Theorem: Any (real) $(n\times n)$ matrix M_n having (real) e-vals can be upper-triangularized with orthonormal matrix U_n

Note: "real" is for convenience only.
i.e. $U_n^T M_n U_n = T_n \leftarrow$ (upper-triangular)

Proof: (By induction)

[Proof by induction: Suppose we have a statement S_n that depends on integer values of n: $n = 1, 2, 3, \ldots$. To prove by induction, we have to show:

- S_1 is true
- For any $k \geq 1$, if we assume S_k to be true, then S_{k+1} is also true.]

Back to theorem: Take statement of theorem as S_n.

- Show S_1 is true: true since scalars are trivially upper-triangular
- Show if S_k is true, then S_{k+1} is true.

(†) Inductive hypothesis: Assume any real $(k\times k)$ matrix M_k having real e-vals can be upper-triangularized with orthonormal U_k.

1. Let $M_{k+1} \in \mathbb{R}^{(k+1)\times(k+1)}$ be a square matrix & let $(\lambda_1, \vec{v}_1)$ be an e-val/e-vec pair for M_{k+1}. We will assume WLOG that $\|\vec{v}_1\| = 1$.

(2.) Choose an ON basis for $\mathbb{R}^{k+1}$ that includes $\vec{v}_1$: $\{\vec{v}_1, \vec{r}_1, \vec{r}_2, \ldots \vec{r}_k\}$.

[Q. How do we find $\vec{r}_1, \vec{r}_2, \ldots \vec{r}_k$?

A. Use G-S on set $\{\vec{v}_1, \vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\}$!

1st vec. in G-S procedure]

(3.) Then, $\boxed{\tilde{U}_{k+1} = [\vec{v}_1 \; \underbrace{\vec{r}_1 \; \vec{r}_2 \ldots \vec{r}_k}_{R_k}]}$ (A) is a $(k+1)\times(k+1)$ ON matrix

Check: $\tilde{U}_{k+1}^T \tilde{U}_{k+1} = \begin{bmatrix} \vec{v}_1^T \\ R_k^T \end{bmatrix} [\vec{v}_1 \;\; R_k]$

$= \begin{bmatrix} \vec{v}_1^T\vec{v}_1 & \vec{v}_1^T R_k \\ R_k^T \vec{v}_1 & R_k^T R_k \end{bmatrix} = \begin{bmatrix} 1 & \boxed{0} \\ \boxed{0} & \boxed{\underbrace{R_k^T R_k}_{I_k}} \end{bmatrix}$

$= I_{k+1}$

$\Rightarrow M_{k+1}\tilde{U}_{k+1} = [M_{k+1}\vec{v}_1 \;\; M_{k+1}\vec{r}_1 \; \ldots \; M_{k+1}\vec{r}_k]$

$= [\lambda_1\vec{v}_1 \;\; M_{k+1}\vec{r}_1 \ldots \; M_{k+1}\vec{r}_k]$

$\Rightarrow \tilde{U}_{k+1}^T M_{k+1} \tilde{U}_{k+1} = \begin{bmatrix} \vec{v}_1^T \\ \vec{r}_1^T \\ \vdots \\ \vec{r}_k^T \end{bmatrix} [\lambda_1\vec{v}_1 \;\; M_{k+1}\vec{r}_1 \ldots \; M_{k+1}\vec{r}_k]$

λ_1 ; $v_1^T M_{k+1} R_k$ $= [\vec{v}_1^T M_{k+1}\vec{r}_1 \ldots \vec{v}_1^T M_{k+1}\vec{r}_k]$

$= \begin{bmatrix} \lambda_1 \vec{v}_1^T\vec{v}_1 & \star & \star & \cdots & \star \\ \lambda_1 \vec{r}_1^T\vec{v}_1 & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \lambda_1 \vec{r}_k^T\vec{v}_1 & \star & \star & & \star \end{bmatrix}$

0

$R_k^T M_{k+1} R_k = \begin{bmatrix} \vec{r}_1^T \\ \vdots \\ \vec{r}_k^T \end{bmatrix} M [\vec{r}_1 \ldots \vec{r}_k]$

$$\tilde{U}_{k+1}^T M U_{k+1} = \left[\begin{array}{c|c} \lambda_1 & v_1^T M_{k+1} \\ \hline \vec{0}_{k\times 1} & M_k \end{array}\right]$$

(B) $\boxed{M_k = R_k^T M_{k+1} R_k}$ is not necessarily UT but is $(k\times k)$

(C) $\boxed{\begin{array}{l}\text{By our inductive hypothesis assumption } (\dagger), \\ U_k^T M_k U_k = T_k \to \text{(upper-triangular)} \\ M_k \to (k\times k)\text{ matrix}\end{array}}$

$$U_k^T M_k U_k = U_k^T \underbrace{(R_k^T M_{k+1} R_k)}_{M_k} U_k = (R_k U_k)^T M_{k+1} (R_k U_k)$$

replace R_k in (A) by $R_k U_k$

(A)

$$\tilde{U}_{k+1} = [\vec{v}_1 \quad R_k] \Longrightarrow U_{k+1} = [\vec{v}_1 \quad R_k U_k]$$

U_{k+1} is a $(k+1)\times(k+1)$ ON matrix.

$$\boxed{\underline{\text{Check:}} \quad U_{k+1}^T U_{k+1} = \begin{bmatrix} \vec{v}_1^T \\ (R_k U_k)^T \end{bmatrix} \begin{bmatrix} \vec{v}_1 & R_k U_k \end{bmatrix} = \begin{bmatrix} \overset{1}{\vec{v}_1^T \vec{v}_1} & \overset{0_{1\times k}}{\vec{v}_1^T R_k U_k} \\ \underset{0_{k\times 1}}{U_k^T R_k^T \vec{v}_1} & \underset{I_{k\times k}}{U_k^T R_k^T R_k U_k} \end{bmatrix}}$$

$\underline{\text{Note}}$: $$U_{k+1}^T M_{k+1} U_{k+1} = \begin{bmatrix} \vec{v}_1^T \\ U_k^T R_k^T \end{bmatrix} M_{k+1} \begin{bmatrix} \vec{v}_1 & R_k U_k \end{bmatrix}$$

$$U_{k+1}^T M_{k+1} U_{k+1} = \begin{bmatrix} \vec{v}_1^T \\ U_k^T R_k^T \end{bmatrix} \begin{bmatrix} \underbrace{M_{k+1} \vec{v}_1}_{\lambda_1 \vec{v}_1} & M_{k+1} R_k U_k \end{bmatrix}$$

$$= \left[\begin{array}{c|c} \lambda_1 \overbrace{\vec{v}_1^T \vec{v}_1}^{1} & v_1^T M_{k+1} R_k U_k \\ \hline \lambda_1 \underbrace{U_k^T R_k^T \vec{v}_1}_{0_{k\times 1}} & U_k^T \underbrace{R_k^T M_{k+1} R_k}_{M_k \text{ (by defn (B))}} U_k \end{array}\right]$$

But, by our inductive hypothesis assumption (C),

$$U_k^T M_k U_k = T_k \leftarrow \text{upper-triangular}$$

$$\Rightarrow \quad U_{k+1}^T M_{k+1} U_{k+1} = \left[\begin{array}{c|cccccc} \lambda_1 & * & * & \cdots & \cdots & * \\ \hline 0 & * & * & \cdots & \cdots & * \\ 0 & 0 & * & & & \\ \vdots & \vdots & 0 & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & 0 & \\ 0 & 0 & 0 & \cdots & 0\ \ 0 & * \end{array}\right] = T_{k+1}$$

Upper-triangular $k \times k$

upper triangular $(k+1)\times(k+1)$

$\square$

We have to check one more thing: that e-vals. of M_k are real (for our induction hypothesis). It can be shown that the e-vals of M_{k+1} (real by assumption) are $\{\lambda_1, \text{e-vals}(M_k)\} \Rightarrow$ e-vals(M_k) are therefore real.

$$\boxed{M_{k+1} = \left[\begin{array}{c|c} \lambda_1 & * \ * \ \cdots \ * \\ \hline 0 & \\ \vdots & M_k \\ 0 & \end{array}\right]}$$