u_d[0] + e[0]$$

$$x_d[2] = \lambda_d x_d[1] + b_d u_d[1] + e[1]$$

$$x_d[3] = \lambda_d x_d[2] + b_d u_d[2] + e[2]$$

$$\vdots$$

Least squares!

$$\boxed{\vec{s} = D\vec{p} + \vec{e}}$$

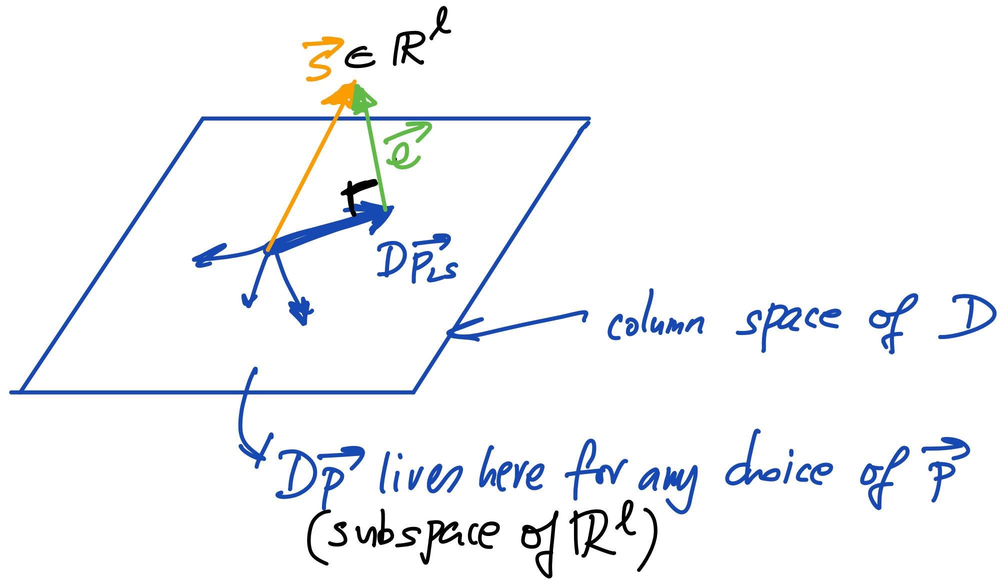
$l > n$

$$\underbrace{\begin{bmatrix} s[1] \\ s[2] \\ \vdots \\ s[l] \end{bmatrix}}_{\vec{s}_{l\times 1}} = \underbrace{\begin{bmatrix} \uparrow & & \uparrow \\ \vec{d}_1 & \cdots & \vec{d}_n \\ \downarrow & & \downarrow \end{bmatrix}}_{D_{l\times n}} \underbrace{\begin{bmatrix} p[1] \\ \vdots \\ p[n] \end{bmatrix}}_{\vec{p}_{n\times 1}} + \underbrace{\begin{bmatrix} e[1] \\ e[2] \\ \vdots \\ e[l] \end{bmatrix}}_{\vec{e}_{l\times 1}}$$

Least-squares: Find $\vec{p}_{LS}$ s.t. $D\vec{p}_{LS}$ is as close to $\vec{s}$ as possible in the sense that $\|\vec{e}\|^2 = \|\vec{s} - D\vec{p}\|^2$ is minimized when $\vec{p} = \vec{p}_{LS}$.

$$\sum_{i=1}^{l} e^2[i]$$

Let's look at the geometry of the Least-Squares problem:



$\|\vec{e}\|_2^2$ is minimized when $\vec{e}$ is $\perp$ to col. space(D)

$$\vec{d}_1^T\vec{e} = 0 \qquad \text{where } D = \begin{bmatrix} \vec{d}_1 & \vec{d}_2 & \cdots & \vec{d}_n \end{bmatrix}$$

$$\vec{d}_2^T\vec{e} = 0$$

$$\vdots$$

$$\underbrace{\vec{d}_n^T\vec{e} = 0}_{D^T\vec{e} = 0} \qquad D^T = \begin{bmatrix} \vec{d}_1^T \\ \vdots \\ \vec{d}_n^T \end{bmatrix}$$

$$\vec{e} = \vec{s} - D\vec{p} \implies D^T(\vec{s} - D\vec{p}_{LS}) = 0$$

$$D^T D\vec{p}_{LS} = D^T\vec{s}$$

$$\boxed{\text{If } D^T D \text{ is invertible, LS soln. is: } \quad \vec{p}_{LS} = (D^T D)^{-1} D^T\vec{s}}$$

Scalar Diff. Eq.: Least-Squares

Unknowns: λ_d, b_d

Goal: "Learn" λ_d, b_d from data

$$x_d[1] = \lambda_d x_d[0] + b_d u_d[0] + e[0]$$

$$x_d[2] = \lambda_d x_d[1] + b_d u_d[1] + e[1]$$

$$\vdots$$

$$\underbrace{\begin{bmatrix} x_d[0] & u_d[0] \\ x_d[1] & u_d[1] \\ \vdots & \\ x_d[l-1] & u_d[l-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \lambda_d \\ b_d \end{bmatrix}}_{\vec{p}} = \underbrace{\begin{bmatrix} x_d[1] \\ x_d[2] \\ \vdots \\ x_d[l] \end{bmatrix}}_{\vec{s}}$$

$$\vec{p}_{LS} = (D^T D)^{-1} D^T \vec{s}$$

if $(D^T D)$ is invertible!

Vector Least-Squares:

Ex. $n = 2$; $m = 1$ (2-dim state vector, 1-dim. control)

$$\underset{(2\times1)}{\vec{x}[i+1]} = \underset{(2\times2)}{A}\ \underset{(2\times1)}{\vec{x}[i]} + \underset{(\ \ )\,2\times1}{B}\ \underset{(1\times1)}{u[i]} + \underset{(2\times1)}{\vec{e}[i]}$$

$$\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix} + \underbrace{\begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix}}_{B} u[i] + \begin{bmatrix} e_1[i] \\ e_2[i] \end{bmatrix}$$

How many unknowns?

A: $n \times n$ (n^2 unknowns)

B: $n \times m$ (nm ″)

$n = 2$, $m = 1$

Total # unknowns: $n(n+m) = 6$