## Announcements:

- Student Support Hours: Mon 1-3pm
  Everyone welcome! Schedule an appt. on course calendar on class website!
- 0.25 EC point for each lecture you attend for rest of the term.
  links.eecs16b.org/lecture-ec

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**Last time:**

- upper-triangularization : Schur decomp.
- Proof by induction (see extra notes on class website)

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**Today:**

1. Recap of implications of upper-triangularization
2. Symmetric real matrices $\longleftrightarrow$ Spectral theorem
   - eigenvalues & eigenvectors of symmetric real matrices
3. Min. energy control : motivation

$$\vec{x}[k+1] = A\,\vec{x}[k] + B\,\vec{u}[k]$$

- ANY SQUARE MATRIX CAN BE TRANSFORMED INTO AN UPPER TRIANGULAR MATRIX!

$$U^T M U = T$$ (upper triangular) is also called the SCHUR DECOMPOSITION of a square matrix

- Proof: by induction (see extra notes on class webpage)

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- If $U^T M U = T$ (upper triangular),

(i) eigenvalues$(M) \equiv$ eigenvalues$(T)$

(ii) eigenvalues$(T) \equiv$ diagonal entries of $T$

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- If $T = \begin{bmatrix} \lambda_1 & & \diagdown\diagdown \\ & \lambda_2 & \\ 0 & & \ddots & \lambda_n \end{bmatrix}$ and $U^T A U = T$,

then system $\vec{x}[k+1] = A\,\vec{x}[k] + B\,\underbrace{u[k]}_{\text{input}}$

is BIBO stable if and only if all the eigenvalues of $A$ are inside the unit circle.

i.e. $|\underbrace{\lambda_i(A)}_{=\lambda_i(T)}| < 1$ even if $A$ is not diagonalizable!

BIBO stability of systems with

non-diagonalizable matrices.

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}}_{\text{not diagonalizable}} \vec{x}[i] + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} u[i]$$

Under what condition is this system BIBO stable?

$$\vec{x}[i] = \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}.$$

$$x_2[i+1] = \lambda \cdot x_2[i] + \beta \cdot u[i] \quad (1)$$

$\hookrightarrow$ is scalar sys? BIBO stable?

if $|\lambda| < 1$, then bounded $u \Rightarrow$ bounded $x_2$.

$$x_1[i+1] = \lambda x_1[i] + \underbrace{x_2[i] + \alpha u[i]}_{\text{general input}} \quad (2)$$

$$x_1[i+1] = \lambda x_1[i] + \text{input}.$$

if $|\lambda| < 1$, is this BIBO stable?

Because $x_2[i]$ is bounded, we know input to ② is bounded!

$\Longrightarrow$ $x_1[i]$ is bounded!

$$\begin{bmatrix} \lambda_1 & & \text{stuff} \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \Longrightarrow$$ if $|\lambda_i| < 1$ for all $i$ then BIBO stable!

So, $|\text{eigenvalue}_i| < 1$ $\forall i$ is true for all BIBO stable system matrices!

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We have seen that $T = U^T M U$ ("Schur" form)

$T$: upper-triangular

$U$: orthonormal

Q) What if we had a real symmetric matrix $S$?

- When $A$ is diagonalizable, we can find $V$ such that

$$V^{-1} A V = \Lambda \rightarrow \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$V$ is the eigenbasis of $A$,
but $V$ is not necessarily orthogonal!

- If we upper-triangularize $A$ we get an orthogonal $U$ such that

$$U^{-1} A U = U^T A U = T \rightarrow \begin{bmatrix} \lambda_1 & * & * \cdots * \\ & \lambda_2 & * \;\; * \\ & & \ddots \;\; \vdots \\ 0 & & \lambda_n \end{bmatrix}$$

(Schur decomposition)

Q) What if we had a real symmetric matrix $S$?

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} \qquad \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

$$S = S^T : \quad S_{ij} = S_{ji}$$

Symmetric matrices get the best of both worlds!

i.e. Symmetric matrices are diagonalizable + eigenvectors of symm. matrices are orthogonal!

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Ex.

$$S = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 0 \end{bmatrix} \qquad (S^T = S)$$

$$\boxed{\begin{array}{l} \det(S - \lambda I) = 0 \\ \text{Solve for } \lambda \end{array}}$$

E-values of $S$ are $\underbrace{6}_{\lambda_1}, \underbrace{-3}_{\lambda_2}, \underbrace{-1}_{\lambda_3}$.

Corresponding e-vectors are: $\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_3}$

Remarks:

(1) E-values of $S$ are all real

(2) E-vectors of $S$ are orthogonal

$$\langle \vec{v}_1, \vec{v}_2 \rangle = 0; \langle \vec{v}_1, \vec{v}_3 \rangle = 0; \langle \vec{v}_2, \vec{v}_3 \rangle = 0$$

## Spectral Theorem:

Let $S \in \mathbb{R}^{n\times n}$ be a (real) symmetric matrix. Then,

(i) $S$ can be diagonalized (whether or not the e-vectors of $S$ form a basis)

(ii) The e-vectors of $S$ form an orthonormal basis: the diagonalizing basis for $S$.

(iii) The e-vals. of $S$ are all real.

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(i) $\boxed{U^T S U = T} \Rightarrow \boxed{S = U T U^T}$

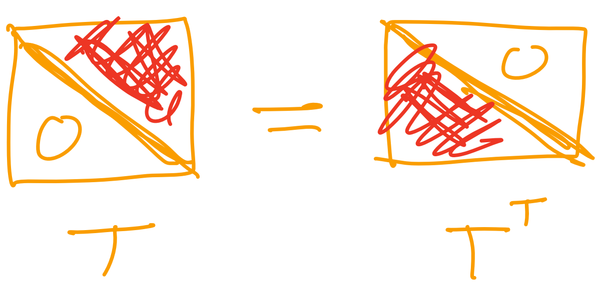
Upper-triangular $S$:

$$S = U T U^T$$

Taking transpose, $S^T = (U T U^T)^T = U T^T U^T$

$\underline{S = S^T} \Rightarrow U T U^T = U T^T U^T$

$$\Rightarrow \boxed{T = T^T}$$




$\Rightarrow$ All non-diagonal entries of $T$ must be 0

$\Rightarrow T$ must be diagonal.

(ii) If $S$ is a symmetric matrix, $U$, the basis for $UT$ actually diagonalizes $S$!

$$S = U T U^{-1}$$

$$S = U \boxed{\diagdown} U^{-1}$$

(If a matrix has distinct e-vals, it can be diagonalized $\rightarrow$ e-vectors form a basis)

1. Symmetric matrix can be diagonalized. ALWAYS!

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(ii) $S = U D U^T$ (D is used to emphasize that $T = D$)

$$S U = U D \underbrace{U^T U}_{I}$$

$$S U = U D$$

$$S \begin{bmatrix} \vec{u_1} & \vec{u_2} & \cdots & \vec{u_n} \end{bmatrix} = \begin{bmatrix} \vec{u_1} & \vec{u_2} & \cdots & \vec{u_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

$$\left.\begin{aligned} S\vec{u_1} &= \lambda_1 \vec{u_1} \\ S\vec{u_2} &= \lambda_2 \vec{u_2} \\ &\vdots \\ S\vec{u_n} &= \lambda_n \vec{u_n} \end{aligned}\right\}$$ $\vec{u_1}, \vec{u_2}, \ldots \vec{u_n}$ must be the e-vectors of $S$!

② The diagonalizing basis/matrix, $U$, is made up of the e-vectors of $S$!

Because $U$ was orthonormal, $\vec{u_1}, \vec{u_2}, \ldots \vec{u_n}$ are orthonormal

$\Rightarrow$ eigenvectors of $S$ are orthonormal!

(iii) $\boxed{S\vec{v} = \lambda\vec{v}}$ ① [$(\lambda, \vec{v})$ are an e-value/e-vector pair for $S$.]

Let $\lambda = \lambda_r + j\lambda_i$

Let's try to show that $\lambda = \lambda^*$
(i.e. $\lambda_i = 0$)

Take conjugates of both sides of ①

$$S^*\vec{v}^* = \lambda^*\vec{v}^* \quad [\text{Since } (AB)^* = A^*B^*]$$

$$S\vec{v}^* = \lambda^*\vec{v}^* \quad [\text{Since } S = S^* \text{ (real)}]$$

Take transpose:

$$\vec{v}^{*T}S^T = \vec{v}^{*T}\lambda^{*T} = \lambda^*\vec{v}^{*T}$$

$$\boxed{\vec{v}^{*T}S = \lambda^*\vec{v}^{*T}} \quad (\text{Since } S^T = S)$$

Multiply on the right by $\vec{v}$

$$\boxed{\vec{v}^{*T} S \vec{v} = \lambda^{*} \vec{v}^{*T} \vec{v}} \quad (A)$$

$$\boxed{S\vec{v} = \lambda \vec{v}} \quad (1)$$

Multiply (1) by $\vec{v}^{*T}$ on the left:

$$\boxed{\vec{v}^{*T} S \vec{v} = \lambda \vec{v}^{*T} \vec{v}} \quad (B)$$

(A) & (B) have the same LHS
$\Rightarrow$ they must have the same RHS

$$\lambda^{*} \cancel{\vec{v}^{*T} \vec{v}} = \lambda \cancel{\vec{v}^{*T} \vec{v}}$$

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

$\vec{v}^{*T} = (v_1^* \ v_2^* \ldots \ v_n^*)$

$\vec{v}^{*T} \vec{v}$
$= v_1^* v_1 + v_2^* v_2 + \ldots + v_n^* v_n$
$= |v_1|^2 + |v_2|^2 + \ldots + |v_n|^2$
$\neq 0$

since $\vec{v}$ is an e-vec.

$\Rightarrow \lambda^* = \lambda$

$\Rightarrow \lambda_r + j\lambda_i = \lambda_r - j\lambda_i$

$\Rightarrow \lambda_i = 0$

$$\boxed{\lambda = \lambda_r}$$

③ All e-vals of $S$ are real.

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CONTROL

- Stability (states blow up?)
- Controllability: Can I get where I want?
- Efficiency:

Control system:

$$\vec{x}[k+1] = A\vec{x}[k] + \vec{b}u[k]$$

$$\vec{x}[k] = A^k\vec{x}[0] + A^{k-1}\vec{b}u[0] + \dots + \vec{b}u[k-1]$$

Consider $k = 100$

$\vec{x}[0] = 0.$

$$\vec{x}[100] = A^{99}\vec{b}u[0] + \dots + \vec{b}u[99]$$

$$\vec{x}[100] = \underbrace{\begin{bmatrix} A^{99}\vec{b} & A^{98}\vec{b} & \dots & A\vec{b} & \vec{b} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} u[0] \\ \vdots \\ u[99] \end{bmatrix}}_{\vec{u}}$$

$\underbrace{\vec{x}[100]}_{\vec{d}} = C\vec{u}$

$\min \|\vec{u}\|^2$ "Minimum Energy" control

s.t. $\vec{x}[100] = Cu$

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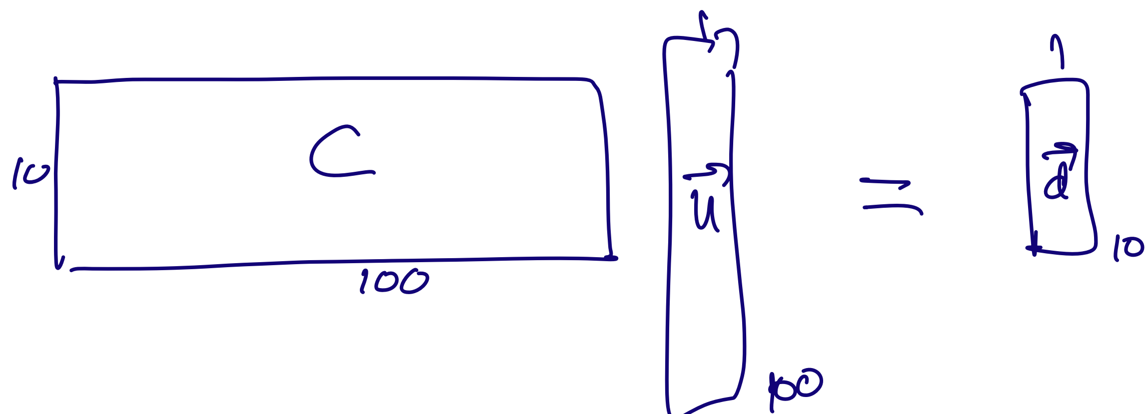
$$C_{(10\times 100)}\,\vec{u}_{(10\times 1)} = \vec{d}_{(10\times 1)}$$




$$\boxed{\text{Solve} \quad \min \|\vec{u}\|^2 \quad \text{s.t. } C\vec{u} = \vec{d}} \quad (*)$$

Solution to optimization problem (*) is $\vec{u}^*$ called the minimum norm solution.