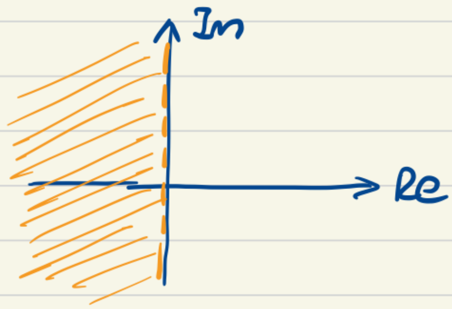
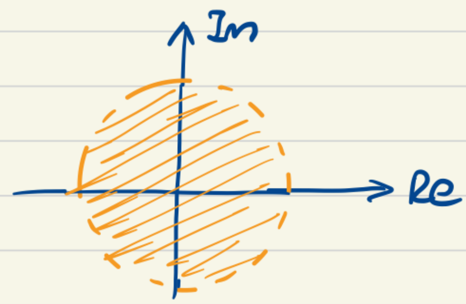
Lec. 17 : (module 2, Lec. 5)

Today:

Last time	Today
• Stability • Feedback Control.: "eigenvalue placement" using $\vec{u} = F\vec{x}$	• Wrap up FB control • Controllability

Notes 9 (stability + FB control), 10 (controllability) posted.

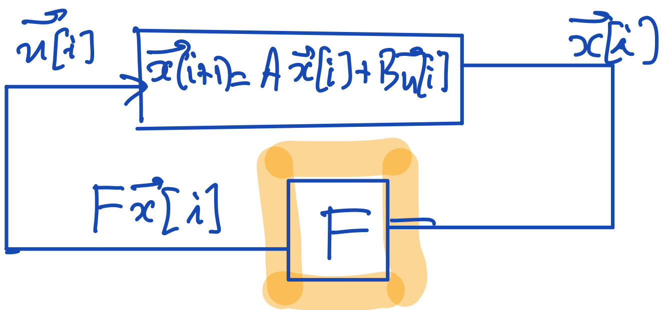
Summary: (BIBO stability)

	continuous-time	discrete-time
	$\frac{d}{dt}\vec{x}(t) = A_c\vec{x}(t) + \vec{w}(t)$	$\vec{x}[i+1] = A_d\vec{x}[i] + \vec{w}[i]$
		
stability condition	$\mathrm{Re}\,\lambda_k < 0$ for each eigenvalue of A_c, $k = 1, \ldots, n$	$\lvert\lambda_k\rvert < 1$ for each eigenvalue of A_d, $k = 1, \ldots n$

i.e. eigenvalues must be in the shaded regions above.

Recap of Feedback Control:

$$\boxed{\begin{array}{l}\text{Stabilization by Feed Back Control} \\ \vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i]\end{array}} \quad (*)$$



Substitute $\vec{u}[i] = F\vec{x}[i]$ in (*):

$$\vec{x}[i+1] = \underbrace{(A+BF)}_{A_{CL}}\vec{x}[i] + \vec{e}[i]$$

"closed-loop system"

$$\boxed{A_{CL} = A + BF}$$

achievable using linear FB control policy: $\vec{u} = F\vec{x}$

closed-loop system matrix

open-loop system matrix

EXAMPLE-1 (SCALAR) $x[i+1] = 2x[i] + u[i]$

unstable in "open-loop" system

$$u(i) = f\,x(i)$$

Closed-loop: $x(i+1) = \overbrace{(2+f)}^{a+bf}\,x(i)$

For stability, we want $|2+f| < 1$

If we want an e-val @ $\lambda_0 \Rightarrow 2+f = \lambda_0$

$$\boxed{f = \lambda_0 - 2}$$

EXAMPLE-2 $\vec{x}(i+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}}_{A} \vec{x}(i) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(i)$

$(m=1,\ n=2)$

$$A_{CL} = A + BF = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} [f_1 \ \ f_2]}_{\begin{bmatrix} 0 & 0 \\ f_1 & f_2 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 1 \\ 3+f_1 & 2+f_2 \end{bmatrix}$$

E-vals of A : (roots of characteristic polynomial)

$\det(\lambda I - A) = 0$

$\det\begin{bmatrix} \lambda & -1 \\ -3 & \lambda-2 \end{bmatrix} = 0$

$\Rightarrow \lambda(\lambda-2) - 3 = 0$

$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$

$\Rightarrow (\lambda-3)(\lambda+1) = 0$

$\lambda = 3, -1$

unstable!

E-vals of A_{CL}

$\det\left[\begin{pmatrix} \lambda & -1 \\ -3-f_1 & \lambda-2-f_2 \end{pmatrix}\right] = 0$

$$\boxed{\lambda^2 - (2+f_2)\lambda - (3+f_1) = 0} \quad (1)$$

If we want e-vals. of A_{CL} at λ_1, λ_2.

$$\boxed{(\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2 = 0} \quad (2)$$

By pattern-matching (1) & (2) : $-3-f_1 = \lambda_1\lambda_2$

$$\boxed{f_1 = -\lambda_1\lambda_2 - 3} \quad (3)$$

Also, $2+f_2 = \lambda_1+\lambda_2$

$$\boxed{f_2 = \lambda_1+\lambda_2-2} \quad (4)$$

We can place e-vals. λ_1, λ_2 anywhere we want by designing f_1, f_2 according to (3), (4).!!! ooo

Q) Does this always work?

Ex. 3

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \vec{x}[i] + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u[i]$$

E-vals. of A are $1, 2$ $\Rightarrow$ unstable!

$$A_{CL} = A + BF = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [f_1 \; f_2]$$

$$= \begin{bmatrix} 1+f_1 & 1+f_2 \\ 0 & 2 \end{bmatrix}$$

e-vals of A_{CL} are $(1+f_1)$ & 2.

still unstable!

No choice of f_1 & f_2 can make the closed-loop system stable!

$\boxed{\text{What makes Example 2 work and Example 3 fail?}}$

Controllability:

$$\vec{x}[i+1] = A\vec{x}[i] + B\,u[i] \qquad \left(\begin{array}{c}\text{assume}\\ \text{single}\\ \text{control}\\ \text{input}\end{array}\right)$$

$\underline{i=0}$: $\quad \vec{x}[1] = A\vec{x}[0] + B\,u[0]$

$\underline{i=1}$: $\quad \vec{x}[2] = A\vec{x}[1] + B\,u[1]$

$$= A^2\vec{x}[0] + AB\,u[0] + B\,u[1]$$

$\vdots$

$\underline{i=l-1}$: $\quad \vec{x}[l] = A^l\vec{x}[0] + A^{l-1}B\,u[0] + \dots + AB\,u[l-2] + B\,u[l-1]$

$$\boxed{\vec{x}[l] - A^l\vec{x}[0] = \underbrace{\left[A^{l-1}B \,\middle|\, A^{l-2}B \,\middle|\, \dots \,\middle|\, AB \,\middle|\, B\right]}_{\triangleq\, C_l} \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[l-1] \end{bmatrix}}$$

"controllability matrix"

Q) Can we find sequence $\{u[0], u[1], \dots, u[l-1]\}$ that brings $\vec{x}[0]$ from any state @ time 0 to a target $\vec{x}_{target}$ @ time l?

Yes! if $\vec{x}_{target} - A^{\ell}\vec{x}[0]$ lies in the COLUMN SPAN of

$$C_\ell \triangleq \left[A^{\ell-1}B \,|\, A^{\ell-2}B \,|\, \dots \,|\, AB \,|\, B\right]$$

"controllability": ability to reach any target state $\vec{x}_{target}$ from any initial state $\vec{x}[0]$.

Defn:
A system is called controllable if given any target state $\vec{x}_{target}$ & I.C. $\vec{x}[0]$, we can find an ℓ & an input sequence $\{u[0], \dots u[\ell-1]\}$ s.t. $\vec{x}[\ell] = \vec{x}_{target}$.

So, in a word, you can go anywhere in $\mathbb{R}^n$ ($\vec{x}$ is n-dimensional)

Q) How do we check this? If C_ℓ has n linearly indep. columns for some ℓ, then the column space is $\mathbb{R}^n$. This means that we can make "$C_\ell \begin{bmatrix} u(0) \\ \vdots \\ u(\ell-1) \end{bmatrix}$"

(RHS of (*)) anything in $\mathbb{R}^n$ by choosing input control sequence $\{u[0], .. u[\ell-1]\}$.

If we assign $C_\ell \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[\ell-1] \end{bmatrix} = \vec{x}_{target} - A^\ell \vec{x}[0]$,

then in (*),

$$\vec{x}[\ell] = A^\ell \vec{x}[0] + C_\ell \begin{bmatrix} u[0] \\ \vdots \\ u[\ell-1] \end{bmatrix}$$

$$= \cancel{A^\ell \vec{x}[0]} + \vec{x}_{target} - \cancel{A^\ell \vec{x}[0]}$$

$$= \vec{x}_{target}$$

as desired.

Q) How do we check this? If C_ℓ has n linearly indep. columns for some ℓ, then the column space is $\mathbb{R}^n$. This means that we can make "$C_\ell \begin{bmatrix} u(0) \\ \vdots \\ u(\ell-1) \end{bmatrix}$"

(RHS of (*)) anything in $\mathbb{R}^n$ by choosing input control sequence $\{u[0], .. u[\ell-1]\}$.

Ex. 2

$n=2$
$m=1$

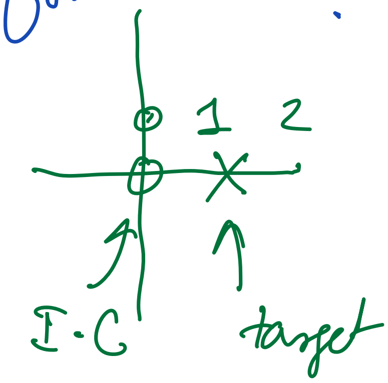
$$\vec{x}(i+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}}_{A} \vec{x}[i] + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u[i]$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad ; \quad AB = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$C_1 = B$ (dim $= 1$) $\qquad C_2 = [AB \mid B]$

$$C_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad \text{dim} = 2$$

controllable for $l = 2$.



ex: $\vec{x}[0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\vec{x}_{target} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Can $\vec{x}[1] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

Can $\vec{x}[2] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

$$\underbrace{\vec{x}[2]}_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} u[0] \\ u[1] \end{bmatrix}}_{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

If $\begin{bmatrix} u[0] \\ u[1] \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, then $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, as desired!

Ex. 3

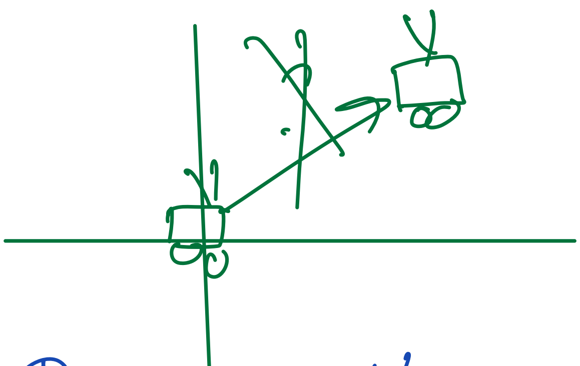
$$\vec{x}[i+1] = \underbrace{\begin{bmatrix}1&1\\0&2\end{bmatrix}}_{A}\vec{x}[i] + \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{B}u[i]$$

$C_1 = B$ (dim = 1)

$C_2 = [AB \mid B] = \begin{bmatrix}1&1\\0&0\end{bmatrix}$ (dim = 1)

$C_3 = [A^2B \mid AB \mid B] = \begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}$ (dim = 1)

not controllable for any ℓ!



Dim. can't increase further:
stuck @ $1 < n = 2$.

What we observe about getting stuck is due to the following lemma:

<u>Lemma:</u> If $A^l B$ is linearly dependent on $\{A^{l-1}B, A^{l-2}B, \dots AB, B\}$, then $A^{l+1}B$ is also linearly dependent on $\{A^{l-1}B, \dots AB, B\}$

<u>Proofs:</u> $A^l B = \alpha_{l-1} A^{l-1} B + \dots + \alpha_1 AB + \alpha_0 B$ for some $\alpha_{l-1}, \dots \alpha_1, \alpha_0$ by linear dependence

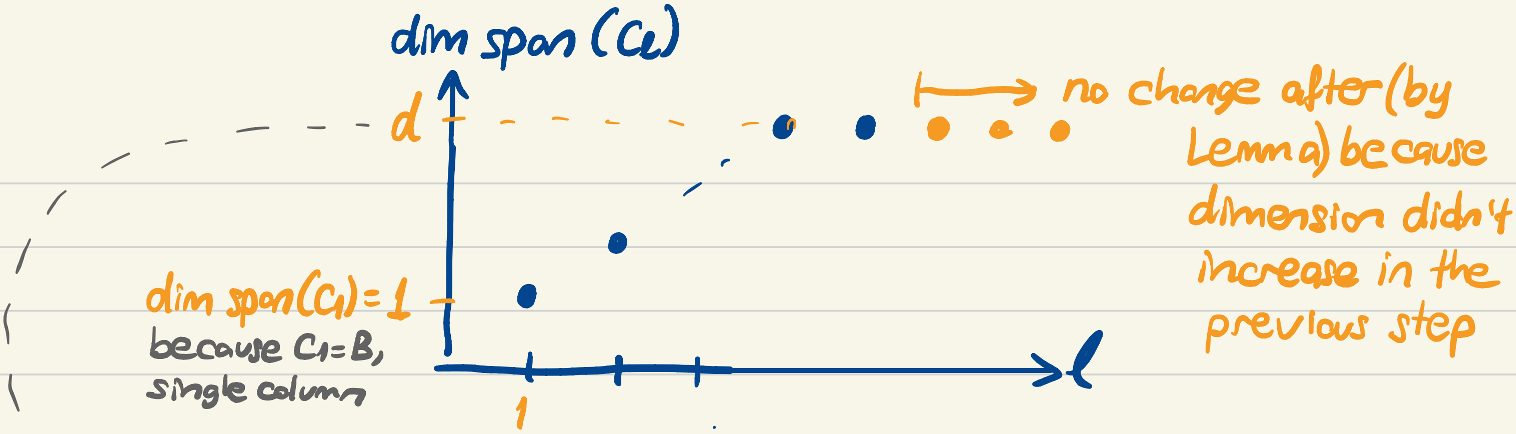
Then, $A^{l+1}B$

$$
\begin{aligned}
&= A \cdot A^l B \\
&= A\left[\alpha_{l-1} A^{l-1} B + \dots + \alpha_1 AB + \alpha_0 B\right] \\
&= \alpha_{l-1} A^l B + \alpha_{l-2} A^{l-1} B + \dots + \alpha_1 A^2 B + \alpha_0 AB \\
&= \alpha_{l-1}\left[\alpha_{l-1} A^{l-1} B + \dots + \alpha_1 AB + \alpha_0 B\right] \\
&= * A^{l-1} B + * A^{l-2} B + \dots + * AB \\
&\quad + * B
\end{aligned}
$$

□

Lemma implies: if $\dim \operatorname{span}(C_{\ell+1}) = \dim \operatorname{span}(C_\ell) = d$, then $\dim \operatorname{span}(C_{\ell+2}) = \dim \operatorname{span}(C_{\ell+1}) = d$.

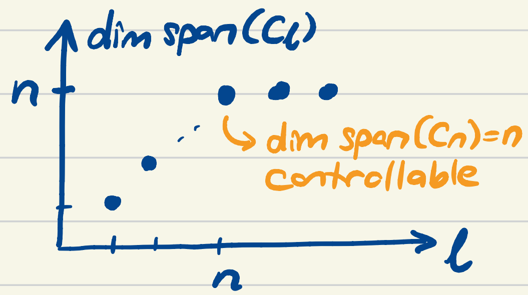
<u>Once the dimension stops growing, it ~~stops~~ stops growing forever!</u>

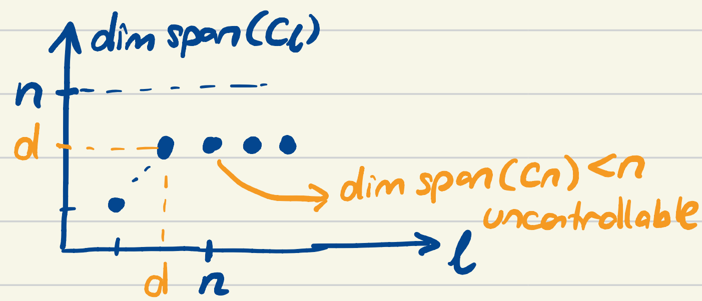


Let d denote the dimension at which we get stuck.
If $d = n$: CONTROLLABLE
If $d < n$: UNCONTROLLABLE (will never reach n).

We could write a code that increments ℓ by one as long dim span (C_ℓ) keeps growing and terminates once growth stops (it has to stop because dimension can't exceed n). Then we can apply the test above with the dimension d reached to check controllability...

OR we can be smarter: dimension grows by one at each step as long as it grows (we are adding a single column when we increment ℓ by 1). If we are able to reach n, we will reach it at $\ell = n$. Otherwise, growth will have stopped before n, so dim span(C_n) will be $d < n$. Thus, all we have to do is check if dim span$(C_n) = n$. Controllable if so, uncontrollable if not. One-shot test!





Controllability condition: $\dim \operatorname{span}(C_n) = n$
i.e., $C_n = [A^{n-1}B \;\cdots\; AB \;\; B]$ has n linearly indep. columns.

FACT: Any controllable system

$$\vec{x}[n+1] = A\,\vec{x}[n] + B\,u[n];$$

(i.e. one having controllability matrix

$$C_n = \left[\, A^{n-1}B \;\middle|\; \cdots \;\middle|\; A^2B \;\middle|\; AB \;\middle|\; B \,\right]$$ that

separates columns of C_n

is rank-n & therefore invertible)

can be "transformed" to a canonical form.

$$\vec{y}[n+1] = A_y\,\vec{x}[n] + B_y\,u[n]$$

where $A_y = \left[\begin{array}{c|ccc} 0 & & & \\ \vdots & & I_{n-1} & \\ \hline -\alpha_1 & -\alpha_2 & \cdots & -\alpha_n \end{array}\right]$ & $B_y = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

where $\vec{y}[n] = T\,\vec{x}[n]$ for a carefully chosen invertible transformation matrix T.