EECS 16B Lecture 19 (Module 2, Lecture 7)

Last time:

- Gram-Schmidt orthogonalization of linearly independent vector set.

Today:

- Recap of G-S orthogonalization
- G-S w/ linearly dependent vector set
- Revisit BIBO stability:

— Upper-triangularization: what to do when system matrix is not diagonalizable?

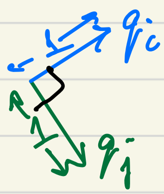
Check out G-S 3-D animation on youtube:

https://youtu.be/79Ss_HkwthF

RECAP:

Orthonormal bases and Gram-Schmidt Procedure:

Column vectors $\vec{q}_1, \dots \vec{q}_k$ are called orthonormal if



$$\vec{q}_i^{\,T} \vec{q}_j = \begin{cases} 0 & \text{if } i \neq j \quad (\text{ortho}) \\ 1 & \text{if } i = j \quad (\text{normal}) \end{cases} \quad --(1)$$

A matrix $Q = [\vec{q}_1 \cdots \vec{q}_k]$ with orthonormal columns

satisfies:

$\vec{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{q}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

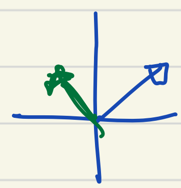
$Q = [\vec{q}_1 \ \vec{q}_2] \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$; $Q^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$Q^T Q = \begin{bmatrix} \vec{q}_1^{\,T} \\ \vdots \\ \vec{q}_k^{\,T} \end{bmatrix} [\vec{q}_1 \ \cdots \ \vec{q}_k] = \begin{bmatrix} \vec{q}_1^{\,T}\vec{q}_1 & \cdots & \vec{q}_1^{\,T}\vec{q}_k \\ \vdots & & \\ \vec{q}_k^{\,T}\vec{q}_1 & \cdots & \vec{q}_k^{\,T}\vec{q}_k \end{bmatrix}$$

$$= I_{k \times k} \text{ by def'n } (1).$$

$$\boxed{Q^T Q = I}$$

If Q is square $Q^T Q = I$, means:



$$Q^T = Q^{-1}.$$

(Q is called orthogonal.)

Gram-Schmidt Algorithm / Procedure

Given $\{\vec{s_1}, \vec{s_2}, \dots \vec{s_n}\}$

Convert this to a set

$$\{\vec{q_1}, \vec{q_2}, \dots \vec{q_n}\}.$$

such that: $\langle \vec{q_i}, \vec{q_i} \rangle = \|\vec{q_i}\|^2 = 1$

$$\langle \vec{q_i}, \vec{q_j} \rangle = 0$$

<u>and:</u>

$$\text{span}\{\vec{s_1}\} = \text{span}\{\vec{q_1}\}$$

$$\text{span}\{\vec{s_1}, \vec{s_2}\} = \text{span}\{\vec{q_1}, \vec{q_2}\}.$$

$$\vdots$$

$$\text{span}\{\vec{s_1}, \vec{s_2}, \dots \vec{s_n}\} = \text{span}\{\vec{q_1}, \vec{q_2}, \dots \vec{q_n}\}.$$

Consider: linearly independent set.

$$\{\vec{s_1}, \vec{s_2}, \dots \vec{s_n}\} \quad \text{lin indep.}$$

Gram-Schmidt - Alg.

$\{\vec{s_1}, \vec{s_2}, \vec{s_3}\}$.

(1) $\{\vec{s_1}\} \rightarrow \vec{q_1}$ find.

$$\frac{\vec{s_1}}{\|\vec{s_1}\|} = \vec{q_1} \longrightarrow \text{unit norm.}$$

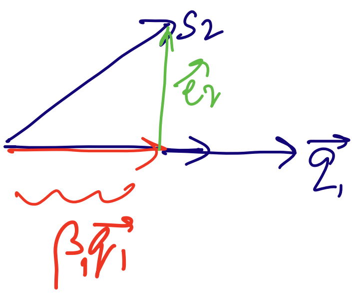
$\text{span}\{\vec{q_1}\} = \text{span}\{\vec{s_1}\}$ ✓

(2) $\{\vec{s_1}, \vec{s_2}\}$

$$\boxed{\vec{q_1} = \frac{\vec{s_1}}{\|\vec{s_1}\|}}$$

What is new in $\vec{s_2}$, that is not captured by $\vec{q_1}$.

Remove from $\vec{s_2}$, the projection of $\vec{s_2}$ onto $\vec{q_1}$



$$\vec{e_2} = \vec{s_2} - \overbrace{\boxed{\frac{\langle \vec{s_2}, \vec{q_1} \rangle}{\|\vec{q_1}\|^2}}}^{\beta_1} \cdot \vec{q_1}$$

$$\vec{e_2} = \vec{s_2} - \langle \vec{s_2}, \vec{q_1} \rangle \cdot \vec{q_1}$$

$$\vec{q_2} = \frac{\vec{e_2}}{\|\vec{e_2}\|} \longrightarrow \underline{\underline{\text{unit norm}}}$$

<u>Check</u>: $\langle \vec{q_2}, \vec{q_1} \rangle = \left\langle \frac{\vec{e_2}}{\|\vec{e_2}\|}, \vec{q_1} \right\rangle$

$$= \left\langle \frac{\vec{s}_2 - \langle \vec{s}_2, \vec{q}_1 \rangle \cdot \vec{q}_1}{\|\vec{e}_2\|}, \vec{q}_1 \right\rangle$$

$$= \frac{1}{\|\vec{e}_2\|} \left[\langle \vec{s}_2, \vec{q}_1 \rangle - \langle \vec{s}_2, \vec{q}_1 \rangle \underbrace{\langle \vec{q}_1, \vec{q}_1 \rangle}_{1} \right]$$

$$= 0$$

Furthermore: $\text{span}\{s_1, \vec{s}_2\} = \text{span}\{\vec{q}_1, \vec{q}_2\}$.

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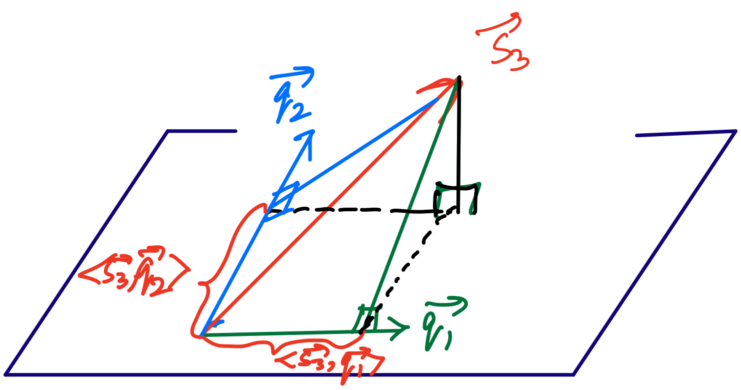
③ $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$. $\boxed{\vec{q}_1, \vec{q}_2}$



Project $\vec{s}_3$ onto span $\{\vec{q}_1, \vec{q}_2\}$

$$\text{proj} = \begin{bmatrix} \langle \vec{s}_3, \vec{q}_1 \rangle \\ \langle \vec{s}_3, \vec{q}_2 \rangle \end{bmatrix} \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix}$$

$$= \underbrace{\langle \vec{s}_3, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{s}_3, \vec{q}_2 \rangle \vec{q}_2}_{\text{true by projection formula \& the fact that } \vec{q}_1, \vec{q}_2 \text{ are orthonormal}}$$

$$\vec{e_3} = \vec{s_3} - \left(\langle \vec{s_3}, \vec{q_1} \rangle \vec{q_1} + \langle \vec{s_3}, \vec{q_2} \rangle \vec{q_2} \right)$$

$$\vec{q_3} = \frac{\vec{e_3}}{\|\vec{e_3}\|} \quad \checkmark$$

Check: $\text{span}\{\vec{q_1}, \vec{q_2}, \vec{q_3}\} = \text{span}\{\vec{s_1}, \vec{s_2}, \vec{s_3}\}$

Check: $\langle \vec{q_3}, \vec{q_1} \rangle = \langle \vec{q_3}, \vec{q_2} \rangle = 0$.

$$\underbrace{\overbrace{\begin{bmatrix} | & | & | \\ \vec{s_1} & \vec{s_2} & \vec{s_3} \\ | & | & | \end{bmatrix}}^{A}}_{\text{Columnspace}} \longleftrightarrow \underbrace{\begin{bmatrix} | & | & | \\ \vec{q_1} & \vec{q_2} & \vec{q_3} \\ | & | & | \end{bmatrix}}_{\text{Basis for columnspace of } A.}$$

$$\boxed{\vec{q_1}, \vec{q_2}, \vec{q_3}}$$

Check out 3-D animation on:

https://youtu.be/79Ss_HkwthF

What if $\{\vec{s_1}, \vec{s_2} \ldots \vec{s_n}\}$ is not independent?

① $\{\vec{s_1}, \vec{s_2}, \vec{s_3}\}$

Suppose $\vec{s_2} = 2\vec{s_1}$. $\longrightarrow \{\vec{q_1}, \vec{q_2}, \vec{q_3}\}$?

Step ①: $\vec{s_1} \longrightarrow$ normalize

$$\vec{q_1} = \frac{\vec{s_1}}{\|\vec{s_1}\|}$$

Step ②: $\vec{e_2} = \vec{s_2} - \mathrm{Proj}_{\vec{q_1}}(\vec{s_2})$

$$= \vec{s_2} - \langle \vec{s_2}, \vec{q_1} \rangle \vec{q_1}$$

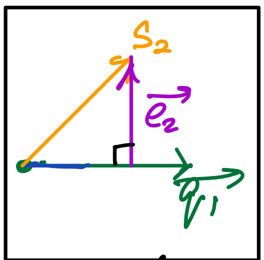
since $\vec{q_1}$ is a unit vector

$$= \vec{s_2} - \|\vec{s_2}\| \vec{q_1}$$

$\vec{s_2} = \|\vec{s_2}\| \vec{q_1}$

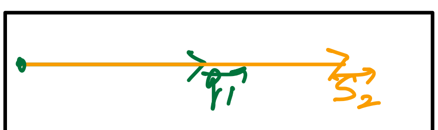
$$= \vec{s_2} - \vec{s_2}$$

$$= \boxed{0}$$



Normal geometry

No "new" dimension in $\vec{s_2}$ compared to $\vec{s_1}$.



Geometry when $\vec{s_2} = 2\vec{s_1}$

What do we do?

Ignore $\vec{s_2}$! Don't add a $\vec{q_2}$ corresponding to $\vec{s_2}$

② Gram-Schmidt for building a basis:

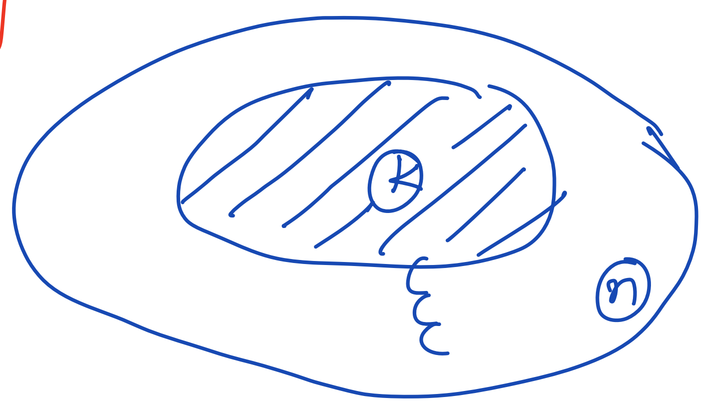
$\underbrace{\vec{s_1}, \vec{s_2}, \ldots \vec{s_k}}_{\text{not a basis: why?}} \in \mathbb{R}^n$ $\quad k < n$

Q) Can we create an $\perp$-basis (orthonormal basis) using $\vec{s_1}, \vec{s_2}, \ldots \vec{s_k}$?

Consider: $\{\vec{s_1}, \vec{s_2}, \ldots \vec{s_k}, \underbrace{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}}\}$

"standard" basis set for $\mathbb{R}^n$

$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$; $\vec{e_i} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ith position



→ Do G-S in this order

$\frac{\vec{s_1}}{\|\vec{s_1}\|} = \vec{q_1}$.

$\vdots$

Q) Max # of linearly indep. vectors that can come out = ?

A) n

$\{\vec{q_1}, \vec{q_2}, \ldots \vec{q_n}\}$ → Guaranteed that all other vectors are linear combinations

Important: First vector $\vec{q_1}$ is a scalar multiple of $\vec{s_1}$.

Example :

$\vec{s_1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$; $\vec{s_2} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$;

$\vec{q_1} = \dfrac{\vec{s_1}}{\|\vec{s_1}\|} = \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ✓ $\boxed{q_1}$

$\vec{r_2} = \vec{s_2} - \langle \vec{s_2}, \vec{q_1} \rangle \vec{q_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} - \langle (4, 0), (1, 0) \rangle \vec{q_1}$

$= \begin{pmatrix} 4 \\ 0 \end{pmatrix} - 4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ✗ $\boxed{q_1}$

Add $\vec{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\vec{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{r_3} = \vec{e_1} - \langle \vec{e_1}, \vec{q_1} \rangle \vec{q_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ✗ $\boxed{q_1}$

$\vec{r_4} = \vec{e_2} - \langle \vec{e_2}, \vec{q_1} \rangle \vec{q_1}$

$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ✓ $\boxed{q_1, q_2}$

$\{ \vec{s_1}, \vec{s_2}, \vec{e_1}, \vec{e_2}, \vec{e_3} \} \longrightarrow \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

ON basis for $\mathbb{R}^2$!

BIBO stability

$$\vec{x}[k+1] = A\vec{x}[k] + \vec{u}[k]$$

Diagonalizable if linearly indep. e-vectors

System is stable if all e-vals of

A : $|\lambda| < 1$

Q? What if A is not diagonalizable?

Ex. $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

e-vals of A : λ, λ

e-vectors of A are <u>not</u> independent

$$A\vec{v} = \lambda\vec{v} \Rightarrow \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Solve: $\lambda v_1 + v_2 = \lambda v_1 \Rightarrow v_2 = 0$

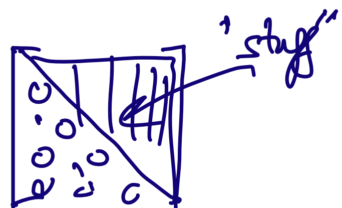
$\lambda v_2 = \lambda v_2$

$\Rightarrow$ e-vec. are $\alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Next Best Thing: Upper-triangular Matrix

$$\frac{d\vec{x}(t)}{dt} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix}\vec{x}(t)$$

$\frac{dx_2(t)}{dt} = \lambda_2 x_2(t)$ ✓



$\frac{dx_1}{dt} = \lambda_1 x_1(t) + x_2(t)$

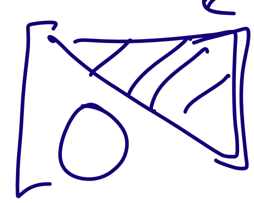
Q) How can we convert a square matrix into UT (upper-triangular) form using a change of basis?

(Similar to a "diagonalizing" basis or a "CCF-generating" basis that we have seen earlier).

Q) If M is not diagonalizable, can we find a U s.t. $U^{-1} M U = T$

$U = [\vec{v}_1 \ \vec{v}_2 \cdots \vec{u}_n]$

ON basis



A) YES!

ANY square matrix can be UTized

Simplest case: $M = 1 \times 1$ matrix

$M = m$.

upper-triangular!

Let's build some intuition.

$M = 2\times 2$ case:

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

Assume M has all real e-vals (for convenience only)

$$U = \begin{bmatrix} ? & ? \end{bmatrix}$$

Try e-vectr of M maybe?

① Say $\vec{v}_1$ is one e-vec of M

$M\vec{v}_1 = \lambda_1 \vec{v}_1$ $\qquad \vec{v}_1 \in \mathbb{R}^2$

$\longrightarrow$ Assume WLOG that $\|\vec{v}_1\| = 1$

$$U = \begin{bmatrix} \vec{v}_1 & \bullet \end{bmatrix}$$

② Build out the ON basis using G-S

Say $\vec{r}_1$ is the vector that "completes" the G-S procedure with $\vec{v}_1$ as the first or "anchor" vector.

$\rightarrow$ We know that $[\vec{v}_1 \ \vec{r}_1]$ form an ON basis.

$\vec{v}_1 \perp \vec{r}_1$ $\qquad \langle \vec{v}_1, \vec{r}_1 \rangle = 0$

To check if U works try $U^{-1}MU$.
(Recall U is an ON basis by G-S construction)

$$U_2^{-1}MU_2 = U_2^T M U_2$$
$$= \begin{bmatrix} - v_1^T - \\ - \vec{r}_1^T - \end{bmatrix} \begin{bmatrix} | & | \\ M\vec{v}_1 & M\vec{r}_1 \\ | & | \end{bmatrix}$$
$$= \begin{bmatrix} \vec{v}_1^T M \vec{v}_1 & \vec{v}_1^T M \vec{r}_1 \\ \vec{r}_1^T M \vec{v}_1 & \vec{r}_1^T M \vec{r}_1 \end{bmatrix}$$

<u>Check:</u> $\vec{r}_1^T M \vec{v}_1 = \lambda_1 \vec{r}_1^T \vec{v}_1 = 0$

$$\vec{v}_1^T M \vec{v}_1 = \lambda \vec{v}_1^T \vec{v}_1 = \lambda$$

$$U_2^{-1} M U_2 = \begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix} = T_2 \checkmark$$

Let's build up from the 2×2 case:

<u>Case:</u> 3×3

$$U_3^{-1} M U_3 = T_3$$

$$U_3 = \begin{bmatrix} \vec{v}_1 & \bullet & \bullet \end{bmatrix}$$

(1) $M\vec{v}_1 = \lambda_1 \vec{v}_1$ ($\vec{v}_1$ is an e-vec of M)

(2) Use $\vec{v}_1$ to start the G-S procedure as before to "complete" an ON basis for $\mathbb{R}^3$.

$$\{\vec{v}_1, \vec{r}_1, \vec{r}_2\}$$

Define $\underbrace{R}_{3\times 2} = \begin{bmatrix} \vec{r}_1 & \vec{r}_2 \end{bmatrix}$

$$\underbrace{U_3}_{(3\times 3)} = \begin{bmatrix} \underbrace{\vec{v}_1}_{(3\times 1)} & \underbrace{R}_{(3\times 2)} \end{bmatrix}$$

U_3 is an ON matrix

Consider $\begin{bmatrix} \vec{v}_1 & R \end{bmatrix}^T M \begin{bmatrix} \vec{v}_1 & R \end{bmatrix}$

$$= \begin{bmatrix} \vec{v}_1^T \\ R^T \end{bmatrix} M \begin{bmatrix} \vec{v}_1 & R \end{bmatrix}$$

$$= \begin{bmatrix} \vec{v}_1^T M \vec{v}_1 & v_1^T M R \\ R^T M \vec{v}_1 & R^T M R \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & \boxed{v_1^T M R} \\ \boxed{\begin{matrix} 0 \\ 0 \end{matrix}} & \boxed{R^T M R} \end{bmatrix}$$

$$R^T M \vec{v}_1 = \lambda_1 R^T \vec{v}_1 = 0$$

Need $R^T M R$ to also be UT!