# Note 2: Transient Analysis and Inputs

## 1 Piecewise Constant Inputs

### 1.1 Motivation

Oftentimes, we encounter functions that vary with time but are constant over certain intervals of time. Suppose we have a circuit as in fig. 1.

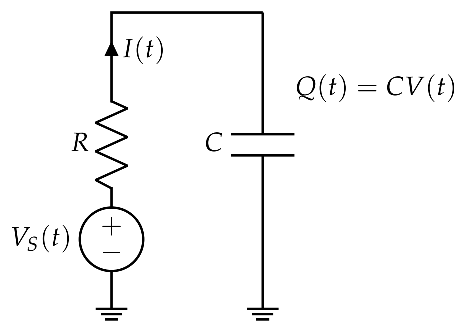



**Figure 1:** Capacitor charging through a circuit with a resistor.

In the previous note, we covered instances where $V_S(t)$ is a constant in time. Now, suppose we have $V_S(t)$ as in fig. 2:

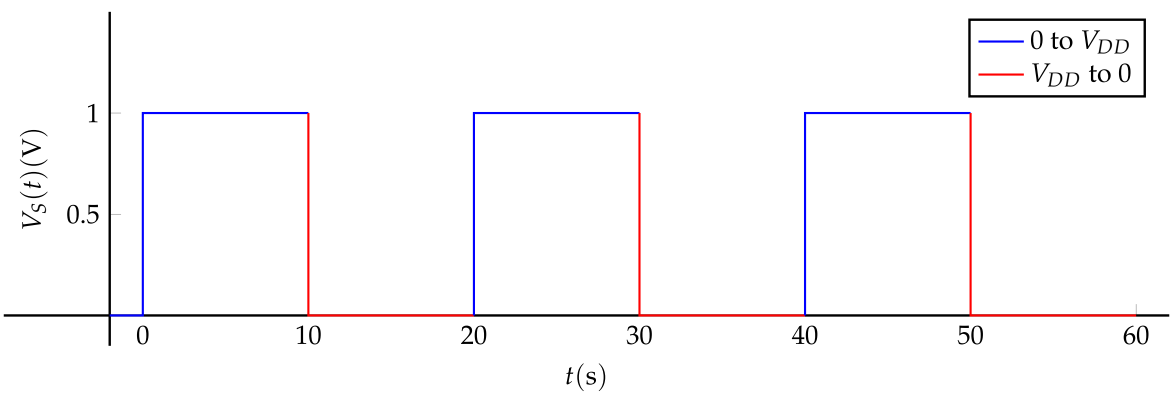



**Figure 2:** Example of $V_S(t)$

Given an input like this, we would want to model the voltage across the capacitor as a function of time, i.e. $V(t)$. These types of inputs are quite common in practice (e.g. voltage sources controlled by switches), but more importantly, they will help us understand how to approach more general types of inputs.

## 1.2 Differential Equations with Piecewise Constant Inputs

**Definition 1** (Piecewise Constant Inputs)

Suppose $u(t)$ is a piecewise constant input. This means that there are a sequence of indices $i \in \{1, 2, 3, \ldots\}$ and corresponding times $t_1, t_2, t_3, \ldots$ such that $0 \leq t_1 < t_2 < t_3 < \ldots$ and $u(t)$ is constant for $t \in [t_i, t_{i+1})$.[a]

[a] For the purposes of this note, we wil primarily focus our attention on *right continuous* piecewise constant functions, which is described by this definition.

An example of a piecewise constant input is shown in fig. 2. Here, we have $t_1 = 0$, $t_2 = 10$, $t_3 = 20$, etc. Over each interval $t \in [t_i, t_{i+1})$, $V_S(t)$ is constant (either 1 or 0).

**Theorem 2** (Solving Differential Equations with Piecewise Constant Inputs)

Consider a differential equation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda x(t) + u(t) \tag{1}$$

for $\lambda \neq 0$. Let $u(t)$ be a piecewise function with time indices $t_1, t_2, t_3, \ldots$ such that $u(t)$ is constant for $t \in [t_i, t_{i+1})$. The solution to this differential equation is characterized by the *recurrence equation*

$$x(t) = x(t_{i-1}) \mathrm{e}^{\lambda(t - t_{i-1})} + \frac{\left(\mathrm{e}^{\lambda(t - t_{i-1})} - 1\right) u(t_{i-1})}{\lambda} \tag{2}$$

where $t_{i-1} \leq t < t_i$. If $\lambda = 0$, then the recurrence equation is

$$x(t) = u(t_{i-1})(t - t_{i-1}) + x(t_{i-1}) \tag{3}$$

*Proof.* Case 1. Suppose $\lambda \neq 0$. Since $t \in [t_{i-1}, t_i)$, we know $u(t) = u(t_{i-1})$ will be constant. Hence, we can consider $x(t_{i-1})$ as an "initial condition" and apply the formula for a differential equation with constant input (Theorem 20 of Note 1), namely

$$\begin{aligned} x(t) &= \left(k + \frac{u(t_{i-1})}{\lambda}\right) \mathrm{e}^{\lambda(t - t_{i-1})} - \frac{u(t_{i-1})}{\lambda} && (4) \\ &= k \mathrm{e}^{\lambda(t - t_{i-1})} + \frac{\left(\mathrm{e}^{\lambda(t - t_{i-1})} - 1\right) u(t_{i-1})}{\lambda} && (5) \end{aligned}$$

where $k = x(t_{i-1})$ represents our "initial condition".

Case 2. Suppose $\lambda = 0$. Then, the differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = u(t) \tag{6}$$

Again, since $t \in [t_{i-1}, t_i)$, we know $u(t) = u(t_{i-1})$ will be constant. Hence, we can apply Theorem 13 of Note 1 and obtain

$$x(t) = u(t_{i-1})(t - t_{i-1}) + x(t_{i-1}) \tag{7}$$

□

**Key Idea 3** (Solving Recurrence Equations)
When provided a recurrence equation as in eq. (2), we often do not know the value of $x(t_{i-1})$, i.e. suppose we know the initial condition $x(t_0)$. We can find $x(t_{i-1})$ by applying the recurrence equation again, namely

$$x(t_{i-1}) = x(t_{i-2})e^{\lambda(t_{i-1}-t_{i-2})} + \frac{\left(e^{\lambda(t_{i-1}-t_{i-2})} - 1\right)u(t_{i-2})}{\lambda} \tag{8}$$

which will give us $x(t_{i-1})$ in terms of $x(t_{i-2})$[a]. Apply the recurrence repeatedly until all of the terms on the RHS are known. This *recursive* procedure is the reason equations like eq. (2) are called "recurrence equations".

[a]Note that $t_{i-1} \notin [t_{i-2}, t_{i-1})$, which was a crucial part of the proof of Theorem 2. However, we can assume that $x(t)$ will be continuous at $t_{i-1}$, so the recurrence will still hold.

### 1.2.1 Example

Consider the circuit in fig. 1 and piecewise voltage input in fig. 2. Suppose we wish to find $V(t)$. Using KCL and properties of capacitors, we can model $V(t)$ with the following differential equation:

$$\frac{d}{dt}V(t) = -\frac{V(t)}{RC} + \frac{V_S(t)}{RC} \tag{9}$$

Now, we can derive the recurrence equation. Suppose that, for $t \in [t_{i-1}, t_i)$, $V_S(t) = 0$. Thus,

$$V(t) = V(t_{i-1})e^{-\frac{t-t_{i-1}}{RC}} \tag{10}$$

where $\lambda = -\frac{1}{RC}$ and $u(t_{i-1}) = 0$. If, instead, $V_S(t) = 1$, then $V(t)$ will be

$$V(t) = (V(t_{i-1}) - 1)e^{-\frac{t-t_{i-1}}{RC}} + 1 \tag{11}$$

where $\lambda = -\frac{1}{RC}$ and $u(t_{i-1}) = \frac{1}{RC}$.

Suppose we wanted to find $V(25)$, knowing the initial condition $V(0) = 0$. We can apply the recurrence equation as follows:

$$\begin{aligned} V(15) &= V(10)e^{-\frac{5}{RC}} && (12) \\ &= \underbrace{\left((V(0) - 1)e^{\frac{10}{RC}} + 1\right)}_{V(10)} e^{-\frac{5}{RC}} && (13) \\ &= \left(1 - e^{\frac{10}{RC}}\right)e^{-\frac{5}{RC}} && (14) \end{aligned}$$

If we were to plot $V(t)$, then we would see a graph similar to fig. 3.

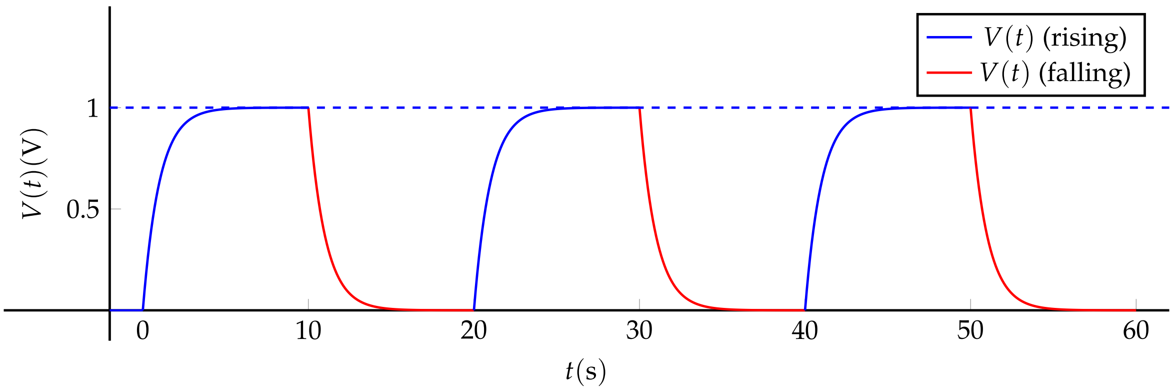



**Figure 3:** Plot of $V(t)$

# 2 Differential Equations with General Time-Varying Inputs

## 2.1 Motivation

Suppose that now we would like to deal with general functions $u(t)$. In particular, let's say that we want to find a solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu(t) \tag{15}$$

for $\lambda \in \mathbb{R}$, $b \in \mathbb{R}$, and $u(t) : \mathbb{R} \to \mathbb{R}$. We can further assume that $u(t)$ is integrable and differentiable everywhere. This is called an inhomogeneous, first order, linear differential equation. These types of differential equations allow us to model more general types of voltage inputs to our system, such as sinusoidal voltage inputs provided by an oscilloscope.

## 2.2 Solution with $\lambda = 0$

We can first consider the case of $\lambda = 0$.

**Theorem 4** (Inhomogeneous Solution with $\lambda = 0$)
If $\lambda = 0$, then the solution to eq. (15) is

$$x(t) = x(t_0) + b\int_{t_0}^{t} u(\theta)\,\mathrm{d}\theta \tag{16}$$

where $x(t_0)$ is a given initial condition.

*Proof.* If $\lambda = 0$, then we can rewrite eq. (15) as

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = bu(t) \tag{17}$$

From here, we can take integrals on both sides, from $t_0$ to $t$. Furthermore, introduce a dummy variable $\theta$ for integration:

$$\int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\theta}x(\theta)\,\mathrm{d}\theta = \int_{t_0}^{t} bu(\theta)\,\mathrm{d}\theta \tag{18}$$

Applying the fundamental theorem of calculus, we obtain

$$x(t) - x(t_0) = b\int_{t_0}^{t} u(\theta)\,\mathrm{d}\theta \tag{19}$$

$$x(t) = x(t_0) + b\int_{t_0}^{t} u(\theta)\,\mathrm{d}\theta \tag{20}$$

□

## 2.3 Solution with $\lambda \neq 0$ (Integrating Factor Method)

In the earlier case, we could solve the differential equation using separation of variables (i.e., isolate all the $x$ terms to the left-hand side of the equation and integrate both sides). However, if $\lambda \neq 0$, this is not immediately possible. To accomplish a similar form, we need to introduce an *integrating factor*.

**Definition 5** (Integrating Factor)
Consider the following differential equation for $x(t)$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + bu(t) \tag{21}$$

$$\iff \frac{\mathrm{d}x(t)}{\mathrm{d}t} - \lambda x(t) = bu(t) \tag{22}$$

with $\lambda \neq 0$. We define an integrating factor $\mu(t)$ such that

$$\mu(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t} - \lambda\mu(t)x(t) = bu(t)\mu(t) \tag{23}$$

$$\iff \frac{\mathrm{d}}{\mathrm{d}t}(\mu(t)x(t)) = bu(t)\mu(t) \tag{24}$$

From the definition above, we can see that choosing a valid integrating factor allows us to obtain a differential equation of similar form to the one in the previous case, with $\lambda = 0$.

**Theorem 6** (Integrating Factor for First Order, Linear Differential Equations)
The integrating factor for a first order, linear differential equation of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} - \lambda x(t) = bu(t) \tag{25}$$

is

$$\mu(t) = \mathrm{e}^{-\lambda t} \tag{26}$$

*Proof.* The derivation of the integrating factor from first principles is out of scope for this class. However, we can prove that $\mu(t) = \mathrm{e}^{-\lambda t}$ is a valid integrating factor.

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} - \lambda x(t) = bu(t) \tag{27}$$

$$\mu(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t} - \lambda\mu(t)x(t) = b\mu(t)u(t) \tag{28}$$

$$\mathrm{e}^{-\lambda t}\frac{\mathrm{d}x(t)}{\mathrm{d}t} - \lambda\mathrm{e}^{-\lambda t}x(t) = b\mathrm{e}^{-\lambda t}u(t) \tag{29}$$

Now, notice that, by the product rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\lambda t}x(t)\right) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}\mathrm{e}^{-\lambda t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\lambda t}\right)x(t) \tag{30}$$

$$= \frac{\mathrm{d}x(t)}{\mathrm{d}t}\mathrm{e}^{-\lambda t} - \lambda \mathrm{e}^{-\lambda t}x(t) \tag{31}$$

so plugging back into eq. (29), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\lambda t}x(t)\right) = b\mathrm{e}^{-\lambda t}u(t) \tag{32}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu(t)x(t)) = b\mu(t)u(t) \tag{33}$$

which precisely satisfies the definition of an integrating factor as explained in Definition 5. □

Now, we can proceed to solve the differential equation using the given integrating factor.

**Theorem 7** (Inhomogeneous Solution with $\lambda \neq 0$ (Integrating Factor Method))
If $\lambda \neq 0$, then the solution to eq. (15) is

$$x(t) = x(t_0)\mathrm{e}^{\lambda(t-t_0)} + b\mathrm{e}^{\lambda t}\int_{t_0}^{t} \mathrm{e}^{-\lambda\theta}u(\theta)\,\mathrm{d}\theta \tag{34}$$

*Proof.* Rewriting eq. (32), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\lambda t}x(t)\right) = b\mathrm{e}^{-\lambda t}u(t) \tag{35}$$

We can define an integration dummy variable $\theta$ and integrate both sides from $t_0$ to $t$, and apply the fundamental theorem of calculus as follows:

$$\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\mathrm{e}^{-\lambda\theta}x(\theta)\right)\mathrm{d}\theta = \int_{t_0}^{t} b\mathrm{e}^{-\lambda\theta}u(\theta)\,\mathrm{d}\theta \tag{36}$$

$$\mathrm{e}^{-\lambda t}x(t) - \mathrm{e}^{-\lambda t_0}x(t_0) = \int_{t_0}^{t} b\mathrm{e}^{-\lambda\theta}u(\theta)\,\mathrm{d}\theta \tag{37}$$

$$\mathrm{e}^{-\lambda t}x(t) = \mathrm{e}^{-\lambda t_0}x(t_0) + b\int_{t_0}^{t}\mathrm{e}^{-\lambda\theta}u(\theta)\,\mathrm{d}\theta \tag{38}$$

$$x(t) = \mathrm{e}^{\lambda t}\cdot\mathrm{e}^{-\lambda t_0}x(t_0) + \mathrm{e}^{\lambda t}\cdot b\int_{t_0}^{t}\mathrm{e}^{-\lambda\theta}u(\theta)\,\mathrm{d}\theta \tag{39}$$

$$x(t) = \mathrm{e}^{\lambda(t-t_0)}x(t_0) + b\mathrm{e}^{\lambda t}\int_{t_0}^{t}\mathrm{e}^{-\lambda\theta}u(\theta)\,\mathrm{d}\theta \tag{40}$$

where we apply the fundamental theorem of calculus to arrive at eq. (37). □

## 2.4 (OPTIONAL) Solution with $\lambda \neq 0$ (Piecewise Approximation Method)

The case with $\lambda \neq 0$ is more difficult and will require us to include our approach for dealing with piecewise constant inputs. Namely, we can write any function as a piecewise constant function with the constant intervals being $t_i = t_{i-1} + \Delta$, and then take $\Delta \to 0$. This is illustrated in fig. 4, where the red function denotes the piecewise constant approximation.

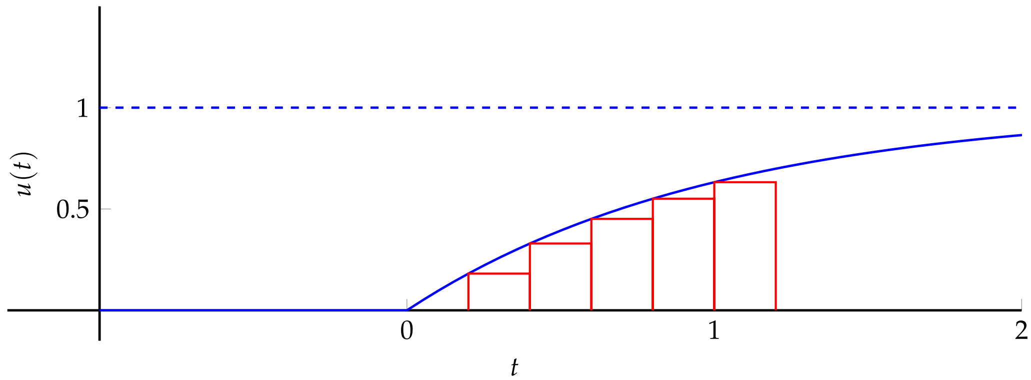



**Figure 4:** Our style of approximating a general function by a piecewise constant function.

To approach the problem of general, time-varying functions $u(t)$, we must make use of a lemma involving the piecewise constant approximation of $u(t)$ alluded to previously.

**Lemma 8** (Piecewise Constant Approximation Solution)
Consider the differential equation in eq. (15) with $\lambda \neq 0$. Let $u(t)$ be a piecewise constant function with time indices $0, \Delta, 2\Delta, \ldots$ such that $u(t)$ is constant for $t \in [i\Delta, (i+1)\Delta)$. Furthermore, assume that $x(0)$ is a known initial condition, and define $N := \lfloor \frac{t}{\Delta} \rfloor$. Therefore, the solution to eq. (15) is

$$x(t) = x\left(\left\lfloor \frac{t}{\Delta} \right\rfloor \Delta\right) \mathrm{e}^{t - \lfloor \frac{t}{\Delta} \rfloor \Delta} + \frac{b\left(\mathrm{e}^{\lambda\left(t - \lfloor \frac{t}{\Delta} \rfloor \Delta\right)} - 1\right)}{\lambda} u\left(\left\lfloor \frac{t}{\Delta} \right\rfloor \Delta\right) \tag{41}$$

where

$$x\left(\left\lfloor \frac{t}{\Delta} \right\rfloor \Delta\right) = x(0)\mathrm{e}^{\lambda \lfloor \frac{t}{\Delta} \rfloor \Delta} + \frac{b(\mathrm{e}^{\lambda\Delta} - 1)}{\lambda} \sum_{k=1}^{\lfloor \frac{t}{\Delta} \rfloor} u((k-1)\Delta)\mathrm{e}^{\left(\lfloor \frac{t}{\Delta} \rfloor - k\right)\Delta} \tag{42}$$

*Proof.* From Theorem 2, we know that

$$x(i\Delta) = x((i-1)\Delta)\mathrm{e}^{\lambda\Delta} + \frac{b\left(\mathrm{e}^{\lambda\Delta} - 1\right)}{\lambda} u((i-1)\Delta) \tag{43}$$

$$\implies x(i\Delta) - x((i-1)\Delta)\mathrm{e}^{\lambda\Delta} = \frac{b\left(\mathrm{e}^{\lambda\Delta} - 1\right)}{\lambda} u((i-1)\Delta) \tag{44}$$

Now, define $N := \lfloor \frac{t}{\Delta} \rfloor$. Note that we can write

$$\begin{aligned} x(N\Delta) - x(0)\mathrm{e}^{\lambda N\Delta} &= x(N\Delta) - x((N-1)\Delta)\mathrm{e}^{\lambda\Delta} && (45)\\ &\quad + x((N-1)\Delta)\mathrm{e}^{\lambda\Delta} - x((N-2)\Delta)\mathrm{e}^{2\lambda\Delta} + \ldots && (46)\\ &= \sum_{k=1}^{N} \left(x(k\Delta) - x((k-1)\Delta)\mathrm{e}^{\lambda\Delta}\right)\mathrm{e}^{(N-k)\lambda\Delta} && (47)\\ &\underset{\text{eq. (44)}}{\overset{\uparrow}{=}} \frac{b\left(\mathrm{e}^{\lambda\Delta} - 1\right)}{\lambda} \sum_{k=1}^{N} u((k-1)\Delta)\mathrm{e}^{(N-k)\lambda\Delta} && (48) \end{aligned}$$

so

$$x(N\Delta) = x(0)\mathrm{e}^{\lambda N\Delta} + \frac{b\left(\mathrm{e}^{\lambda\Delta} - 1\right)}{\lambda} \sum_{k=1}^{N} u((k-1)\Delta)\mathrm{e}^{(N-k)\lambda\Delta} \tag{49}$$

Since $N := \lfloor \frac{t}{\Delta} \rfloor$, we know $t \in [N\Delta, (N+1)\Delta)$. Applying Theorem 2 again, we have

$$x(t) = \left(x(N\Delta) + \frac{u(N\Delta)}{\lambda}\right)\mathrm{e}^{t-N\Delta} - \frac{u(N\Delta)}{\lambda} \tag{50}$$

$$= x(N\Delta)\mathrm{e}^{t-N\Delta} + \frac{b\left(\mathrm{e}^{t-N\Delta} - 1\right)}{\lambda} u(N\Delta) \tag{51}$$

□

**Theorem 9** (Inhomogeneous Solution with $\lambda \neq 0$ (Piecewise Approximation Method))
If $\lambda \neq 0$, then the solution to eq. (15) is

$$x(t) = x(t_0)\mathrm{e}^{\lambda(t-t_0)} + b\mathrm{e}^{\lambda t}\int_{t_0}^{t} \mathrm{e}^{-\lambda\theta} u(\theta)\,\mathrm{d}\theta \tag{52}$$

*Proof.* As mentioned before, we can take $\Delta \to 0$ to make it such that our piecewise constant approximation exactly matches the original $u(t)$. Again, define $N := \lfloor \frac{t}{\Delta} \rfloor$. Note that, as $\Delta \to 0$, $N \to \infty$ and $N\Delta = \lfloor \frac{t}{\Delta} \rfloor \Delta \to t$. First, we can consider the case where $t_0 = 0$, and later generalize to arbitrary $t_0$. Using the result from Lemma 8, we have

$$\lim_{\Delta\to 0} x(t) = \left(\lim_{\Delta\to 0} x(N\Delta)\right)\underbrace{\left(\lim_{\Delta\to 0} \mathrm{e}^{t-N\Delta}\right)}_{=1} + \frac{b}{\lambda}\underbrace{\left(\lim_{\Delta\to 0}\left(\mathrm{e}^{t-N\Delta} - 1\right)\right)}_{=0}\left(\lim_{\Delta\to 0} u(N\Delta)\right) \tag{53}$$

$$= \lim_{\Delta\to 0} x(N\Delta) \tag{54}$$

To compute this limit, we can substitute eq. (42)

$$\lim_{\Delta\to 0} x(N\Delta) = x(0)\lim_{\Delta\to 0} \mathrm{e}^{\lambda N\Delta} + \frac{b}{\lambda}\lim_{\Delta\to 0}\left(e^{\lambda\Delta} - 1\right)\sum_{k=1}^{N} u((k-1)\Delta)\mathrm{e}^{(N-k)\lambda\Delta} \tag{55}$$

$$= x(0)\mathrm{e}^{\lambda t} + \frac{b}{\lambda}\lim_{\Delta\to 0}\left(\sum_{n=1}^{\infty}\frac{(\lambda\Delta)^n}{n!}\right)\sum_{k=1}^{N} u((k-1)\Delta)\mathrm{e}^{(N-k)\lambda\Delta} \tag{56}$$

$$= x(0)\mathrm{e}^{\lambda t} + b\lim_{\Delta\to 0}\left(\sum_{k=1}^{N} u((k-1)\Delta)\mathrm{e}^{(N-k)\lambda\Delta}\Delta\right)\left(1 + \sum_{n=2}^{\infty}\frac{\lambda^{n-1}\Delta^n}{n!}\right) \tag{57}$$

$$= x(0)\mathrm{e}^{\lambda t} + b\underbrace{\left(\lim_{\Delta\to 0} \mathrm{e}^{\lambda N\Delta}\right)}_{e^{\lambda t}}\left(\lim_{\Delta\to 0}\sum_{k=1}^{N} u((k-1)\Delta)\mathrm{e}^{-k\lambda\Delta}\Delta\right)\underbrace{\left(\lim_{\Delta\to 0} 1 + \sum_{n=2}^{\infty}\frac{\lambda^{n-1}\Delta^n}{n!}\right)}_{=1} \tag{58}$$

$$= x(0)\mathrm{e}^{\lambda t} + b\mathrm{e}^{\lambda t}\left(\lim_{\Delta\to 0}\sum_{k=1}^{N} u((k-1)\Delta)\mathrm{e}^{-k\lambda\Delta}\Delta\right) \tag{59}$$

$$= x(0)\mathrm{e}^{\lambda t} + b\mathrm{e}^{\lambda t}\underbrace{\left(\lim_{\Delta\to 0} \mathrm{e}^{-\lambda\Delta}\right)}_{=1}\left(\lim_{\Delta\to 0}\sum_{k=1}^{N} u((k-1)\Delta)\mathrm{e}^{-(k-1)\lambda\Delta}\Delta\right) \tag{60}$$

$$= x(0)\mathrm{e}^{\lambda t} + b\mathrm{e}^{\lambda t}\left(\lim_{\Delta\to 0}\sum_{k=1}^{N} u((k-1)\Delta)\mathrm{e}^{-(k-1)\lambda\Delta}\Delta\right) \tag{61}$$

where we use the Taylor expansion of $\mathrm{e}^{\lambda\Delta}$ to obtain eq. (56). Note that the $\sum_{k=1}^{N} u((k-1)\Delta)\mathrm{e}^{-(k-1)\lambda\Delta}\Delta$ term is a left Riemann sum of the function $u(\theta)\mathrm{e}^{-\lambda\theta}$. So, as we take $\Delta \to 0$, this Riemann sum becomes an integral from 0 to $t$ and we end up with

$$\lim_{\Delta\to 0} x(N\Delta) = x(0)e^{\lambda t} + be^{\lambda t}\int_0^t u(\theta)e^{-\lambda\theta}\,d\theta \tag{62}$$

$$\implies \lim_{\Delta\to 0} x(t) = x(t) = x(0)e^{\lambda t} + be^{\lambda t}\int_0^t u(\theta)e^{-\lambda\theta}\,d\theta \tag{63}$$

where $\theta$ is a dummy variable in the integration.

For arbitrary initial conditions (e.g. $x(t_0)$), we can define $\tau := t - t_0$ (or equivalently, $t = \tau + t_0$), $\widetilde{x}(\tau) := x(\tau + t_0)$, and $\widetilde{u}(\tau) := u(\tau + t_0)$. We can redefine the differential equation in eq. (15) as follows:

$$\frac{d}{d\tau}\widetilde{x}(\tau) = \frac{d}{dt}x(\tau + t_0)\underbrace{\frac{d\tau}{dt}}_{=1} \tag{64}$$

$$= \lambda x(\tau + t_0) + bu(\tau + t_0) \tag{65}$$

$$= \lambda\widetilde{x}(\tau) + b\widetilde{u}(\tau) \tag{66}$$

with initial condition $\widetilde{x}(0) = x(t_0)$. We can derive the following solution for $\widetilde{x}(\tau)$ and back substitute to find $x(t)$

$$x(t) = \widetilde{x}(\tau) = \widetilde{x}(0)e^{\lambda\tau} + be^{\lambda t}\int_0^\tau \widetilde{u}(\widetilde{\theta})e^{-\lambda\widetilde{\theta}}\,d\widetilde{\theta} \tag{67}$$

$$= x(t_0)e^{\lambda(t-t_0)} + be^{\lambda(t-t_0)}\int_0^{t-t_0} u(\widetilde{\theta} + t_0)e^{-\lambda\widetilde{\theta}}\,d\widetilde{\theta} \tag{68}$$

$$= x(t_0)e^{\lambda(t-t_0)} + be^{\lambda(t-t_0)}\int_{t_0}^{t} u(\theta)e^{-\lambda(\theta-t_0)}\,d\theta \tag{69}$$

$$= x(t_0)e^{\lambda(t-t_0)} + be^{\lambda t}\int_{t_0}^{t} u(\theta)e^{-\lambda\theta}\,d\theta \tag{70}$$

where we obtain eq. (69) by performing u-substitution, defining $\theta := \widetilde{\theta} + t_0$. □

**Concept Check:** Show that this solution is unique for a given differential equation and initial condition.

## 2.5 Example

Consider the circuit in fig. 1, with $V_S(t) = e^{-t}$, with the capacitor initially discharged (i.e. $V(0) = 0$). The differential equation that models the voltage across the capacitor is

$$\frac{d}{dt}V(t) = -\frac{1}{RC}V(t) + \frac{1}{RC}e^{-t} \tag{71}$$

Here, we can perform the pattern matching with $\lambda = -\frac{1}{RC}$, $b = \frac{1}{RC}$, and $u(t) = e^{-t}$. Applying the result of Theorem 7 (or equivalently Theorem 9) and plugging into eq. (34) (or equivalently eq. (52)), we obtain

$$V(t) = \frac{1}{RC}e^{-\frac{t}{RC}}\int_0^t e^{-\frac{\theta}{RC}}e^{-\theta}\,d\theta \tag{72}$$

$$= \frac{e^{-\frac{t}{RC}}\left(1 - e^{-\left(1+\frac{1}{RC}\right)t}\right)}{RC + 1} \tag{73}$$

A plot of $V(t)$ would resemble the graph in fig. 5.

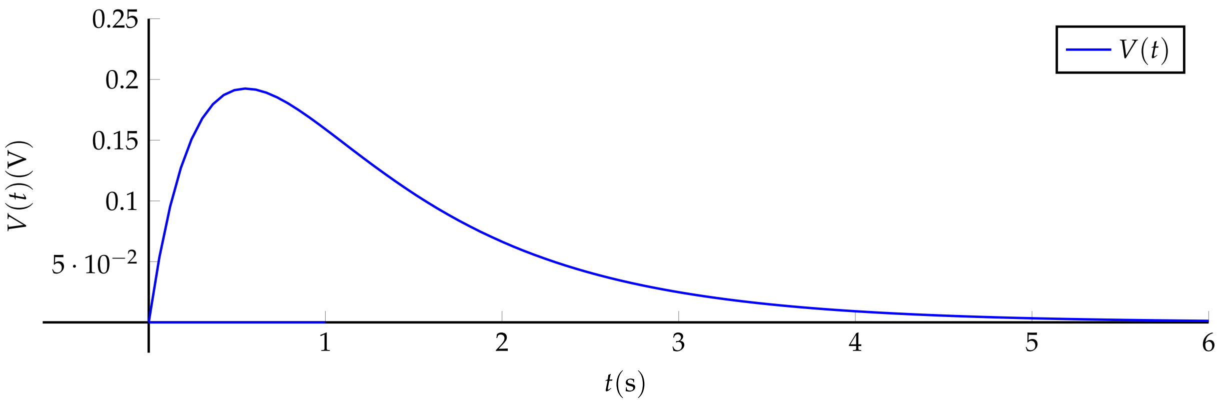
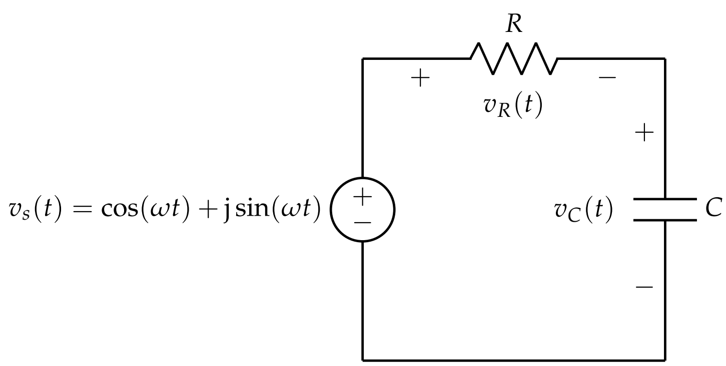



**Figure 5:** Plot of $V(t)$

## 2.6 Example with Sinusoidal Functions

Consider the following circuit:




where $v_c(0) = 0$. Applying KCL, we have

$$i_R(t) = i_c(t) \tag{74}$$

$$\frac{v_R(t)}{R} = C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} \tag{75}$$

and applying KVL, we have

$$v_s(t) = v_c(t) + v_R(t) \tag{76}$$

Hence, our differential equation governing the system is

$$\frac{\mathrm{d}v_c(t)}{\mathrm{d}t} = \underbrace{-\frac{1}{RC}}_{\lambda} v_c(t) + \underbrace{\frac{1}{RC}}_{b} \underbrace{v_s(t)}_{u(t)} \tag{77}$$

We can simplify $v_s(t)$ using Euler's formula as follows:

$$v_s(t) = \cos(\omega t) + \mathrm{j}\sin(\omega t) = \mathrm{e}^{\mathrm{j}\omega t} \tag{78}$$

Furthermore, let us define the R-C time constant $\tau := RC$. This simplifies the differential equation as follows:

$$\frac{\mathrm{d}v_c(t)}{\mathrm{d}t} = \underbrace{-\frac{1}{\tau}}_{\lambda} v_c(t) + \underbrace{\frac{1}{\tau}}_{b} \underbrace{\mathrm{e}^{\mathrm{j}\omega t}}_{u(t)} \tag{79}$$

From the initial condition, we have $t_0 = 0$, and $v_c(t_0) = 0$. Now, let us apply eq. (34) (or equivalently eq. (52)):

$$
\begin{aligned}
v_c(t) &= \frac{\mathrm{e}^{-\frac{t}{\tau}}}{\tau} \int_0^t \mathrm{e}^{\mathrm{j}\omega\theta} \mathrm{e}^{\frac{\theta}{\tau}} \,\mathrm{d}\theta && (80)\\
&= \frac{\mathrm{e}^{-\frac{t}{\tau}}}{\tau} \cdot \frac{\mathrm{e}^{\mathrm{j}\omega t + \frac{t}{\tau}} - 1}{\mathrm{j}\omega + \frac{1}{\tau}} && (81)\\
&= \frac{\mathrm{e}^{\mathrm{j}\omega t} - \mathrm{e}^{-\frac{t}{\tau}}}{\mathrm{j}\omega\tau + 1} && (82)\\
&= \frac{\mathrm{e}^{\mathrm{j}\omega t}}{\mathrm{j}\omega\tau + 1} - \frac{\mathrm{e}^{-\frac{t}{\tau}}}{\mathrm{j}\omega\tau + 1} && (83)
\end{aligned}
$$

Suppose we want to find $|v_c(t)|$ (the magnitude of $v_c(t)$) at steady state, i.e., when $t \to \infty$. We can compute the following:

$$
\begin{aligned}
\lim_{t\to\infty} |v_c(t)| &= \lim_{t\to\infty} \left| \frac{\mathrm{e}^{\mathrm{j}\omega t}}{\mathrm{j}\omega\tau + 1} - \frac{\mathrm{e}^{-\frac{t}{\tau}}}{\mathrm{j}\omega\tau + 1} \right| && (84)\\
&= \left| \frac{1}{\mathrm{j}\omega\tau + 1} \right| \lim_{t\to\infty} \left| \mathrm{e}^{\mathrm{j}\omega t} - \mathrm{e}^{-\frac{t}{\tau}} \right| && (85)
\end{aligned}
$$

Now, we can compute the magnitude inside the limit expression, namely

$$
\begin{aligned}
\left| \mathrm{e}^{\mathrm{j}\omega t} - \mathrm{e}^{-\frac{t}{\tau}} \right| &= \sqrt{\left(\mathrm{e}^{\mathrm{j}\omega t} - \mathrm{e}^{-\frac{t}{\tau}}\right)\overline{\left(\mathrm{e}^{\mathrm{j}\omega t} - \mathrm{e}^{-\frac{t}{\tau}}\right)}} && (86)\\
&= \sqrt{\left(\mathrm{e}^{\mathrm{j}\omega t} - \mathrm{e}^{-\frac{t}{\tau}}\right)\left(\mathrm{e}^{-\mathrm{j}\omega t} - \mathrm{e}^{-\frac{t}{\tau}}\right)} && (87)\\
&= \sqrt{1 - \mathrm{e}^{-\frac{t}{\tau}}\mathrm{e}^{-\mathrm{j}\omega t} - \mathrm{e}^{-\frac{t}{\tau}}\mathrm{e}^{\mathrm{j}\omega t} + \mathrm{e}^{-2\frac{t}{\tau}}} && (88)
\end{aligned}
$$

Notice that $\mathrm{e}^{\mathrm{j}\omega t}$ is some number on the complex unit circle, so $\left|\mathrm{e}^{\mathrm{j}\omega t}\right| = 1$. Furthermore, $\mathrm{e}^{-\frac{t}{\tau}} = 0$ as $t \to \infty$. Thus,

$$
\begin{aligned}
\lim_{t\to\infty} \left| \mathrm{e}^{\mathrm{j}\omega t} - \mathrm{e}^{-\frac{t}{\tau}} \right| &= \lim_{t\to\infty} \sqrt{1 - \mathrm{e}^{-\frac{t}{\tau}}\mathrm{e}^{-\mathrm{j}\omega t} - \mathrm{e}^{-\frac{t}{\tau}}\mathrm{e}^{\mathrm{j}\omega t} + \mathrm{e}^{-2\frac{t}{\tau}}} && (89)\\
&= \sqrt{\lim_{t\to\infty} \left(1 - \mathrm{e}^{-\frac{t}{\tau}}\mathrm{e}^{-\mathrm{j}\omega t} - \mathrm{e}^{-\frac{t}{\tau}}\mathrm{e}^{\mathrm{j}\omega t} + \mathrm{e}^{-2\frac{t}{\tau}}\right)} && (90)\\
&= \sqrt{1} = 1 && (91)
\end{aligned}
$$

Combining the above steps, we have

$$
\lim_{t\to\infty} |v_c(t)| = \left| \frac{1}{\mathrm{j}\omega\tau + 1} \right| \tag{92}
$$

So, if we have a very high-frequency input, i.e. $\omega \to \infty$, the magnitude of the capacitor's voltage at steady state is 0. On the other hand, if we have a very low-frequency input, i.e. $\omega \to 0$, the magnitude of the capacitor's voltage at steady state is 1.

**Contributors:**
- Anish Muthali.
- Neelesh Ramachandran.
- Nikhil Shinde.
- Anant Sahai.
- Aditya Arun.