Lec. 16 : (module 2, Lec. 4)

Today :
- Stability (contd.)
  - Discrete-time
  - Continuous-time
- Feedback Control
  - Stabilization with "eigenvalue placement"

Announcements

1) MT on Mon (8-10pm)
- Check ED for logistics

2) No lecture next Tues. (March 14)

Recap of stability :

$$x[t+1] = \lambda x[t] + \underbrace{u[t]}_{\text{engineer's choice}} + \underbrace{e[t]}_{\text{nature's choice}}$$

BIBO stable: System is BIBO stable iff when $|u[t]| \leq E$ and $|e(t)| \leq E$ $\forall t$, $\exists$ a $K (< \infty)$ s.t. $|x(t)| \leq K$ $\forall t$.

ex. $x(t+1) = \lambda x(t) + u(t)$

$|\lambda| > 1$ $\qquad |u(t)| \leq E$

Pf : Find some u(t) for which the sys. state blows up.

Try : $u(0) = E$, $u(t) = 0$, $t > 0$

$x(0) = x(0)$

$x(1) = \lambda x(0) + u(0) = \lambda x(0) + E$

$x(2) = \lambda(\lambda x(0) + E) + 0$

$\vdots$

$x(t) = \lambda^{t-1}(\lambda x(0) + E)$

$|x(t)| = \underbrace{|\lambda^{t-1}|}_{\to \infty} \; |\lambda x(0) + E|$

BIBO unstable

$\to \infty$ as $t \to \infty$ since $|\lambda| > 1$

Ex:

$$x(t+1) = \lambda x(t) + u(t)$$

$|\lambda| < 1$ $\qquad$ $|u(t)| < E$

Guess : BIBO stable

Proof: Want to be bounded for ALL $u(t)$!

$$x[1] = \lambda x(0) + u(0)$$

$$x(2) = \lambda x(1) + u(1)$$

$$= \lambda \left(\lambda x(0) + u(0)\right) + u(1)$$

$$= \lambda^2 x(0) + \lambda u(0) + u(1)$$

$$x(3) = \lambda^3 x(0) + \left(\lambda^2 u(0) + \lambda u(1) + u(2)\right)$$

$$\vdots$$

$$\boxed{x(t) = \lambda^t x(0) + \sum_{i=0}^{t-1} \lambda^i u(t-1-i)} \quad (*)$$

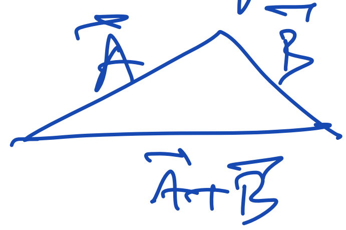
We want $|x(t)| < K \quad \forall t$

Take abs. value of both sides of $(*)$

$$|x(t)| = \left| \lambda^t x(0) + \sum_{i=0}^{t-1} \lambda^i u(t-1-i) \right|$$

(Use: $|A+B| \le |A| + |B|$)

a.k.a. Triangle-ineqn



$$|x(t)| \le \underbrace{|\lambda^t x(0)|}_{\le |x(0)|} + \underbrace{\left| \sum_{i=0}^{t-1} \lambda^i u(t-1-i) \right|}$$

$$\le |x(0)| + \underbrace{\sum_{i=0}^{t-1} |\lambda^i| \underbrace{|u(t-1-i)|}_{\le E}}_{\text{by repeated application of Triangle-inequality}}$$

$$\le |x(0)| + E \underbrace{\sum_{i=0}^{t-1} |\lambda^i|}$$

$$= |x(0)| + E \left[1 + |\lambda| + |\lambda|^2 + \dots + |\lambda|^{t-1} \right]$$

$$\leq |x(0)| + E \underbrace{\sum_{i=0}^{\infty} |\lambda|^{i}}_{\text{Geom. Series: } = \frac{1}{1-|\lambda|}\ ;\ |\lambda|<1}$$

$$\underbrace{\leq |x(0)| + \frac{E}{1-|\lambda|}}_{=K:\ \text{Bound!}} \quad ; \ |\lambda|<1$$

$\square$

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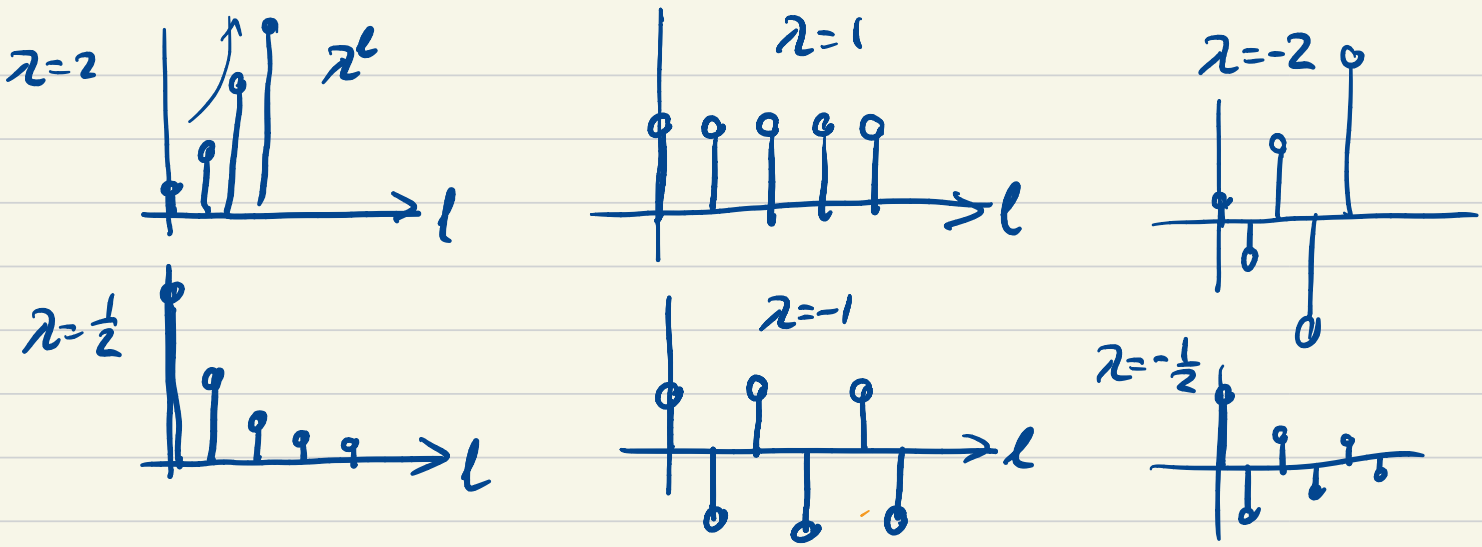
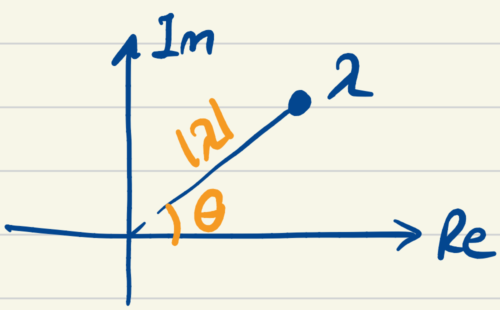
is $x[\ell] = \lambda^\ell x[0]$. Bounded if $|\lambda| \leq 1$.



What about complex λ? When does λ^ℓ remain bounded?

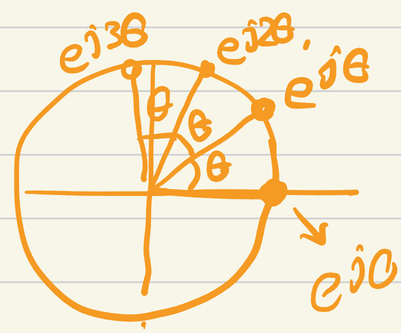


$$\lambda = |\lambda| e^{j\theta}$$

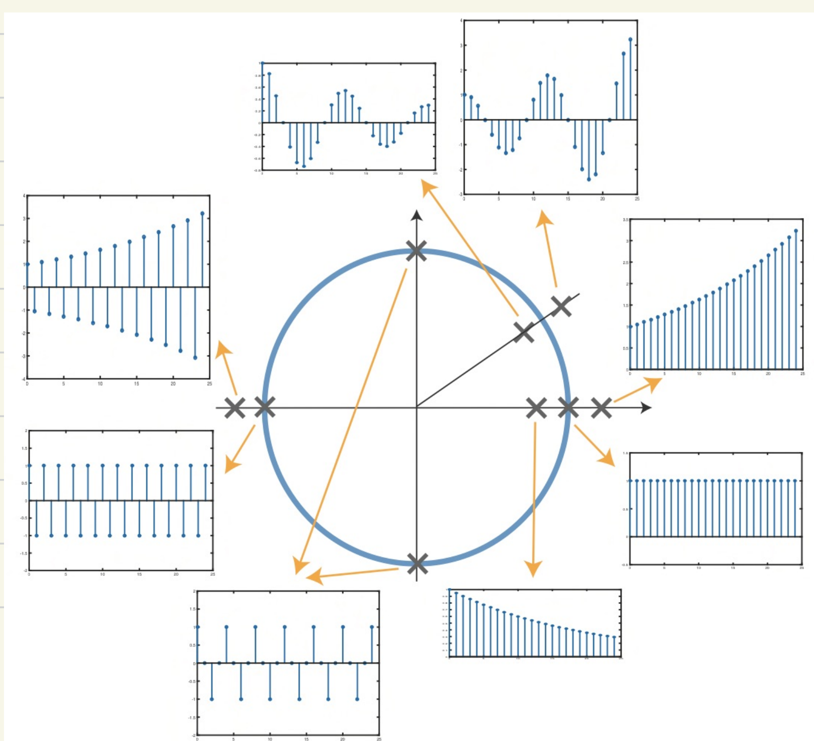
$$\lambda^\ell = |\lambda|^\ell \underbrace{e^{j\theta\ell}}$$

$$|e^{j\theta\ell}| = 1$$

$$\Rightarrow |\lambda^\ell| = |\lambda|^\ell$$



Therefore, $|\lambda| \leq 1$ required for boundedness of λ^ℓ.



λ^ℓ, $\ell = 0, 1, 2, 3, \ldots$ for various choices of λ in the complex plane, each marked with a cross. Only the real part of λ^ℓ is shown when λ is complex.

Cases: $|\lambda| > 1$: UNSTABLE

$|\lambda| < 1$: STABLE

$|\lambda| = 1$: ??

$|u(t)| < E \ \forall t$

Case: $|\lambda| = 1$.

$$x(t) = \lambda^t x(0) + \sum_{k=0}^{t-1} \lambda^k u(t-1-k)$$

$\lambda = 1$

$$x(t) = x(0) + \underbrace{\sum_{k=0}^{t-1} \underbrace{u(t-1-k)}_{=E \ \forall t}}$$

Choose $u(t-1-k) = E \ \forall t$

$$\Rightarrow \quad x(t) = x(0) + \underbrace{E.t}_{\text{Blows up as } t \to \infty}$$

Unstable!

"MARGINALLY UNSTABLE"

> Exercise: Do $\lambda = -1$
>
> Find $u(t)$ s.t. $x(t) \to \infty$ as $t \to \infty$

Vector system:

Ex.

$$\vec{x}(t+1) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \vec{x}(t) + \vec{u}(t)$$

$\hookrightarrow$ BIBO unstable

BIBO stability: All $|\text{eigenvalues}| < 1$

$$\vec{x}(t+1) = A\vec{x}(t) + \vec{u}(t) + \vec{e}(t)$$

$$A = V \Lambda V^{-1}$$

$$\Lambda = \operatorname{diag}\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Eigenvalues of A are $\{\lambda_1, \lambda_2, \ldots \lambda_n\}$

$$\vec{x}(t+1) = V \Lambda V^{-1} \vec{x}(t) + \vec{u}(t) + \vec{e}(t)$$

$$\underbrace{V^{-1}\vec{x}(t+1)}_{\vec{\tilde{x}}(t+1)} = \Lambda \underbrace{V^{-1}\vec{x}(t)}_{\vec{\tilde{x}}(t)} + \underbrace{V^{-1}\vec{u}(t)}_{\vec{\tilde{u}}(t)} + \underbrace{V^{-1}\vec{e}(t)}_{\vec{\tilde{e}}(t)}$$

$$\vec{\tilde{x}}(t+1) = \Lambda \vec{\tilde{x}}(t) + \vec{\tilde{u}}(t) + \vec{\tilde{e}}(t)$$

↗ diagonal!

→ Find the e-vals. of A — if all have $|\lambda| < 1$, then BIBO stable!

Else, not.

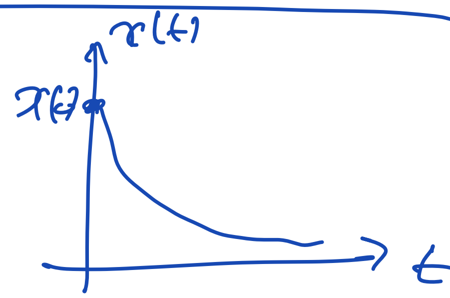
Continuous-time systems

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t)$$

$$x(t) = e^{\lambda t}x(0) + \int_0^t e^{\lambda(t-\tau)}u(\tau)\,d\tau$$

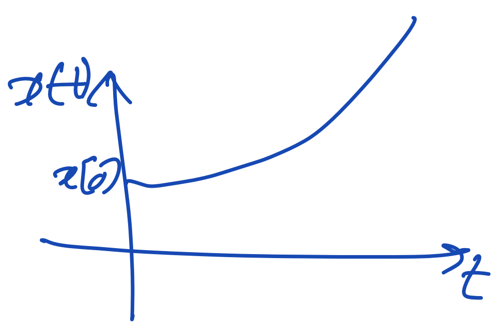
$$\frac{d}{dt}x(t) = -2x(t)$$

$$x(t) = e^{-2t}x(0)$$



$$\frac{d}{dt}x(t) = 2x(t)$$

$$x(t) = e^{2t}x(0)$$



If λ is complex

$$e^{\lambda t} = e^{(Re\{\lambda\} + j\,Im\{\lambda\})t}$$

$$= e^{Re\{\lambda\}t}\,e^{j\,Im\{\lambda\}t}$$

Continuous-time: (complex λ case)

$Re\{\lambda\} > 0$ unstable

$Re\{\lambda\} < 0$ stable

$Re\{\lambda\} = 0$ marginally unstable

Recap:

1. Identify the system: sys-ID: learnt the model parameters
2. once we have the model, is the sys. stable or not?
3. What can we do if our system is unstable? Can we make it stable? → Feedback Control
4. Can we get our system to go where we want to go? → controllability

Stabilization by FB

$$\boxed{\vec{x}[i+1] = A\,\vec{x}[i] + B\,\underbrace{\vec{u}[i]}_{\text{input}} + \underbrace{\vec{e}[i]}_{\text{disturbance}}} \quad (*)$$

OPEN-LOOP DYNAMICS

Q) What if A has an e-val. with $|\lambda| > 1$?

Can we achieve stability by designing $\vec{u}$ appropriately?

There are many choices for $\vec{u}$, let's try

$$\underbrace{\vec{u}}_{(m\times 1)} = \underbrace{F}_{(m\times n)}\underbrace{\vec{x}}_{(n\times 1)}$$

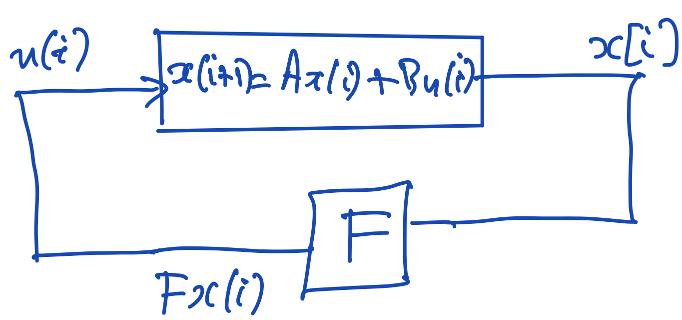
$(m=1)$

$u = F\vec{x}$

If $m=1$, $F \in \mathbb{R}^{1\times n}$

$$\boxed{u = [f_1 \; f_2 \ldots f_n]\begin{bmatrix} x_1[i] \\ x_2[i] \\ \vdots \\ x_n[i] \end{bmatrix}}$$

$$\boxed{u[i] = Fx[i] = f_1\,x_1[i] + f_2\,x_2[i] + \ldots + f_n\,x_n[i]}$$



Subst: $u[i] = Fx[i]$ in $\circledast$

$$\vec{x}(i+1) = \underbrace{(A+BF)}_{A_{CL}}\vec{x}(i) + \vec{e}(i)$$

"closed-loop system"

Q) Can we design F s.t. e-vals. of $A_{CL} = A + BF$ are inside the unit circle (u.c.).

Let's try some examples:

Ex 1 (SCALAR) $x(i+1) = 2x(i) + u(i)$

unstable without feedback

$$u(i) = f\,x(i)$$

Closed-loop: $x(i+1) = (2+f)\,x(i)$

For stability, we want $|2+f| < 1$

If we want an e-val @ $\lambda_0 \Rightarrow 2+f = \lambda_0$

$$\boxed{f = \lambda_0 - 2}$$

Ex. 2

$\vec{x}(i+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}}_{A} \vec{x}(i) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(i)$

$(m=1,\ n=2)$

$A_{CL} = A + BF = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} [f_1 \ \ f_2]}_{\begin{bmatrix} 0 & 0 \\ f_1 & f_2 \end{bmatrix}}$

$= \begin{bmatrix} 0 & 1 \\ 3+f_1 & 2+f_2 \end{bmatrix}$

E-vals of A:

$\det(\lambda I - A) = 0$

$\det \begin{bmatrix} \lambda & -1 \\ -3 & \lambda-2 \end{bmatrix} = 0$

$\Rightarrow \lambda(\lambda-2) - 3 = 0$

$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$

$\Rightarrow (\lambda-3)(\lambda+1) = 0$

$\lambda = 3, -1$

unstable!

E-vals of A_{CL}

$\Rightarrow \det\left[\begin{pmatrix} \lambda & -1 \\ -3-f_1 & \lambda-2-f_2 \end{pmatrix}\right] = 0$

$\Rightarrow \boxed{\lambda^2 - (2+f_2)\lambda - (3+f_1) = 0}$ (1)

If we want e-vals (A_{CL}) @ λ_1, λ_2.

$$\boxed{(\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2 = 0}$$ (2)

We want: $-3 - f_1 = \lambda_1\lambda_2$

$$\boxed{f_1 = -\lambda_1\lambda_2 - 3}$$ (3)

Also, $2 + f_2 = \lambda_1 + \lambda_2$

$$\boxed{f_2 = \lambda_1 + \lambda_2 - 2}$$ (4)

e.g. if we want $\lambda_1 = \frac{1}{2}$, $\lambda_2 = -\frac{1}{4}$ (stable!)

$$f_1 = -3 - \frac{1}{8} = -3\tfrac{1}{8}$$

$$f_2 = -1\tfrac{3}{4}.$$

Almost magical. We can "place eigenvalues" from $3, -1$ to λ_1, λ_2.

Q) Does this always work?

Ex. 3

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \vec{x}[i] + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u[i]$$

E-vals of A are $1, 2 \Rightarrow$ unstable!

$$A_{CL} = A + BF = \begin{bmatrix} 1+f_1 & 1+f_2 \\ 0 & 2 \end{bmatrix}.$$

e-vals of A_{CL} are $(1+f_1)$ & 2.

still unstable!

No choice of f_1 & f_2 can make the closed-loop system stable!