EECS 16B Lecture 20 (Module 2, Lecture 8)

Announcements:

- Student Support Hours: Mon 1-3 pm
  Everyone welcome! Schedule an appt. on course calendar on class website!

- 0.25 EC point for each lecture you attend for rest of the term.
  links.eecs16b.org/lecture-ec

Last time:

- Upper-triangularization

Today:

- Recap of upper triangularization:
  any square matrix M can be upper-triangularized $(U^T M U = T)$ also known as SCHUR DECOMPOSITION
- Proof of upper-triangularization:
- Eigenvalues of upper-triangular matrices
- Revisiting BIBO stability: non-diagonalizable system matrix

(Read Note 13 for details)

Recap: (from last time)

$$\vec{x}[k+1] = A\,\vec{x}[k] + B\,\vec{u}[k]$$

$\vec{x}$:	$n \times 1$
A:	$n \times n$
B:	$n \times m$
$\vec{u}$:	$m \times 1$

- A diagonalizable if all n e-vectors of A are independent.
- BIBO stable if all e-vals of A: $|\lambda_i(A)| < 1$ $\forall i \in 1, 2, \ldots n$

Q) What if A is <u>not</u> diagonalizable?

Example: $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

- e-vals of A are λ, λ.
- e-vectors of A are not independent

$$A\vec{v} = \lambda\vec{v} \Rightarrow \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

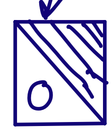
$\lambda v_1 + v_2 = \lambda v_1 \Rightarrow v_2 = 0$

$\lambda v_2 = \lambda v_2$

$\Rightarrow$ e-vecs. are $\propto \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

not independent!

Next best thing: Upper-triangular matrix



$$\frac{d\vec{x}(t)}{dt} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix} \vec{x}(t)$$

Start w/ second eq. $\frac{dx_2(t)}{dt} = \lambda_2 x_2(t)$ ✓

Then, $\frac{dx_1(t)}{dt} = \lambda_1 x_1(t) + x_2(t)$ ✓

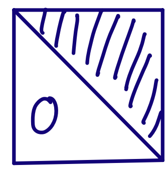
Can solve/analyze system by <u>back-substitution</u>.

Q) Can we convert any matrix M into upper-triangular (UT) form using a basis transformation?

A) Yes!

We can always find a change-of-basis:

$$\underbrace{U^{-1}}_{U^T} M U = T$$



where U is an orthonormal (ON) basis.

i.e. $U^T U = I \Rightarrow U^{-1} = U^T$

$$U = \left[\vec{u}_1 \; \vec{u}_2 \; \cdots \; \vec{u}_n\right]$$

... form an ON basis

$$\langle \vec{u}_i, \vec{u}_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \quad (\text{for } i,j = 1,2,\ldots n)$$

$$\left[\begin{array}{l} \vec{u}_i^T \vec{u}_j = 0 \text{ for all } i \neq j \\ \vec{u}_i^T \vec{u}_i = 1 \text{ for } i = 1,2 \ldots n \end{array}\right]$$

ANY SQUARE MATRIX CAN BE TRANSFORMED INTO AN UPPER TRIANGULAR MATRIX!

$U^T M U = T$ (upper-triangular) is also called the **SCHUR DECOMPOSITION** of a square matrix

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Simplest Case : M: 1×1 "matrix"

$M = m$ ✓ upper-triangular

Let's build some intuition:

M = 2x2 case:

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

Assume M has real eigenvalues.
(for convenience only)

$U = [\ ? \quad ?\ ]$ Try eigenvectors maybe?

① Say $\vec{v}_1$ is one e-vector of M

$M\vec{v}_1 = \lambda_1 \vec{v}_1$ $\quad \vec{v}_1 \in \mathbb{R}^2$

(Assume $\|\vec{v}_1\| = 1$)

$$U = [\vec{v}_1 \quad \bullet\ ]$$

② Build out the basis using G-S

Say $\vec{r}_1$ is the vector that "completes"
Gram-Schmidt (GS) procedure with $\vec{v}_1$ as first basis vector.

We know that $[\vec{v}_1 \ \vec{r}_1]$ forms an $\perp$-basis

$\vec{v}_1 \perp \vec{r}_1 \quad \langle \vec{v}_1, \vec{r}_1 \rangle = 0$

To check if U works, try $U^{-1} M U = U^T M U$.

(Recall U is an orthonormal transform by G-S construction!)

$$U^T M U$$

$$= \underbrace{[\vec{v}_1 \;\; \vec{r}_1]^T}_{2\times 2} \underbrace{M}_{2\times 2} \underbrace{[\vec{v}_1 \;\; \vec{r}_1]}_{2\times 2}$$

$$= \begin{bmatrix} - \vec{v}_1^T - \\ - r_1^T - \end{bmatrix} M \begin{bmatrix} \vec{v}_1 & \vec{r}_1 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{v}_1^T M \vec{v}_1 & \vec{v}_1^T M \vec{r}_1 \\ r_1^T M \vec{v}_1 & \vec{r}_1^T M \vec{r}_1 \end{bmatrix}$$

<u>Check:</u> $\vec{r}_1^T M \vec{v}_1 = \vec{r}_1^T \lambda_1 \vec{v}_1$

$= \lambda_1 \vec{r}_1^T \vec{v}_1 = 0$ ✓

$$\vec{v}_1^T M \vec{v}_1 = \vec{v}_1^T \lambda_1 \vec{v}_1 = \lambda_1$$

(upper-triangular)

$$U^T M U = \begin{bmatrix} \lambda_1 & * \\ 0 & * \end{bmatrix} = T$$ ✓

Ex. of Schur-decomposition for a 2x2 matrix M

(see Note 13 Sec. 6)

$$M = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \Rightarrow \text{Eigenvalues of } A \text{ and corresponding e-vectors}$$

$$\det(\lambda I - M) = \det\begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix}$$
$$= \lambda(\lambda+3) + 2$$
$$= \lambda^2 + 3\lambda + 2$$
$$= (\lambda+2)(\lambda+1)$$
$$\lambda_1 = -2; \ \lambda_2 = -1$$

are $\vec{v}_1 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$; $\lambda_1 = -2$

and $\vec{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$; $\lambda_2 = -1$

Applying the algorithm with $\vec{v}_1$ initially, we can extend $\vec{v}_1$ to an ON basis in $\mathbb{R}^2$ using G-S algorithm on $\{\vec{v}_1, \binom{1}{0}, \binom{0}{1}\}$:

getting $U = [\vec{v}_1 \ \vec{u}_1]$, where $\vec{u}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

Thus, $U = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

Computing $U^T M U$, we get:

$$U^T M U = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix} \qquad (\blacksquare)$$

Plan: Let's build up from the 2x2 case

Case: M : 3×3 (having real e-vals)

$$U^T M U = T$$

Try $\tilde{U} = \begin{bmatrix} \vec{v}_1 & \underline{\quad} & \underline{\quad} \end{bmatrix}$

Note we are using $\tilde{U}$ instead of U to show our uncertainty about whether our idea will really work!

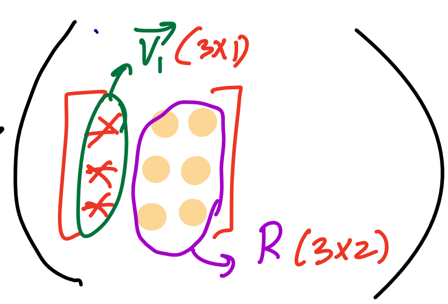
① $M\vec{v}_1 = \lambda_1 \vec{v}_1$ $\quad$ $\vec{v}_1$ is an e-vector of M.

② Use $\vec{v}_1$ to do G-S & fill out an orthonormal basis.

$$\{\vec{v}_1, \vec{r}_1, \vec{r}_2\}$$

Define: $[\vec{r}_1 \;\; \vec{r}_2] = R$

$$\underbrace{\tilde{U}}_{(3\times3)} = \begin{bmatrix} \underbrace{\vec{v}_1}_{(3\times1)} & \underbrace{R}_{(3\times2)} \end{bmatrix}$$



(Convince yourself that $\tilde{U}$ is ON.
$\vec{v}_1^T \vec{r}_1 = 0$; $\vec{v}_1^T \vec{r}_2 = 0$; $\vec{v}_1^T R = 0$; $R^T R = I$)

Consider $\tilde{U}^T M \tilde{U}$

$$= \begin{bmatrix} \vec{v}_1^T \\ R^T \end{bmatrix} M \begin{bmatrix} \vec{v}_1 & R \end{bmatrix}$$

$\vec{v}_1^T M \vec{v}_1 = \vec{v}_1^T \lambda_1 \vec{v}_1 = \lambda_1 \vec{v}_1^T \vec{v}_1 = \lambda_1$

$$= \left[\begin{array}{c|c} \lambda_1 & \underbrace{\vec{v}_1^T M R}_{(1\times 2)} \\ \hline 0_{2\times 1} & \underbrace{R^T M R}_{(2\times 2)} \end{array}\right]$$

$R^T M \vec{v}_1 = R^T \lambda_1 \vec{v}_1 = \lambda_1 R^T \vec{v}_1 = \vec{0}$

$$\begin{matrix} \lambda_1 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{matrix}$$

Q) What do we need of $\boxed{R^T M R}$?

A) $\boxed{R^T M R}$ should also be UT ( )!

Looking at $\tilde{U}^T M \tilde{U} = \begin{bmatrix} \lambda_1 & * & * \\ 0 & \bullet & \bullet \\ 0 & \bullet & \bullet \end{bmatrix}$, we have good news/bad news.

☹ : $\tilde{U} = \begin{bmatrix} \vec{v}_1 & R \end{bmatrix}$ doesn't quite work for our 3x3 case

☺ : we have managed to convert the 3x3 UT problem into a 2x2 UT problem!

Reduced problem: 2×2 $\quad R^T M R$

From earlier 2×2 case, we know that there exists an ON U_2 such that

$$U_2^T \underbrace{(R^T M R)}_{2\times 2\ \text{matrix}} U_2 = T_2 \qquad \blacksquare$$

$$\boxed{\begin{aligned} &U_2^T R^T M R U_2 \\ &= \underbrace{(R U_2)}_{\text{matrix}}^T M \underbrace{(R U_2)}_{\text{matrix}} = T_2 \end{aligned}} \qquad (\bigstar)$$

Idea, instead of trying $U_3 = [\vec{v}_1 \; R]$ as we did as our first try, (it didn't quite work), based on above observation, let's modify our try to:

$$\boxed{U_3 = [\vec{v}_1 \;\; R U_2] = \underbrace{[\vec{v}_1 \;\; R]}_{\tilde{U}_3} \left[\begin{array}{c|c} 1 & \vec{0}_2^T \\ \hline \vec{0}_2 & U_2 \end{array}\right]}$$

$U_3^T M U_3$

$$= \begin{bmatrix} \vec{v}_1^T \\ (R U_2)^T \end{bmatrix} M \begin{bmatrix} | & | \\ \vec{v}_1 & R U_2 \\ | & | \end{bmatrix}$$

$$= \begin{bmatrix} \vec{v}_1^T M \vec{v}_1 & \vec{v}_1^T M R U_2 \\ U_2^T R^T M \vec{v}_1 & \boxed{U_2^T R^T M R U_2} \end{bmatrix}$$

$= T_2$ (from ⊛)

$$= \begin{bmatrix} \lambda_1 & v_1^T M U_2 \\ 0 & \\ 0 & 0 \quad \end{bmatrix}$$

$$U_2^T R^T M \vec{v}_1 = U_2^T R^T \lambda_1 \vec{v}_1 = \lambda_1 U_2^T \underbrace{R^T \vec{v}_1}_{0} = 0.$$

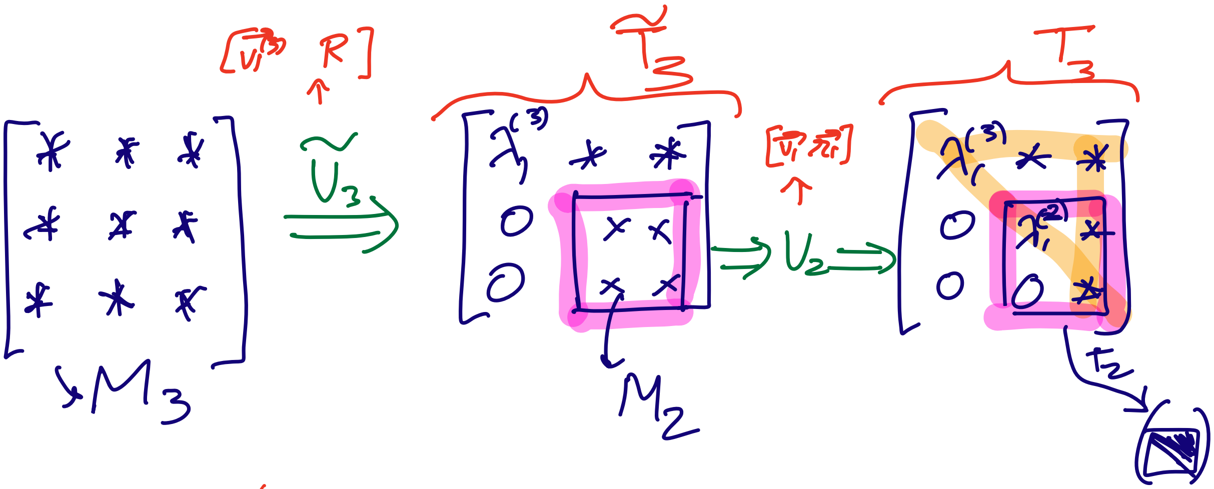
$$\boxed{\text{So, } U_3 = \begin{bmatrix} \vec{v}_1 & R U_2 \end{bmatrix} \text{ is an ON basis that upper triangularizes } M_{3\times 3}\text{; i.e. } U_3^T M U_3 = T_3}$$

Check: Is $U_3 = [\vec{v}_1 \ \ R U_2]$ orthonormal?

$\langle \vec{v}_1, R U_2 \rangle = \underbrace{\vec{v}_1^T R}_{0} U_2 = \underline{0}.$

R: 3×2
R^T: 2×3

$\langle R U_2, R U_2 \rangle = (R U_2)^T R U_2 = U_2^T \underbrace{R^T R}_{I_{2\times 2}} U_2 = I_{2\times 2}$



$$\tilde{U}_3^T M_3 \tilde{U}_3 = \tilde{T}_3$$

$$U_2^T M_2 U_2 = T_2$$

RECALL:

$$U_3 = [\vec{v}_1 \; R U_2] = \underbrace{[\vec{v}_1 \;\; R]}_{\tilde{U}_3} \left[\begin{array}{c|c} 1 & \vec{0}_2^T \\ \hline \vec{0}_2 & U_2 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1 & \vec{0}_2^T \\ \vec{0}_2 & U_2 \end{bmatrix}^T \tilde{U}_3^T}_{U_3^T} M_3 \underbrace{\tilde{U}_3 \begin{bmatrix} 1 & \vec{0}_2^T \\ \vec{0}_2 & U_2 \end{bmatrix}}_{U_3} = T_3$$

Started with (3×3) $\longrightarrow$ reduced it to the (2×2) case:

used the solution for (2×2) $\longrightarrow$ to construct back the (3×3) solution.

General case: $M \in \mathbb{R}^{(k+1)\times(k+1)}$

$k \geq 1$

$$M\vec{v}_1 = \lambda_1 \vec{v}_1$$

Pick $\tilde{U}_{k+1} = \begin{bmatrix} \vec{v}_1 & R \end{bmatrix}$ R constructed by Gram Schmidt

$$\tilde{U}_{k+1}^T M \tilde{U}_{k+1} = \begin{bmatrix} \vec{v}_1^{\,T} \\ R^T \end{bmatrix} M \begin{bmatrix} \vec{v}_1 & R \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & \vec{v}_1^{\,T} M R \\ \boxed{\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}} & \underbrace{\boxed{R^T M R}}_{(k \times k)} \end{bmatrix}$$

But $R^T M R$ is not necessarily UT.

Let's modify $\tilde{U}_{k+1}$ to U_{k+1}, where

$$U_{k+1} = \left[\vec{v}_1 \;\; R U_k\right],$$

where U_k upper-triangularizes $\underbrace{R^T M R}_{k\times k \text{ matrix}}$, i.e.

$$U_k^T R^T M R U_k = T_k \leftarrow \text{upper triangular matrix}$$

or $(R U_k)^T M (R U_k) = T_k$

By following the exact analysis we did for the (3×3) case being built out of the (2×2) case, we can show that the $(k+1)\times(k+1)$ case can be built out of the $(k\times k)$ case!

i.e

$$U_{k+1} M U_{k+1} = \begin{bmatrix} \lambda & v_1^T M R \\ \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} & \underbrace{\boxed{(R U_k)^T M (R U_k)}}_{\text{upper-triangular}} \end{bmatrix}$$

and we are done!

Reduce to triangularizing a $(k \times k)$ matrix !

We know:

2×2 matrix UT ✓

3×3 matrix UT ✓.

For $k \geq 1$,
if $(k \times k)$ matrix can be upper triangularized, then a $(k+1) \times (k+1)$ matrix can also be upper triangularized.

- Called proof by induction.
- Can also do proof by recursion (Note 13).

- Recursive proof is more intuitive as it lends itself to a Schur Decomposition ALGORITHM: See Note 13, Alg. 10 (go to discussion sections!!!)

called similar matrices

Proof that $M = U^{-1} T U$ and T have the same eigenvalues

If $M = U^{-1} T U$, then e-values$(M) \equiv$ e-values(T)

Pf: Suppose $(\lambda, \vec{v})$ are e-val/e-vector pair assoc. with M.

i.e. $M\vec{v} = \lambda \vec{v}$

$\Rightarrow U^{-1} T U \vec{v} = \lambda \vec{v}$

$\Rightarrow T \underbrace{U\vec{v}}_{\vec{w}} = \lambda \underbrace{U\vec{v}}_{\vec{w}} \Rightarrow T\vec{w} = \lambda \vec{w}$

$\Rightarrow (\lambda, \vec{w})$ are an e-val/e-vector pair for T

(where $\vec{w} = U\vec{v}$)

$\square$

$$\underbrace{\det(M - \lambda I)}_{p_M(\lambda)} = \underbrace{\det(T - \lambda I)}_{p_T(\lambda)}.$$

$\Longrightarrow$ All e-values of M are e-vals of T

" " T " " M

Special property of T:

$$T = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

For T, all entries along diagonal are the eigenvalues!.

To find e-values of T, we want to find λ such that $T - \lambda I$ has a nullspace.

$$T = \begin{bmatrix} \lambda_1 & a_1 & a_2 \\ 0 & \lambda_2 & a_3 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

3x3 case.

$$T - \lambda I = \begin{bmatrix} \lambda_1 - \lambda & a_1 & a_2 \\ 0 & \lambda_2 - \lambda & a_3 \\ 0 & 0 & \lambda_3 - \lambda \end{bmatrix}$$

$\lambda = \lambda_1$

$$T - \lambda_1 I = \underbrace{\begin{bmatrix} 0 & a_1 & a_2 \\ 0 & \lambda_2 - \lambda_1 & a_3 \\ 0 & 0 & \lambda_3 - \lambda_1 \end{bmatrix}}_{\text{HAS a NULLSPACE!}}$$

$T - \lambda_1 I$ has nullspace

$\Rightarrow \lambda_1$ must be an e-val of T!

<u>Choose</u>

$\lambda = \lambda_2$

$$T - \lambda_2 I = \begin{bmatrix} \lambda_1 - \lambda_2 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & \lambda_3 - \lambda_2 \end{bmatrix}$$

$T - \lambda_2 I$ has nullspace

$\Rightarrow \lambda_2$ is an eig. values.

General case: $n \times n$ matrix

$$\begin{bmatrix} \lambda_1 & & & \text{stuff} & \\ 0 & \lambda_2 & & & \\ & & \lambda_3 & & \\ & 0 & & \lambda_4 & \\ & & & & \lambda_5 \end{bmatrix} = T$$

Consider $T - \lambda_3 I$

$$\begin{bmatrix} \lambda_1 - \lambda_3 & & & \text{stuff} & \\ & \lambda_2 - \lambda_3 & & & \\ & & 0 & & \\ & 0 & & \lambda_4 - \lambda_3 & \\ & & & & \lambda_5 - \lambda_3 \end{bmatrix}$$

No pivot in col 3.

$\Rightarrow$ Free variable

$\Rightarrow$ Matrix is not invertible

$\Rightarrow$ It must have a nullspace.

BIBO stability: if systems with non-diagonalizable matrices

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}}_{\text{not diagonalizable}} \vec{x}[i] + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} u[i]$$

Q) Under what conditions is this system BIBO stable?

$$x_2[i+1] = \lambda x_2[i] + \beta u[i] \quad (1)$$

↳ is scalar eqn ; BIBO stable?

If $|\lambda| < 1$, then Bounded input $\Rightarrow$ Bounded output

$$x_1[i+1] = \lambda x_1[i] + \underbrace{x_2[i] + \alpha u[i]}_{\text{general input}}$$

(2)

$x_1[i+1] = \lambda x_1[i] +$ input

If $|\lambda| < 1$, it is BIBO stable!

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ 0 & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$ "stuff"

$\Rightarrow$ If $|\lambda_i| < 1 \;\forall i$ then BIBO stable!

So, $|\lambda_i| < 1 \;\forall i$ is true for all BIBO stable system matrices!