EECS 16B: Lec. 15: [Module 2 (Lecture 3)]

Today:
- System ID (Finish up)
- Stability

Reminders

1. MT next Mon. 4/13 8-10 p.m.
2. Review session tomorrow (7-9pm)
3. No class next Tues.

RECAP:

SYSTEM ID: (SCALAR CASE)

$$x_d[k+1] = \lambda_d x_d[k] + b_d u_d[k] + \underbrace{e[k]}$$

need to be estimated

error or disturbance due to model mismatch, noise, perturbation, etc.

$$\boxed{\text{MODEL PARAMETERS NEED TO BE ESTIMATED}}$$

Goal: "Learn" λ_d, b_d from data

Unknowns: $\lambda_d, b_d \Rightarrow 2$ unknowns

$\begin{bmatrix} \lambda_d \\ b_d \end{bmatrix}$ vector of unknowns

Start system @ time $k=0 \Rightarrow x_d[0]$: observation

$$x_d[k+1] = \lambda_d x_d[k] + b_d u_d[k] + e[k]$$

$$x_d[1] = \lambda_d x_d[0] + b_d u_d[0] + e[0]$$

$$x_d[2] = \lambda_d x_d[1] + b_d u_d[1] + e[1]$$

$$x_d[3] = \lambda_d x_d[2] + b_d u_d[2] + e[2]$$

$\vdots$

$$\boxed{\text{Least Squares!}}$$

Scalar Diff. Eq.: Least-Squares

Unknowns: λ_d, b_d

Goal: "Learn" λ_d, b_d from data

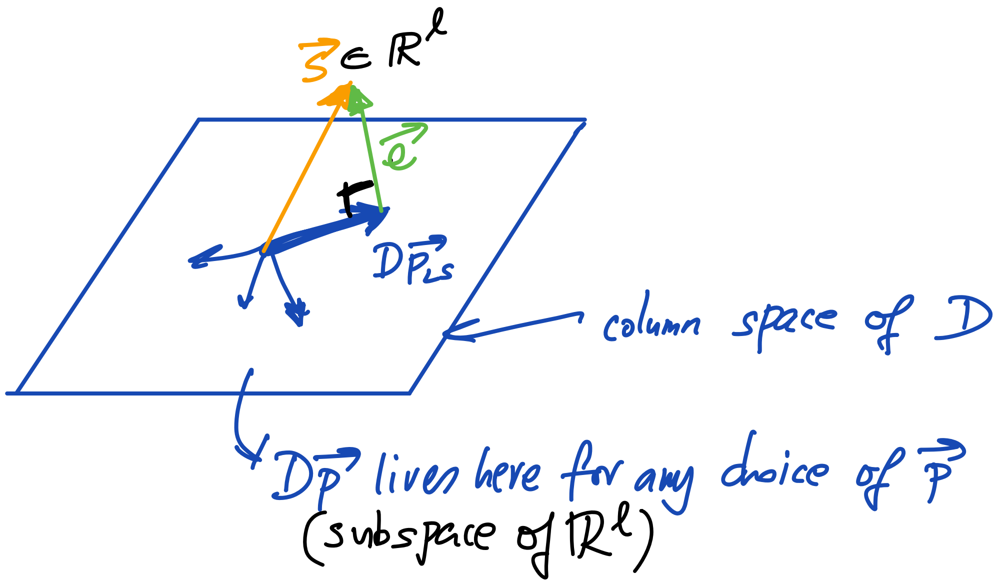
$$x_d[1] = \lambda_d x_d[0] + b_d u_d[0] + e[0]$$

$$x_d[2] = \lambda_d x_d[1] + b_d u_d[1] + e[1]$$

$$\vdots$$

$$\underbrace{\begin{bmatrix} x_d[0] & u_d[0] \\ x_d[1] & u_d[1] \\ \vdots & \\ x_d[l-1] & u_d[l-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \lambda_d \\ b_d \end{bmatrix}}_{\vec{p}} = \underbrace{\begin{bmatrix} x_d[1] \\ x_d[2] \\ \vdots \\ x_d[l] \end{bmatrix}}_{\vec{s}}$$

Solve for $\hat{\vec{p}} = \begin{bmatrix} \hat{\lambda}_d \\ \hat{b}_d \end{bmatrix}$ using Least-Squares.



$$\boxed{\vec{p}_{LS} = (D^T D)^{-1} D^T \vec{s} \qquad \text{(if } [D^T D] \text{ is invertible)}}$$

General vector setting for Least Squares:

$$\underset{(n\times 1)}{\vec{x}[i+1]} = \underset{(n\times n)}{A}\ \underset{(n\times 1)}{\vec{x}[i]} + \underset{(n\times m)}{B}\ \underset{(m\times 1)}{\vec{u}[i]} + \underset{(n\times 1)}{\vec{e}[i]}$$

Example : $n=2$; $m=1$ (2 state vars., 1 control input)

$$\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix} u[i] + \begin{bmatrix} e_1[i] \\ e_2[i] \end{bmatrix}$$

How many unknown? $\left.\begin{array}{l} A: \ n\times n = n^2 \\ B: \ n\times m = nm \end{array}\right\}$ Total of $n(n+m)$

In our ex, $n=2$, $m=1 \Rightarrow$ 6 unknowns

$\boxed{1^{st} \text{ row:}}$ $x_1[i+1] = a_{11} x_1[i] + a_{12} x_2[i] + b_{11} u[i] + e_1[i]$

$i=0$: $x_1[1] = a_{11}\, x_1[0] + a_{12}\, x_2[0] + b_{11}\, u[0] + e[0]$

$i=1$: $x_1[2] = a_{11}\, x_1[1] + a_{12}\, x_2[1] + b_{11}\, u[1] + e[1]$

$\vdots$

$i=l-1$: $x_1[l] = a_{11}\, x_1[l-1] + a_{12}\, x_2[l-1] + b_{11}\, u[l-1] + e[l-1]$

Q) How do we set up a L-S formulation? ($D\vec{p} + \vec{e} = \vec{s}$) unknown

$$\underbrace{\begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1] \\ & \vdots & \\ x_1[\ell-1] & x_2[\ell-1] & u[\ell-1]\end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a_{11}\\ a_{12}\\ b_{11}\end{bmatrix}}_{\vec{P}_1} + \underbrace{\begin{bmatrix} e_1[0]\\ \vdots \\ e_1[\ell-1]\end{bmatrix}}_{\vec{e}} = \underbrace{\begin{bmatrix} x_1[1]\\ x_1[2]\\ \\ x_1[\ell]\end{bmatrix}}_{\vec{s}_1}$$

$$\begin{bmatrix} \hat{a}_{11}\\ \hat{a}_{12}\\ \hat{b}_{11}\end{bmatrix} \qquad \boxed{\hat{\vec{P}}_1 = (D^TD)^{-1}D^T\vec{s}_1 \quad \text{(if } D^TD \text{ is invertible)}}$$

$$\begin{bmatrix} x_1[i+1] \\ x_2[i+1]\end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} x_1[i] \\ x_2[i]\end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{12}\end{bmatrix} u[i] + \begin{bmatrix} e_1[i] \\ e_2[i]\end{bmatrix}$$

Let's est. the second row of parameters a_{21}, a_{22}, b_{12} in the same way!

2nd row: $x_2[i+1] = a_{21}x_1[i] + a_{22}x_2[i] + b_{12}u[i] + e_2[i]$

$$
\begin{array}{l}
i=0 \rightarrow \\
i=1 \rightarrow \\
\\
\\
\end{array}
\underbrace{\begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1] \\ & \vdots & \\ & \vdots & \end{bmatrix}}_{D}
\underbrace{\begin{bmatrix} a_{21} \\ a_{22} \\ b_{12} \end{bmatrix}}_{\vec{p}_2}
+ \underbrace{\begin{bmatrix} e_2[0] \\ \vdots \\ \vdots \end{bmatrix}}_{\vec{e}_2}
= \underbrace{\begin{bmatrix} x_2[1] \\ x_2[2] \\ \vdots \end{bmatrix}}_{\vec{s}_2}
$$

Least-squares!

$$\boxed{\hat{\vec{p}}_2 = (D^T D)^{-1} D^T \vec{s}_2}$$

$$\begin{bmatrix} \hat{a}_{21} \\ \hat{a}_{22} \\ \hat{b}_{12} \end{bmatrix}$$

Note: We need $\ell \geq 3$ obs.

of $\left\{ \begin{array}{l} x_1[i] \\ x_2[i] \end{array} \right\}_{i=1}^{3}$ & input controls

$\{u[i]\}_{i=0}^{2}$ (plus init. state $x_1[0], x_2[0]$)

Q? Do we need to solve separate L-S equ. for each row?

$$\vec{\hat{p}}_1 = (D^T D)^{-1} D^T \vec{s}_1 \quad \text{(1)}$$

$$\vec{\hat{p}}_2 = (D^T D)^{-1} D^T \vec{s}_2 \quad \text{(2)}$$

$$\underbrace{\begin{bmatrix} \vec{\hat{p}}_1 & \vec{\hat{p}}_2 \end{bmatrix}}_{P} = (D^T D)^{-1} D^T \underbrace{\begin{bmatrix} \vec{s}_1 & \vec{s}_2 \end{bmatrix}}_{S}$$

Exactly follows rules of matrix multiplication!

$$\boxed{\hat{P} = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{21} \\ \hat{a}_{12} & \hat{a}_{22} \\ \hat{b}_{11} & \hat{b}_{12} \end{bmatrix} = (D^T D)^{-1} D^T S}$$

General vector case:

$$\underset{(n\times 1)}{\vec{x}(i+1)} = \underset{(n\times n)}{A}\ \underset{(n\times 1)}{\vec{x}(i)} + \underset{(n\times m)}{B}\ \underset{(m\times 1)}{\vec{u}(i)} + \underset{(n\times 1)}{\vec{e}(i)}$$

$$\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \\ \vdots \\ x_n[i+1] \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}}_{\text{unknowns}} \begin{bmatrix} x_1[i] \\ x_2[i] \\ \vdots \\ x_n[i] \end{bmatrix} + \underbrace{\begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \\ b_{n,1} & \cdots & b_{nm} \end{bmatrix}}_{\text{unknowns}} \begin{bmatrix} u_1[i] \\ \vdots \\ u_m[i] \end{bmatrix} + \begin{bmatrix} e_1[i] \\ e_2[i] \\ \vdots \\ e_n[i] \end{bmatrix}$$

$a_1^T = [a_{11}\ a_{12}\ \ldots\ a_{1n}]$ $\qquad$ $b_1^T = [b_{11}\ \ldots\ b_{1m}]$

<u>Goal:</u> Identify (i.e. do L-S est. of) system parameters A, B.

How do we set it up as a L-S set-up?

$$D\underset{\text{unknowns}}{P} = S \Rightarrow \hat{P} = (D^TD)^{-1}D^TS$$

$$x_1[i+1] = a_1^T \vec{x}[i] + b_1^T \vec{u}[i] + \cancel{e_1[i]}$$

Let's drop this term from here on.

$$x_1[i+1] \cong \langle \vec{a}_1, \vec{x}[i] \rangle + \langle \vec{b}_1, \vec{u}[i] \rangle$$

$$\cong \langle \vec{x}[i], \vec{a}_1 \rangle + \langle \vec{u}[i], \vec{b}_1 \rangle$$

$$\boxed{x_1[i+1] \cong \vec{x}^T[i]\, \vec{a}_1 + \vec{u}^T[i]\, b_1}$$

One equation for $i = 0$.

Can get multiple observations by spanning $i = 0, 1, \ldots$ (recall we have to solve for $(\underbrace{n^2}_{A_{n\times n}} + \underbrace{mn}_{B_{n\times m}})$ unknown scalar parameters)

$$\boxed{x_1[i+1] \approxeq \vec{x}^T[i]\,\vec{a}_1 + \vec{u}^T[i]\,\vec{b}_1}$$

One equation for $i = 0$.

$$\begin{array}{c} i=0 \\ i=1 \\ \vdots \\ i=l-1 \end{array}
\underbrace{\begin{bmatrix} \vec{x}^T[0] & \vec{u}^T[0] \\ \vec{x}^T[1] & \vec{u}^T[1] \\ \vdots & \\ \vec{x}^T[l-1] & \vec{u}^T[l-1] \end{bmatrix}}_{D}
\begin{bmatrix} \vec{a}_1 \\ \vec{b}_1 \end{bmatrix}
\approxeq
\underbrace{\begin{bmatrix} x_1[1] \\ x_1[2] \\ \vdots \\ x_1[l] \end{bmatrix}}_{\vec{s}_1}$$

$$\begin{bmatrix} \vec{a}_1 \\ \vec{b}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \\ b_{11} \\ \vdots \\ b_{1m} \end{bmatrix}}_{\vec{p}_1}$$

Let's now estimate $\vec{a}_2, \vec{b}_2$:

$$\boxed{x_2[i+1] \approxeq \vec{x}^T[i]\,\vec{a}_2 + \vec{u}^T[i]\,\vec{b}_2} \qquad i = 0, 1, \ldots, l-1$$

$$
\begin{array}{l}
i=0 \rightarrow \\
i=1 \rightarrow \\
\vdots \\
i=(l-1) \rightarrow
\end{array}
\underbrace{\begin{bmatrix}
\vec{x}^T[0] & \vec{u}^T[0] \\
\vec{x}^T[1] & \vec{u}^T[1] \\
\vdots & \\
\vec{x}^T[l-1] & \vec{u}^T[l-1]
\end{bmatrix}}_{D}
\begin{bmatrix} \vec{a}_2 \\ \vec{b}_2 \end{bmatrix}
\simeq
\underbrace{\begin{bmatrix} x_1[2] \\ x_1[3] \\ \vdots \\ x_1[l+1] \end{bmatrix}}_{\vec{s}_2}
$$

$$
\underbrace{\begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \\ b_{21} \\ \vdots \\ b_{2m} \end{bmatrix}}_{\vec{p}_2}
$$

$$
\left.\begin{array}{l}
D\,\vec{p}_1 \simeq \vec{s}_1 \\
D\,\vec{p}_2 \simeq \vec{s}_2 \\
\vdots \\
D\,\vec{p}_n \simeq \vec{s}_n
\end{array}\right\}
\Rightarrow D \underbrace{\begin{bmatrix} \vec{p}_1 & \vec{p}_2 & \cdots & \vec{p}_n \end{bmatrix}}_{P} \simeq \underbrace{\begin{bmatrix} \vec{s}_1 & \vec{s}_2 & \cdots & \vec{s}_n \end{bmatrix}}_{S}
$$

$$
\Rightarrow \boxed{\hat{P} = \left(D^T D\right)^{-1} D^T S} \quad (*)
$$

L-S est.

$$
\hat{P} = \begin{bmatrix}
\hat{a}_{11} & \hat{a}_{21} & & \hat{a}_{n1} \\
\vdots & \vdots & & \vdots \\
\hat{a}_{1n} & \hat{a}_{2n} & \cdots & \hat{a}_{nn} \\
\hat{b}_{11} & \hat{b}_{21} & & \hat{b}_{n1} \\
\vdots & \vdots & & \vdots \\
\hat{b}_{1m} & \hat{b}_{2m} & & \hat{b}_{nm}
\end{bmatrix}
\quad \vec{p}_1, \ \ldots, \ \vec{p}_n
$$

$$
S = \begin{bmatrix}
x_1[1] & x_2[1] & \cdots & x_n[1] \\
\vdots & \vdots & & \vdots \\
x_1[l+1] & x_2[l+1] & & x_n[l+1]
\end{bmatrix}
\quad \vec{s}_1, \ \ldots, \ \vec{s}_n
$$

Q) How many (vector) observations $\{\vec{x}[i]\}_{i=1}^{\ell}$ and input controls $\{\vec{u}[i]\}_{i=0}^{\ell-1}$ do we need to solve for the unknown sys. ID parameters A, B?

A) Need $\ell \geq (n+m)$.

<u>Note:</u> when doing system ID, we are learning the model param. by trying various input controls $\vec{u}[i]$ and observing the system states $\vec{x}[i]$.

(1) System-ID

$$\boxed{\vec{x}[i+1] = A x[i] + B u[i]} \quad *$$

(2) Use (*) to do control.

STABILITY:

DT: $x(t+1) = \lambda\, x(t) + u(t) + e(t)$

λ: assump known

$u(t)$: input (engineer's choice)

$e(t)$: noise (nature's choice)

e.g. $x(t+1) = -2x(t)$ "unstable"

$x(0) = x(0)$

$x(1) = -2x(0)$

$x(2) = (-2)^2\, x(0)$

$\vdots$

$x(t) = (-2)^t x(0) \xrightarrow{t \to \infty} \pm\infty$

e.g. $x(t+1) = \left(\frac{1}{2}\right)^t x(t)$ "stable"

State-stability:

A system is stable iff $\exists$ a K such that $|x(t)| < K$ for all time t. $(\forall t)$

if & only if

there exists

for all

e.g. $x(t+1) = -2x(t) + u(t)$

$x(0)=1$

is unstable even for $u(t)=0$!

e.g. $x(t+1) = \frac{1}{2}x(t) + u(t)$

New definition:

Bounded-input bounded-output stability (BIBO)

BIBO stable: for every bounded input $\vec{u}$ & init. state $\vec{x}[0]$, the resulting state trajectory $\vec{x}[t]$ is bounded.

$\boxed{\text{BIBO stable iff for } E>0,\ |u(t)| \le E\ \forall t, \text{ then } \exists \text{ a } K (<\infty) \text{ s.t. } |x(t)| \le K\ \forall t}$

(must hold for any initial state $\vec{x}[0]$)

e.g. $x(t+1) = \lambda x(t) + u(t)$

$\lambda > 1$ $\qquad |u(t)| \leq E$

Guess: Not BIBO stable

Proof: Find some $u(t)$ s.t. the system blows up.

Try: $u(0) = E$, $u(t) = 0$, $t > 0$

$x(0) = x(0)$

$x(1) = \lambda x(0) + u(0) = \lambda x(0) + E$

$x(2) = \lambda(\lambda x(0) + E) + 0$

$\vdots$

$x(t) = \lambda^{t-1}(\lambda x(0) + E)$

$|x(t)| = \underbrace{|\lambda^{t-1}|}_{\to \infty} \; |\lambda x(0) + E|$

BIBO unstable

as $t \to \infty$ since $\lambda > 1$

Ex:

$$x(t+1) = \lambda x(t) + u(t)$$

$|\lambda| < 1$ $\qquad$ $|u(t)| < E$

Guess: BIBO stable

Proof: Want to be bounded for all $u(t)$!

$$x[1] = \lambda x(0) + u(0)$$

$$x(2) = \lambda x(1) + u(1)$$

$$= \lambda(\lambda x(0) + u(0)) + u(1)$$

$$= \lambda^2 x(0) + \lambda u(0) + u(1)$$

$$x(3) = \lambda^3 x(0) + (\lambda^2 u(0) + \lambda u(1) + u(2))$$

$$\boxed{x(t) = \lambda^t x(0) + \sum_{i=0}^{t-1} \lambda^i u(t-1-i)}$$

$(*)$

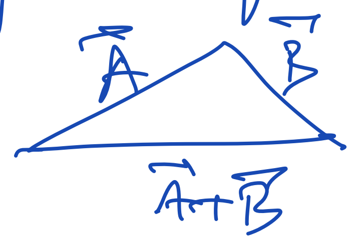
We want $|x(t)| < K \quad \forall t$

Take abs. value of both sides of $(*)$

$$|x(t)| = \left|\lambda^t x(0) + \sum_{i=0}^{t-1} \lambda^i u(t-1-i)\right|$$

(use: $|A+B| \leq |A| + |B|$)

a.k.a. Triangle-ineqn



$$|x(t)| \leq \underbrace{|\lambda^t x(0)|}_{\leq |x(0)|} + \underbrace{\left|\sum_{i=0}^{t-1} \lambda^i u(t-1-i)\right|}$$

$$\leq |x(0)| + \sum_{i=0}^{t-1} |\lambda^i| \underbrace{|u(t-1-i)|}_{\leq E}$$

(by repeated appl. of Δ-in-eq)

$$\leq |x(0)| + E \sum_{i=0}^{t-1} |\lambda|^i$$

$$\sum_{i=0}^{t-1} |\lambda|^i = 1 + |\lambda| + |\lambda|^2 + \cdots$$

(Geom sum)

$$\leq \frac{1}{1-|\lambda|} \quad \text{if } |\lambda| < 1$$

$$\leq |x(0)| + E \frac{1}{1-|\lambda|} = K$$

Bound!