Note 5: Phasors, AC Circuits, and Transfer Functions

1 Introduction to Phasors

1.1 (OPTIONAL) Motivation

So far in this course, we have been looking at inhomogeneous differential equations of the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu(t) \tag{1}$$

for some constant $b \in \mathbb{R}$ and a function $u(t) : \mathbb{R}_+ \to \mathbb{R}$ (where $\mathbb{R}_+$ denotes nonnegative real numbers). In the context of circuits, we often consider the function $u(t)$ term to be some sort of "input" into our circuit. In practice, we often encounter inputs $u(t)$ that fall under a specific class of functions, namely periodic functions.

Definition 1 (Periodic Function)
A function $f(t)$ is periodic if there exists some constant $T \in \mathbb{R}$ such that $f(t) = f(t+T)$ for all t in the domain of f.

An example of a periodic function is $f(t) = \sin(t)$. In this case, $T = 2\pi$. There exists a theorem that says any periodic function can be written as a sum of sinusoidal functions with period T. This theorem and its proof are out of scope for this class, but it is stated below for completeness.

Theorem 2 (Dirichlet's Theorem)
Let f be a periodic, well-behaved[a] function with period T. We can write f as a superposition of sinusoidal functions, namely

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{j2\pi nt}{T}} \tag{2}$$

where

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-\frac{j2\pi nt}{T}}\,\mathrm{d}t \tag{3}$$

Furthermore, it should be noted that $c_{-n} = \overline{c_n}$.

[a] Satisfies the Dirichlet conditions

As we will explore in more depth in this note, it is often easier to reason about periodic functions in terms of the coefficients, c_n. Each c_n (for $n \geq 0$) are the *phasors* for the corresponding sinusoid with frequency $\frac{2\pi n}{T}$. It should be noted that phasors have **no time dependence**. We call this process of analyzing circuits through the phasors as *phasor domain analysis*. When dealing with phasors in this class, we will only calculate phasors for sine or cosine inputs (so $f(t) = V_0 \cos(\omega t + \phi)$ or $f(t) = V_0 \sin(\omega t + \phi)$).

1.2 Magnitude-Phase Representations of Complex Numbers

We can represent any complex number with a magnitude and phase. That is, we can write any c_n in the form $Ae^{j\phi}$ for real values A, ϕ.

Theorem 3 (Magnitude-Phase Representation)
Given a complex number $x = a + \mathrm{j}b$, we can equivalently represent it in the form $x = A\mathrm{e}^{\mathrm{j}\phi}$ where $A = |x| = \sqrt{x\overline{x}} = \sqrt{a^2+b^2}$ and $\phi = \text{atan2}(b, a)$.

Proof. We can set the two representations equal and solve for A and ϕ. That is,

$$a + \mathrm{j}b = A\mathrm{e}^{\mathrm{j}\phi} \tag{4}$$
$$= A\cos(\phi) + \mathrm{j}A\sin(\phi) \tag{5}$$

so we have that $a = A\cos(\phi)$ and $b = A\sin(\phi)$. We have that

$$A^2 = A^2\cos^2(\phi) + A^2\sin^2(\phi) \tag{6}$$
$$= a^2 + b^2 \tag{7}$$

so $A = \sqrt{a^2+b^2}$. Next, we have that

$$\frac{b}{a} = \frac{\sin(\phi)}{\cos(\phi)} \tag{8}$$
$$= \tan(\phi) \tag{9}$$

so $\phi = \text{atan2}(b, a)$[1]. □

1.3 Determining Phasors for Sine and Cosine Functions

First, we should note a corollary of Euler's formula.

Theorem 4 (Euler's Theorem)
The following identities hold:

$$\sin(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}\theta}}{2\mathrm{j}} \tag{10}$$
$$\cos(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}}{2} \tag{11}$$

Concept Check: Show that these identities hold, using the fact that $\mathrm{e}^{\mathrm{j}\theta} = \cos(\theta) + \mathrm{j}\sin(\theta)$.

Now, we can derive a formula for the phasor representation of a sine/cosine. Note that $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$, so it suffices to derive a phasor representation for an arbitrary cosine function.

Theorem 5 (Cosine Phasors)
Suppose we are given an arbitrary, time-varying cosine function of the form $v(t) = V_0\cos(\omega t + \phi)$, where V_0 is the amplitude, ω is the frequency, and ϕ is a phase shift. The function $v(t)$'s phasor for the frequency ω is given by $\widetilde{V} = V_0\mathrm{e}^{\mathrm{j}\phi}$[a].

[a] We denote the phasor for $v(t)$ as $\widetilde{V}$, dropping the time input, capitalizing, and putting a tilde on top.

[1] We choose to use two argument atan (i.e., atan2) because this preserves the sign of the angles and we will not encounter division by 0 this way.

Proof. Using Theorem 4, we have that

$$V_0\cos(\omega t+\phi) = V_0\frac{e^{j(\omega t+\phi)}+e^{-j(\omega t+\phi)}}{2} \tag{12}$$

$$= \underbrace{\frac{V_0 e^{j\phi}}{2}}_{c_\omega} e^{j\omega t} + \underbrace{\frac{V_0 e^{-j\phi}}{2}}_{\overline{c_\omega}} e^{-j\omega t} \tag{13}$$

so we have that c_ω is $\frac{V_0 e^{j\phi}}{2}$ as desired. By convention, we omit the 2 in the denominator when describing the phasor, i.e. we say the phasor is $V_0 e^{j\phi}$. □

Corollary 6 (Sine Phasors)
Suppose we are given an arbitrary, time-varying cosine function of the form $v(t) = V_0\sin(\omega t+\phi)$, where V_0 is the amplitude, ω is the frequency, and ϕ is a phase shift. The function $v(t)$'s phasor for the frequency ω is given by $\widetilde{V} = \frac{V_0 e^{j\phi}}{j}$.

Concept Check: Prove this corollary, using the fact that $\sin(x) = \cos\left(x-\frac{\pi}{2}\right)$ and that $e^{-j\frac{\pi}{2}} = -j = \frac{1}{j}$.
Example:
Suppose $v(t) = 10\cos\left(20t+\frac{3\pi}{2}\right)$. To find the phasor for this function, we can begin by pattern matching $V_0 = 10$ and $\phi = \frac{3\pi}{2}$. Applying this to the resulf of Theorem 5, we have $\widetilde{V} = 10e^{j\frac{3\pi}{2}} = -10j$.

2 Computing Impedances in Phasor Domain

We can look at the phasor domain "resistances" of all passive circuit elements we have learned so far. The technical term for these "resistances" is impedance. Formally, we denote this as

$$Z = \frac{\widetilde{V}}{\widetilde{I}} \tag{14}$$

We are leveraging the I-V relationship of each circuit element in phasor domain so that we can derive their phasor domain impedances.

Theorem 7 (Impedance of a Capacitor)
Suppose we applied an input voltage $v_C(t) = V_0\cos(\omega t+\phi)$ across a capacitor with capacitance C. Its phasor domain impedance is given by $Z_C = \frac{1}{j\omega C}$.

Proof. We can find $i_C(t)$ and then find its phasor domain representation, i.e., $\widetilde{I}_C$. We can apply the equation relating current and voltage across a capacitor, namely

$$i_C(t) = C\frac{d}{dt}v_C(t) \tag{15}$$

$$= C\frac{d}{dt}(V_0\cos(\omega t+\phi)) \tag{16}$$

$$= -\omega C V_0\sin(\omega t+\phi) \tag{17}$$

Using Corollary 6, we have that

$$\widetilde{I}_C = \frac{-\omega C V_0 e^{j\phi}}{j} \tag{18}$$

$$= \mathrm{j}\omega C V_0 \mathrm{e}^{\mathrm{j}\phi} \tag{19}$$

and by Theorem 5, we have that

$$\widetilde{V}_C = V_0 \mathrm{e}^{\mathrm{j}\phi} \tag{20}$$

Hence,

$$Z_C = \frac{\widetilde{V}_C}{\widetilde{I}_C} = \frac{1}{\mathrm{j}\omega C} \tag{21}$$

□

Theorem 8 (Impedance of a Resistor)
Suppose we applied an input voltage $v_R(t) = V_0 \cos(\omega t + \phi)$ across a resistor with resistance R. Its phasor domain impedance is given by $Z_R = R$.

Proof. Using the same technique as the proof of Theorem 7, we find $i_R(t)$ as follows:

$$i_R(t) = \frac{1}{R} v_R(t) = \frac{V_0}{R} \cos(\omega t + \phi) \tag{22}$$

The phasor domain representation of this is

$$\widetilde{I}_R = \frac{V_0}{R} \mathrm{e}^{\mathrm{j}\phi} = \frac{1}{R} V_0 \mathrm{e}^{\mathrm{j}\phi} \tag{23}$$

The expression for $\widetilde{V}_R$ remains the same as the expression for $\widetilde{V}_C$ in Theorem 7. Hence,

$$Z_R = \frac{\widetilde{V}_R}{\widetilde{V}_C} = R \tag{24}$$

□

Theorem 9 (Impedance of an Inductor)
Suppose we applied an input current $i_L(t) = V_0 \cos(\omega t + \phi)$ through an inductor with inductance L. Its phasor domain impedance is given by $Z_L = \mathrm{j}\omega L$.

Proof. We can find $\widetilde{V}_L$ by first finding $v_L(t)$ as follows:

$$v_L(t) = L \frac{\mathrm{d}}{\mathrm{d}t} i_L(t) \tag{25}$$

$$= L \frac{\mathrm{d}}{\mathrm{d}t} (V_0 \cos(\omega t + \phi)) \tag{26}$$

$$= -L\omega V_0 \sin(\omega t + \phi) \tag{27}$$

Now, we can use Corollary 6 to find $\widetilde{V}_L$:

$$\widetilde{V}_L = \frac{-\omega L V_0 \mathrm{e}^{\mathrm{j}\phi}}{\mathrm{j}} \tag{28}$$

$$= \mathrm{j}\omega L V_0 \mathrm{e}^{\mathrm{j}\phi} \tag{29}$$

Here, we have that $\widetilde{I}_L = V_0 \mathrm{e}^{\mathrm{j}\phi}$ so

$$Z_L = \frac{\widetilde{V}_L}{\widetilde{I}_L} = \mathrm{j}\omega L \tag{30}$$

□

Key Idea 10 (Using Phasor Impedances)
Since the phasor impedance represent an I-V relationship in phasor domain, and since the impedance is constant with respect to time, we can treat all components' phasor domain representations as time domain resistors. That is, we can apply the same rules for KCL, NVA, and parallel/series combinations of resistors.

Example:
We can solve for $v_{\text{out}}(t)$ in Figure 1 by using phasor domain conversions.

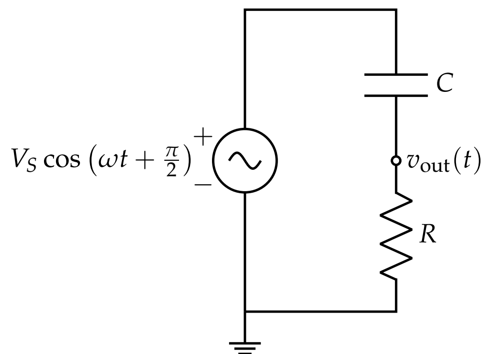

Figure 1: Example Circuit

Here, we can perform the phasor domain conversion on the input voltage since it is a single sinusoid. That is, we have that $v_{\text{in}}(t) := V_S \cos\left(\omega t + \frac{\pi}{2}\right)$ so $\widetilde{V}_{\text{in}} = V_S e^{j\frac{\pi}{2}}$. Using the fact that $Z_C = \frac{1}{j\omega C}$ and $Z_R = R$, we can treat these components as "resistors" in phasor domain. That is, we can apply the resistor voltage divider formula to obtain

$$\widetilde{V}_{\text{out}} = \frac{Z_R}{Z_C + Z_R}\widetilde{V}_{\text{in}} \tag{31}$$

$$= \frac{R}{\frac{1}{j\omega C} + R}\left(V_S e^{j\frac{\pi}{2}}\right) \tag{32}$$

$$= \frac{j\omega RC}{1 + j\omega RC}\left(V_S e^{j\frac{\pi}{2}}\right) \tag{33}$$

$$= \frac{\omega RC e^{j\frac{\pi}{2}}}{\sqrt{1 + (\omega RC)^2} e^{j\,\text{atan2}(\omega RC, 1)}}\left(V_S e^{j\frac{\pi}{2}}\right) \tag{34}$$

$$= \frac{V_S \omega RC}{\sqrt{1 + (\omega RC)^2}} e^{j(\pi - \text{atan2}(\omega RC, 1))} \tag{35}$$

$$= \left(\frac{V_S \omega RC}{\sqrt{1 + (\omega RC)^2}}\right) e^{j(\pi - \text{atan2}(\omega RC, 1))} \tag{36}$$

where we convert to the magnitude-phase representation of the numerator and denominator in eq. (34). Next, we can reverse the steps of Theorem 5 to obtain the time domain output. We can pattern match $V_0 = \frac{V_S \omega RC}{\sqrt{1+(\omega RC)^2}}$ and $\phi = \pi - \text{atan2}(\omega RC, 1)$, so

$$v_{\text{out}}(t) = \frac{V_S \omega RC}{\sqrt{1 + (\omega RC)^2}} \cos(\omega t + \pi - \text{atan2}(\omega RC, 1)) \tag{37}$$

3 Motivation for Transfer Functions

In the previous sections, we introduced phasor domain analysis for circuit elements. We can further expand our analysis of circuits in phasor domain by introducing *transfer functions*. Informally, these are functions that describe the behavior of some system, where the input is the frequency of the input into the system and the output is some representation of the observed input/output behavior of the system. The abstractions we introduce here is somewhat similar to the abstraction of Thevenin equivalent voltage and Norton equivalent current.

4 Introduction to Transfer Functions

Definition 11 (Transfer Function)
Consider the block diagram of a system in Figure 2.

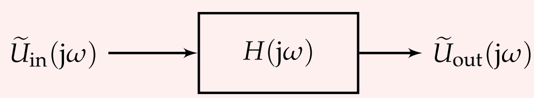

Figure 2: Transfer Function Block Diagram

where $\widetilde{U}_{\text{in}}(j\omega)$ and $\widetilde{U}_{\text{out}}(j\omega)$ are the respective inputs and outputs of the system, and the block is the system itself. The transfer function, $H(j\omega)$, is defined by

$$H(j\omega) = \frac{\widetilde{U}_{\text{out}}(j\omega)}{\widetilde{U}_{\text{in}}(j\omega)} \tag{38}$$

Equivalently, $\widetilde{U}_{\text{out}}(j\omega) = H(j\omega)\widetilde{U}_{\text{in}}(j\omega)$. We use an arbitrary definition of a system above, but we will make this more concrete for the specific case of transfer functions of circuits. When describing the behavior of a transfer function, we typically look at the magnitude and phase of the transfer function, as a function of ω. We will see some examples of how to calculate magnitude and phase of a transfer function in the next section.

5 Transfer Functions of Common Filters

A *filter* is commonly used to block or allow certain ranges of frequencies to pass through as an output, i.e., it allows or restricts certain inputs, based on the frequencies (ω) of the inputs. Generally, filters are written as transfer functions of the form $H(j\omega) = \frac{p(\omega)}{q(\omega)}$, for $p(\cdot)$ and $q(\cdot)$ being polynomials. From this, we can define the concept of the *order* of a filter.

Definition 12 (Filter Order)
Suppose that a filter's transfer function can be written as a simplified fraction of two polynomials, i.e., $H(j\omega) = \frac{p(\omega)}{q(\omega)}$. The order of a transfer function is $\max(\deg(p), \deg(q))$, where $\deg(\cdot)$ denotes the degree of the polynomial.

In circuits, we define the "inputs" to our transfer function to be some sort of input voltage phasor, denoted $\widetilde{V}_{\text{in}}$, and the output as some sort of output voltage phasor $\widetilde{V}_{\text{out}}$. We typically encounter two types of first order filters – a low pass filter and a high pass filter.

5.1 Common First Order Filters

Definition 13 (Low Pass Filter)
A low pass filter is defined by the following transfer function:

$$H(\mathrm{j}\omega) = \frac{1}{1 + \mathrm{j}\frac{\omega}{\omega_c}} \tag{39}$$

where ω_c is known as the *cutoff frequency*[a]. This transfer function attenuates the magnitude of outputs where the inputs have frequency $\omega \gg \omega_c$, and not affect the magnitude for inputs that have frequency $\omega \ll \omega_c$.

[a] The idea of a cutoff frequency will become more concrete when we plot transfer functions.

Proof. We can convert the numerator and denominator of eq. (39) into phasor form, namely:

$$1 = (1)\mathrm{e}^{\mathrm{j}0} \tag{40}$$

and

$$\begin{aligned} 1 + \mathrm{j}\frac{\omega}{\omega_c} &= \left|1 + \mathrm{j}\frac{\omega}{\omega_c}\right| \mathrm{e}^{\mathrm{j}\measuredangle(1+\mathrm{j}\frac{\omega}{\omega_c})} && (41) \\ &= \sqrt{1 + \frac{\omega^2}{\omega_c^2}}\mathrm{e}^{\mathrm{j}\,\mathrm{atan2}\left(\frac{\omega}{\omega_c},1\right)} && (42) \end{aligned}$$

So if we were to combine this altogether, we would have

$$\begin{aligned} H(\mathrm{j}\omega) &= \frac{(1)\mathrm{e}^{\mathrm{j}0}}{\sqrt{1 + \frac{\omega^2}{\omega_c^2}}\mathrm{e}^{\mathrm{j}\,\mathrm{atan2}\left(\frac{\omega}{\omega_c},1\right)}} && (43) \\ &= \frac{1}{\sqrt{1 + \frac{\omega^2}{\omega_c^2}}}\mathrm{e}^{-\mathrm{j}\,\mathrm{atan2}\left(\frac{\omega}{\omega_c},1\right)} && (44) \end{aligned}$$

so $|H(\mathrm{j}\omega)| = \frac{1}{\sqrt{1+\frac{\omega^2}{\omega_c^2}}}$. Note that this is a function of ω, which is expected. Now, we can prove the behavior for $\omega \ll \omega_c$ by taking the limit as $\omega \to 0$:

$$\lim_{\omega \to 0} \frac{1}{\sqrt{1 + \frac{\omega^2}{\omega_c^2}}} = 1 \tag{45}$$

so, as $\omega \to 0$, $\left|\widetilde{U}_{\text{out}}(\mathrm{j}\omega)\right| = \left|\widetilde{U}_{\text{in}}(\mathrm{j}\omega)\right|$. Now, we can show the behavior for $\omega \gg \omega_c$ by taking a limit as $\omega \to \infty$:

$$\lim_{\omega \to \infty} \frac{1}{\sqrt{1 + \frac{\omega^2}{\omega_c^2}}} = 0 \tag{46}$$

so, as $\omega \to \infty$, $\left|\widetilde{U}_{\text{out}}(\mathrm{j}\omega)\right| = 0$. □

Example:
We can implement this kind of transfer function in circuits as follows[2]:

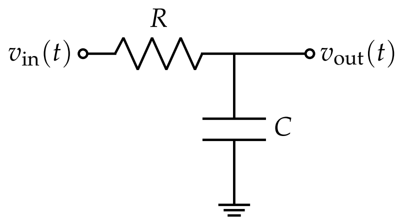

Figure 3: RC Low Pass Circuit

Note that this is not the only way to implement a low pass transfer function in circuit form. Here, the input would be $\widetilde{U}_{\text{in}}(\mathrm{j}\omega) := \widetilde{V}_{\text{in}}(\mathrm{j}\omega)$, the phasor for $v_{\text{in}}(t)$, and the output would be $\widetilde{U}_{\text{out}}(\mathrm{j}\omega) := \widetilde{V}_{\text{out}}(\mathrm{j}\omega)$, the phasor for $v_{\text{out}}(t)$. To show that this circuit is an implementation of a low pass transfer function, we can find $\widetilde{V}_{\text{out}}$ in terms of $\widetilde{V}_{\text{in}}$ and use this to find $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$. Using the voltage divider formula, we have

$$\widetilde{V}_{\text{out}} = \frac{\frac{1}{\mathrm{j}\omega C}}{R + \frac{1}{\mathrm{j}\omega C}}\widetilde{V}_{\text{in}} = \frac{1}{1 + \mathrm{j}\omega RC}\widetilde{V}_{\text{in}} \tag{47}$$

so

$$H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = \frac{1}{1 + \mathrm{j}\omega RC} \tag{48}$$

We can pattern match ω_c, the cutoff frequency, to $\frac{1}{RC}$, and we exactly recover the form of a low pass transfer function.
We can also implement a low pass transfer function using inductors, as follows:

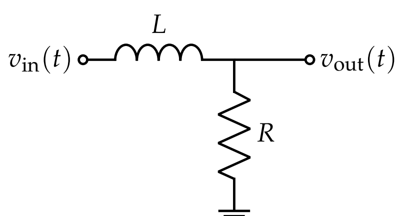

Figure 4: LR Low Pass Circuit

Concept Check: Show that the circuit in Figure 4 implements a low pass transfer function.

Definition 14 (High Pass Filter)
A high pass filter is defined by the following transfer function:

$$H(\mathrm{j}\omega) = \frac{\mathrm{j}\frac{\omega}{\omega_c}}{1 + \mathrm{j}\frac{\omega}{\omega_c}} \tag{49}$$

with ω_c being the cutoff frequency. This transfer function attenuates the magnitude of outputs where the inputs have frequency $\omega \ll \omega_c$, and not affect the magnitude for inputs that have frequency

[2]For all the transfer function implementations described in this note, the circuits themselves are not unique. That is, it is possible to emulate the same transfer function behavior with different circuit components.

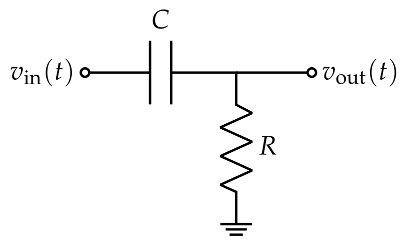

Figure 5: RC High Pass Circuit

$\omega \gg \omega_c$.

Proof. Similar to before, we can separately convert the numerator and denominator to phasors and take limits. For the numerator,

$$\mathrm{j}\frac{\omega}{\omega_c} = \left|\mathrm{j}\frac{\omega}{\omega_c}\right| \mathrm{e}^{\mathrm{j}\measuredangle \mathrm{j}\frac{\omega}{\omega_c}} \tag{50}$$

$$= \frac{\omega}{\omega_c}\mathrm{e}^{\mathrm{j}\frac{\pi}{2}} \tag{51}$$

and for the denominator, the phasor is the same as before, i.e.,

$$1+\mathrm{j}\frac{\omega}{\omega_c} = \left|1+\mathrm{j}\frac{\omega}{\omega_c}\right| \mathrm{e}^{\mathrm{j}\measuredangle(1+\mathrm{j}\frac{\omega}{\omega_c})} \tag{52}$$

$$= \sqrt{1+\frac{\omega^2}{\omega_c^2}}\,\mathrm{e}^{\mathrm{j}\,\mathrm{atan2}\left(\frac{\omega}{\omega_c},1\right)} \tag{53}$$

Hence, the resulting phasor expression for the transfer function is

$$H(\mathrm{j}\omega) = \frac{\frac{\omega}{\omega_c}}{\sqrt{1+\frac{\omega^2}{\omega_c^2}}}\mathrm{e}^{\mathrm{j}\left(\frac{\pi}{2}-\mathrm{atan2}\left(\frac{\omega}{\omega_c},1\right)\right)} \tag{54}$$

Taking limits on the magnitude,

$$\lim_{\omega\to 0}\frac{\frac{\omega}{\omega_c}}{\sqrt{1+\frac{\omega^2}{\omega_c^2}}} = 0 \tag{55}$$

and

$$\lim_{\omega\to\infty}\frac{\frac{\omega}{\omega_c}}{\sqrt{1+\frac{\omega^2}{\omega_c^2}}} = \lim_{\omega\to\infty}\frac{1}{\sqrt{\frac{\omega_c^2}{\omega^2}+1}} = 1 \tag{56}$$

which agrees with the qualitative behavior described above. □

Example:

We can implement this kind of transfer function in circuits as follows: We can show that this implements a high pass transfer function by computing the transfer function $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$ for this circuit as before. Using the voltage divider formula,

$$\widetilde{V}_{\text{out}} = \frac{R}{R+\frac{1}{\mathrm{j}\omega C}}\widetilde{V}_{\text{in}} = \frac{\mathrm{j}\omega RC}{1+\mathrm{j}\omega RC}\widetilde{V}_{\text{in}} \tag{57}$$

which matches the high pass transfer function definition if we pattern match $\omega_c = \frac{1}{RC}$. Another way to implement a high pass transfer function is

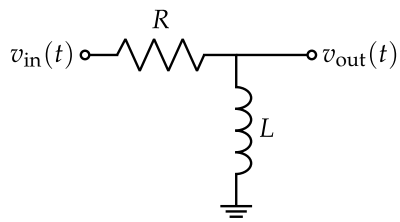

Figure 6: LR High Pass Circuit

Concept Check: Show that the circuit in Figure 6 implements a high pass transfer function. Now we can proceed to discuss filters with more complex behavior by focusing on second order filters.

5.2 Second Order Filters

Based on the transfer functions discussed in the previous subsection, we can define some second order filters to be a product of the filters discussed previously.

Definition 15 (Second Order Low/High Pass)
A second order low/high pass filter is constructed by squaring the transfer function of a first low/high pass filter, i.e.

$$H_{\text{Second Order LP}} = (H_{\text{LP}})^2 \tag{58}$$
$$H_{\text{Second Order HP}} = (H_{\text{HP}})^2 \tag{59}$$

Example:
In practice, we combine filters by connecting them with a unity gain op amp, as shown in Figure 7. The reason for this is that it prevents a loading effect, which would otherwise be present without the unity gain op amp.

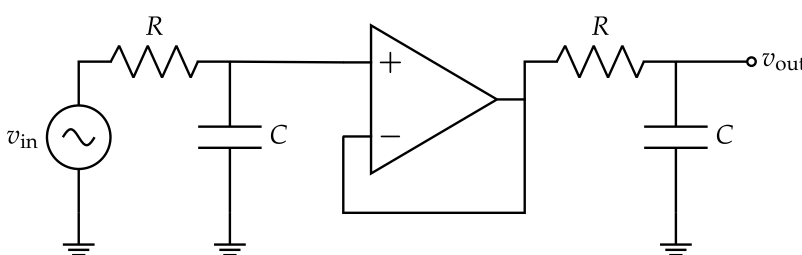

Figure 7: Second Order Low Pass Filter

We can also define a *band pass* filter.

Definition 16 (Band Pass Filter)
A band pass filter does not attenuate the magnitude of inputs with frequencies inside of a certain interval, say $\omega \in [a, b]$, and it attenuates frequencies outside this interval.

Example:
We could implement this in a circuit by combining a low pass filter and high pass filter with a unity gain

op amp (note, this is not the only way to create a band pass filter). Mathematically, we can write this as

$$H_{\text{BP}}(j\omega) = H_{\text{LP}}(j\omega) \cdot H_{\text{HP}}(j\omega) \tag{60}$$

The low pass filter needs to have a cutoff frequency higher than that of the high pass filter. Here, as $\omega \to 0$ and $\omega \to \infty$, $|H(j\omega)| \to 0$ (by virtue of limit multiplication rules) with less attenuation for frequencies somewhere in between. Following the terminology used in the definition above, we would generally set the cutoff frequency of the high pass filter to $\omega_{c,\text{HP}} = a$ and the cutoff frequency of the low pass filter to $\omega_{c,\text{LP}} = b$. A band pass filter implemented in this manner might look like the circuit in Figure 8. Note that we would want $\frac{1}{R_1C_1} > \frac{1}{R_2C_2}$.

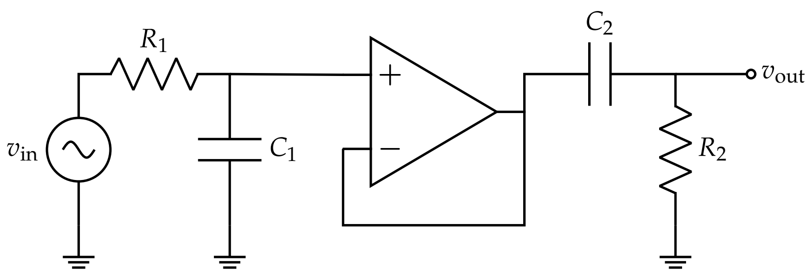

Figure 8: Band Pass Filter

Similarly, we can create a filter that has the opposite effect of a band pass filter, i.e. a notch filter.

Definition 17 (Notch Filter)
A notch filter is the opposite of a band pass filter, in that it attenuates the magnitude of inputs with frequencies inside of a certain interval, say $\omega \in [a, b]$, and it does not attenuate frequencies outside this interval.

Example:
We can implement a notch filter with an LC tank type circuit, as shown in Figure 9.

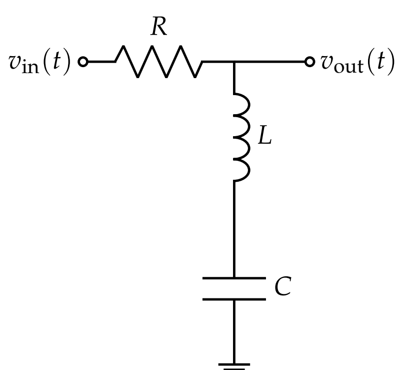

Figure 9: RC Low Pass Circuit

The transfer function, which we derive using voltage dividers, is

$$H(j\omega) = \frac{j\omega L + \frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} \tag{61}$$

$$= \frac{1 - \omega^2 LC}{j\omega RC - \omega^2 LC + 1} \tag{62}$$

$$= \frac{1}{1 + j\frac{\omega RC}{1-\omega^2 LC}} \tag{63}$$

To find $|H(j\omega)|$, we can find the magnitude of the top and bottom separately, i.e.,

$$|1| = 1 \tag{64}$$

and

$$\left|1 + j\frac{\omega RC}{1 - \omega^2 LC}\right| = \sqrt{\left(1 + j\frac{\omega RC}{1 - \omega^2 LC}\right)\left(1 - j\frac{\omega RC}{1 - \omega^2 LC}\right)} \tag{65}$$

$$= \sqrt{1 + \left(\frac{\omega RC}{1 - \omega^2 LC}\right)^2} \tag{66}$$

and divide the two to obtain

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega RC}{1-\omega^2 LC}\right)^2}} \tag{67}$$

To see what frequency is most attenuated, we can see what frequency minimizes $|H(j\omega)|$. To do this, we can maximize the denominator, or equivalently maximize $\frac{\omega RC}{1-\omega^2 LC}$. Notice that this term goes to ∞ when $1 - \omega^2 LC = 0 \iff \omega = \frac{1}{\sqrt{LC}}$. Hence, $|H(j\omega)| = 0$ at $\omega = \frac{1}{\sqrt{LC}}$. **Concept Check:** Show that the transfer function satisfies the remaining requirements of a notch filter by taking limits as $\omega \to 0$ and $\omega \to \infty$.

Key Idea 18 (Choosing Cutoff Frequencies)
When designing transfer functions, you can choose various values for resistance, capacitance, and inductance based on the desired specifications of the system. As we have seen before, the cutoff frequencies in all of these circuit filters are functions of resistance, capacitance, and/or inductance. Hence, it is important to carefully choose values for these based on design requirements and cost constraints.

Note: This section only contains examples of some common transfer functions. It is by no means an exhaustive list of all possible transfer functions, or even an exhaustive list of transfer functions we will cover in the class.

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