Note 9: Stability and Feedback Control

1 Overview and Motivation

Given an LTI model, we know how to determine its behavior at a given time as a function of its initial condition and all inputs. Now, we would like to know how to determine its asymptotics, and get a qualitative idea of long-term system behavior.

We will begin by introducing the concept of *stability*.

Key Idea 1 (Stability)
A model is *stable* if, given sufficiently "nice" inputs, the model won't have "bad" behavior.

Then, we will learn how to tune the long-term system behavior to where we want, by supplying the right inputs. Such inputs will be functions of the current state, which motivates the idea of *feedback control*.

Key Idea 2 (Feedback Control)
Feedback control is when we choose the control input at each time as a function of the current state.

2 Stability

2.1 Definitions

This course uses a particular definition of stability called *bounded-input, bounded-output (BIBO) stability*. We will define it shortly, but first we need to know what *bounded* even means.

Definition 3 (Boundedness)

- A discrete-time function $\vec{z}_d \colon \mathbb{N} \to \mathbb{R}^k$ is *bounded* if there exists some number $R_d \in \mathbb{R}$ such that $\|\vec{z}_d[i]\| \leq R_d$ for all i.
- Similarly, a continuous-time function $\vec{z}_c \colon \mathbb{R}_+ \to \mathbb{R}^k$ is *bounded* if there exists some number $R_c \in \mathbb{R}$ such that $\|\vec{z}_c(t)\| \leq R_c$ for all t.

Note that we need the constants R_d or R_c to be independent of i or t respectively. One can think of them as denoting a radius for the boundary of the region that $\vec{z}_d$ or $\vec{z}_c$ have to stay in for all time.

Now we are ready to define BIBO stability.

Definition 4 (BIBO Stability)
A control model is *(BIBO) stable* if and only if, for *every* bounded input function $\vec{u}$, and *every* initial condition $\vec{x}_0$, the resulting state trajectory $\vec{x}$ is bounded. It is *(BIBO) unstable* if it is not stable.

NOTE: We did not use any continuous-time or discrete-time subscripts because the same definition applies to both kinds of models.

Generally, stability is *desirable* in our control models, because it means that the model will produce well-behaved state trajectories.

A control model is *unstable* if it is not stable. More specifically, this means that we can find a bounded input and an initial condition which results in an unbounded state trajectory (i.e., one without an upper bound R on the norm that holds for all times) as the time(step) goes to ∞. Generally, instability is undesirable in our control models.

Now, we will find some equivalent conditions to BIBO stability in the models we have already explored.

2.2 Asymptotic Stability

Recall from Note 9 the discrete-time model we use:

Model 5 (Discrete-Time LTI Difference Equation Model)
The model is of the form

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] \tag{1}$$

$$\vec{x}[0] = \vec{x}_0 \tag{2}$$

for $\vec{x}\colon \mathbb{N} \to \mathbb{R}^n$ the state vector as a function of timestep, $\vec{u}\colon \mathbb{N} \to \mathbb{R}^m$ the control inputs as a function of timestep, and $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ matrices.

Here is a characterization of stability for this model. Models for which this characterization holds are called *asymptotically stable*.

Theorem 6 (Asymptotic Stability in Discrete-Time LTI Difference Equation Model)
Suppose we are in Discrete-Time LTI Difference Equation Model where $A \in \mathbb{R}^{n\times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$.

(i) If $|\lambda_i| < 1$ for all i, then the model is stable.

(ii) If there is a j such that $|\lambda_j| > 1$, then the model is unstable.

The proof of Theorem 6 is on the longer side and may distract from the overall flow of this note, so it is left to Appendix A.1. We fully expect you to read the proof and understand it. It is completely in-scope for the course.

NOTE: The eigenvalues of A may be real or complex. The $|\cdot|$ refers to the complex magnitude function, i.e., $|x + \mathrm{j}y| = \sqrt{x^2 + y^2}$. This reduces to the more familiar absolute value function when λ is real.

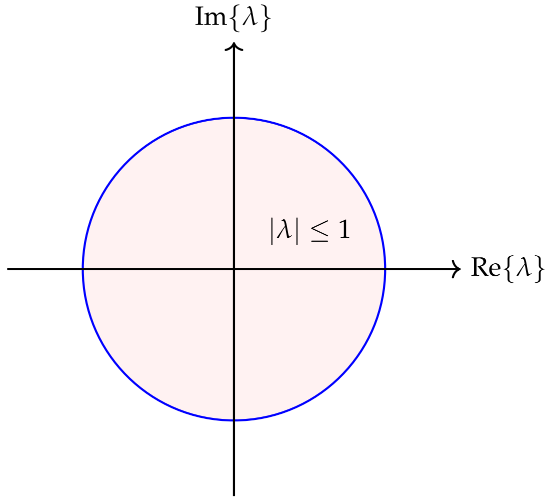

Figure 1: Here, we show the "discrete-time stability region" for the eigenvalues of A. The light red part, $|\lambda| < 1$, is where asymptotic stability is guaranteed, if all eigenvalues are in this region; the blue part, $|\lambda| = 1$, is where so-called *marginal stability* may occur; and the white part, $|\lambda| > 1$ is where asymptotic stability is guaranteed *not* to occur.

We may also look at the stability of continuous-time models. Recall from Note 9 the continuous-time model we use:

Model 7 (Continuous-Time LTI Differential Equation Model)
The model is of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{3}$$

$$\vec{x}(0) = \vec{x}_0 \tag{4}$$

for $\vec{x}\colon \mathbb{R}_+ \to \mathbb{R}^n$ the state vector as a function of time, $\vec{u}\colon \mathbb{R}_+ \to \mathbb{R}^m$ the control inputs as a function of time, and $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ matrices.

Here is a characterization of (asymptotic) stability for this model.

Theorem 8 (Asymptotic Stability in Continuous-Time LTI Differential Equation Model)
Suppose we are in Continuous-Time LTI Differential Equation Model where $A \in \mathbb{R}^{n\times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$.

(i) If $\mathrm{Re}\{\lambda_i\} < 0$ for all i, then the model is stable.

(ii) If there is an j such that $\mathrm{Re}\{\lambda_j\} > 0$, then the model is unstable.

The proof of Theorem 8 is on the longer side and may distract from the overall flow of this note, so it is left to Appendix A.1. We fully expect you to read the proof and understand it. It is completely in-scope for the course.

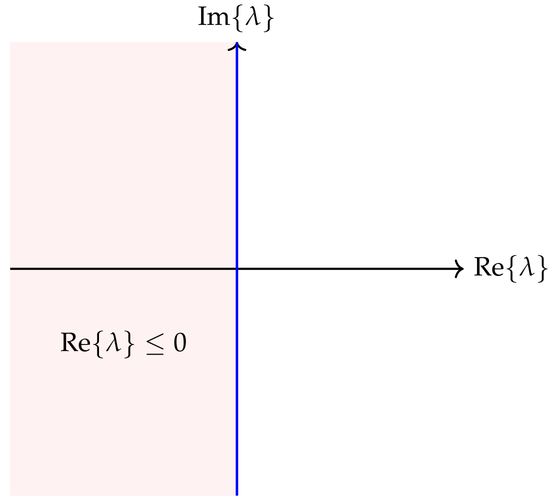

Figure 2: Here, we show the "continuous-time stability region" for the eigenvalues of A. The light red part, $\mathrm{Re}\{\lambda\} < 0$, is where asymptotic stability is guaranteed, if all eigenvalues are in this region; the blue part, $\mathrm{Re}\{\lambda\} = 0$, is where so-called *marginal stability* may occur; and the white part, $\mathrm{Re}\{\lambda\} > 0$, is where asymptotic stability is guaranteed *not* to occur.

2.3 Marginal Stability

In discrete-time, we know whether the system is stable if all eigenvalues of A have magnitude strictly less than 1, and know the system is unstable if any eigenvalues of A has magnitude strictly greater than 1. We *do not* know what happens if all eigenvalues of A have magnitude less than *or equal to* 1, and some eigenvalues have magnitude exactly 1.

Correspondingly, in continuous-time, we know whether the system is stable if all eigenvalues of A have real part strictly less than 0, and know the system is unstable if any eigenvalues of A has real part strictly greater than 0. We *do not* know what happens if all eigenvalues of A have real part less than *or equal to* 0, and some eigenvalues have real part exactly 0.

Where A is diagonalizable, say $A = V\Lambda V^{-1}$, it turns out that there is a general way to check stability in these regimes. We have the following theorems characterizing stability in this context. Models which satisfy the following characterizations, but not the asymptotic stability characterizations, are called *marginally stable*.

Theorem 9 (Marginal Stability in Discrete-Time LTI Difference Equation Model when A is Diagonalizable)
Suppose we are in Discrete-Time LTI Difference Equation Model where $A \in \mathbb{R}^{n\times n}$ is diagonalizable, say $A = V\Lambda V^{-1}$, with eigenvalues $\lambda_1, \ldots, \lambda_n$.

(i) If $|\lambda_i| \leq 1$ for all i, and for every i such that $|\lambda_i| = 1$ we have that the i^{th} row of $V^{-1}B$ is $\vec{0}_m^\top$, then the model is stable.

(ii) If there is a j such that $|\lambda_j| > 1$, or a j such that $|\lambda_j| = 1$ and the j^{th} row of $V^{-1}B$ is nonzero, then the model is unstable.

The proof of Theorem 9 is on the longer side and may distract from the overall flow of this note, so it is left to Appendix A.2. We fully expect you to read the proof and understand it. It is completely in-scope for the course.

Theorem 10 (Marginal Stability in Discrete-Time LTI Difference Equation Model when A is Diagonalizable)
Suppose we are in Continuous-Time LTI Differential Equation Model where $A \in \mathbb{R}^{n\times n}$ is diagonalizable, say $A = V\Lambda V^{-1}$, with eigenvalues $\lambda_1, \ldots, \lambda_n$.

(i) If $\text{Re}\{\lambda_i\} \le 0$ for all i, and for every i such that $\text{Re}\{\lambda_i\} = 0$ we have that the i^{th} row of $V^{-1}B$ is $\vec{0}_m^\top$, then the model is stable.

(ii) If there is a j such that $\text{Re}\{\lambda_j\} > 0$, or a j such that $\text{Re}\{\lambda_j\} = 0$ and the j^{th} row of $V^{-1}B$ is nonzero, then the model is unstable.

The proof of Theorem 10 is on the longer side and may distract from the overall flow of this note, so it is left to Appendix A.2. We fully expect you to read the proof and understand it. It is completely in-scope for the course.

The condition that the i^{th} row of $V^{-1}B$ is $\vec{0}_m$ essentially ensures that the input cannot perturb the i^{th} row of the diagonalizable system in order to send it to infinity. This is what forms the core of the proof.

We conclude this section with a warning; that these conditions only hold when A is diagonalizable. The conditions for general A are much more subtle and mathematically sophisticated, and are best left to later study. Later, we present examples of where the theorems break down.

2.4 Stability Sanity-Checking

Right now, it may not be clear why our conditions for discrete-time and continuous-time stability are different, or even why they are what they are. For intuition about this, it may help to review the scalar cases.

In the scalar case, the Discrete-Time LTI Difference Equation Model has state trajectory

$$x[i] = a^i x_0 + \sum_{k=0}^{i-1} a^{i-1-k} b u[k]. \tag{5}$$

Looking at only the first term, we can see whether its magnitude goes to ∞ ("blows up"). If the first term "blows up", we know that the system is unstable, because we can feed in $u[k] = 0$ for all k, and let the state "blow up". If the first term does not "blow up", then we would need to show that the second term does not "blow up" either; and that is exactly what we do in the proof.

For now, it is important to get an intuitive idea of what is happening with the first term, and in particular the behavior of z^i for a complex number z. More formally, suppose $z := re^{j\omega} \in \mathbb{C}$ is a complex number. Then

$$z^i = r^i e^{j\cdot\omega i} = r^i \cos(\omega i) + jr^i \sin(\omega i). \tag{6}$$

The idea is that r controls the rate of growth of $|z^t|$, and ω controls any oscillatory behavior.

- When $|z| < 1$, the envelope $r^i \to 0$, so z^i decays to 0, although if $\omega \neq 0$ it also has oscillatory behavior due to the sine/cosine.
- When $|z| = 1$, the envelope $r^i = 1$, so if $\omega \neq 0$ then $z^i = \cos(\omega i) + j\sin(\omega i)$ oscillates around the complex unit circle $\{z \in \mathbb{C} \colon |z| = 1\}$.
- When $|z| > 1$, the envelope $r^i \to \infty$, so z^i blows up to ∞, although if $\omega \neq 0$ it also has oscillatory behavior due to the sine/cosine.

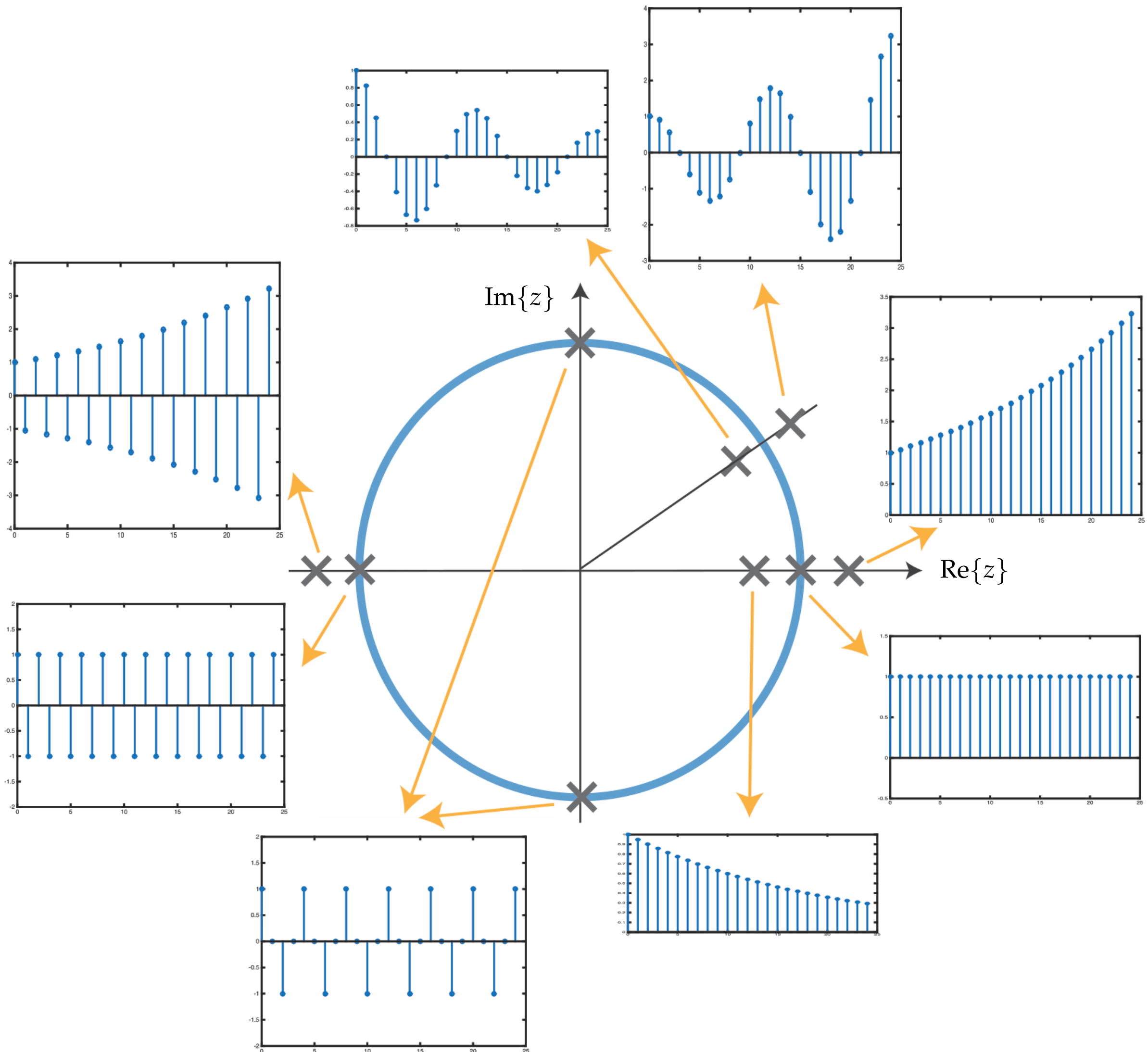

Figure 3: The real part of z^t for various values of z in the complex plane. It grows unbounded when $|z| > 1$, decays to zero when $|z| < 1$, and has constant amplitude when z is on the unit circle ($|z| = 1$).

So when $|z| > 1$ then the first term "blows up", and we can see that this is an indicator of instability.

We can also consider the continuous-time and do a similar analysis. In the scalar case, the Continuous-Time LTI Differential Equation Model has state trajectory

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)\,d\tau. \quad (7)$$

Looking again at only the first term, we can see whether it "blows up". If the first term "blows up", we know that the system is unstable, because we can feed in $u(\tau) = 0$ for all τ, and let the state "blow up". If the first term does not "blow up", then we would need to show that the second term does not "blow up" either; and that is exactly what we do in the proof.

For now, it is important to get an intuitive idea of what is happening with the first term, and in particular the behavior of e^{st} for a complex number s. More formally, suppose $s := \alpha + j\omega \in \mathbb{C}$ is a complex number.

Then

$$e^{st} = e^{(\alpha+j\omega)t} = e^{\alpha t}e^{j\omega t} = e^{\alpha t}\cos(\omega t) + je^{\alpha t}\sin(\omega t). \tag{8}$$

The idea is that α controls the rate of growth of $|e^{st}|$, and ω controls any oscillatory behavior.

- When $\text{Re}\{s\} < 0$, the envelope $e^{\alpha t} \to 0$, so e^{st} decays to 0, although if $\omega \neq 0$ then it also has oscillatory behavior due to the sine/cosine.
- When $\text{Re}\{s\} = 0$, the envelope $e^{\alpha t} = 1$, so if $\omega \neq 0$ then $e^{st} = \cos(\omega t) + j\sin(\omega t)$ oscillates around the complex unit circle $\{z \in \mathbb{C} : |z| = 1\}$.
- When $\text{Re}\{s\} > 0$, the envelope $e^{\alpha t} \to \infty$, so e^{st} blows up to ∞, although if $\omega \neq 0$ it also has oscillatory behavior due to the sine/cosine.

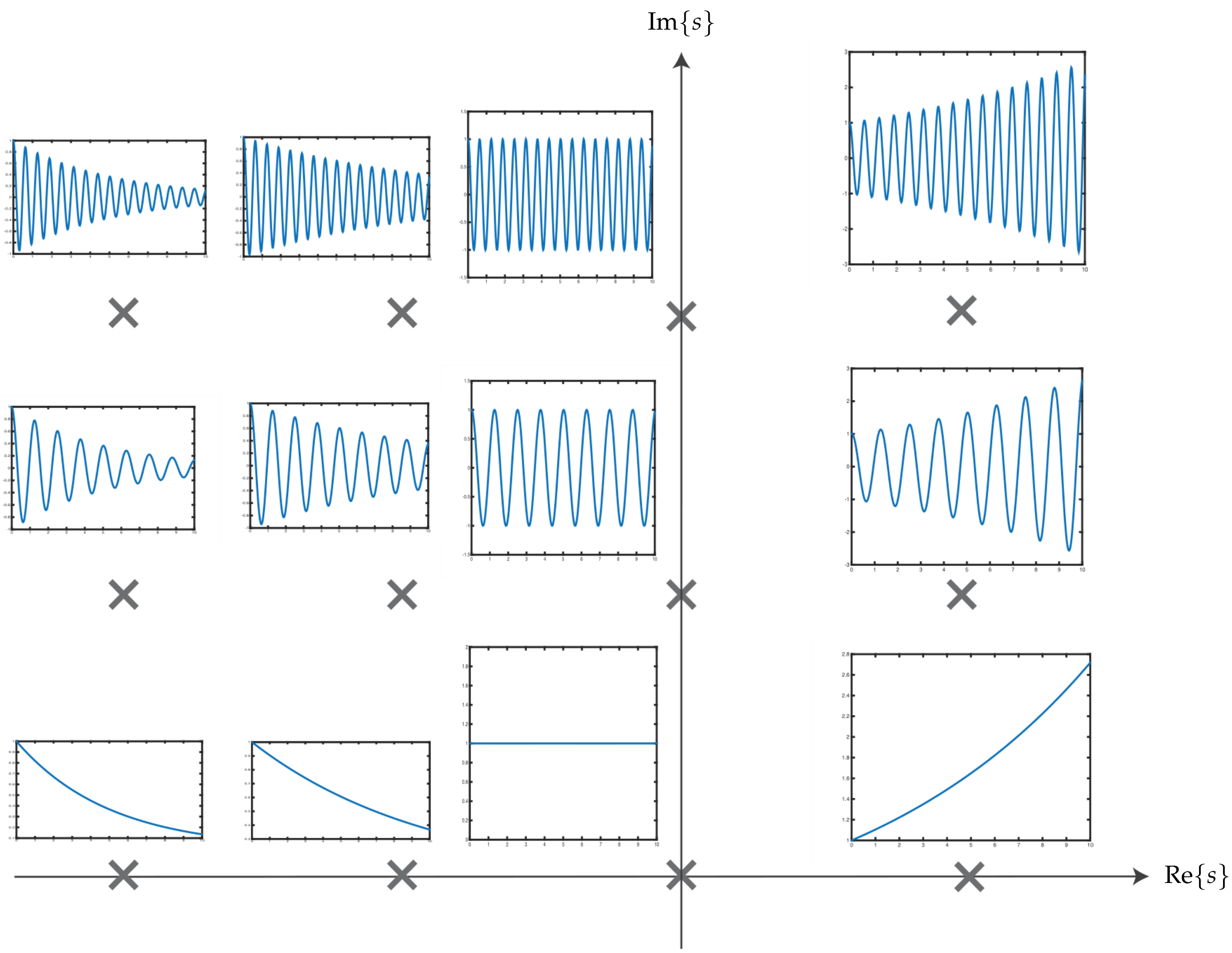

Figure 4: The real part of e^{st} for various values of s in the complex plane. Note that e^{st} is oscillatory when s has an imaginary component. It grows unboundedly when $\text{Re}\{s\} > 0$, decays to 0 when $\text{Re}\{s\} < 0$, and has constant amplitude when $\text{Re}\{s\} = 0$.

When $\text{Re}\{s\} > 0$ then the first term "blows up", and this is an indicator of instability.

2.5 Examples

The discrete-time model

$$x[i+1] = 3x[i] + 2u[i] \tag{9}$$

has the solution

$$x[i] = 3^i x[0] + 2\sum_{k=0}^{i-1} 3^{i-1-k} u[k] \tag{10}$$

and even if $u[k] = 0$ for all k, we still have $x[i] = 3^i x[0]$ which goes off to $\pm\infty$ as long as $x[0] \neq 0$. So this system is unstable.

If instead the model is

$$x[i+1] = \frac{1}{3} x[i] + 2u[i] \tag{11}$$

then the solution is

$$x[i] = \frac{x[0]}{3^i} + 2\sum_{k=0}^{i-1} \frac{u[k]}{3^{i-1-k}}. \tag{12}$$

It can be shown that no matter what $x[0]$ is, and with bounded u, that x is bounded, and therefore stable.

One can come up with continuous-time examples as well. For example, if we have the model

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = 3x(t) + 2u(t) \tag{13}$$

then the solution is

$$x(t) = \mathrm{e}^{3t} x(0) + 2\int_0^t \mathrm{e}^{3(t-\tau)} u(\tau)\,\mathrm{d}\tau\,. \tag{14}$$

Even if $u(\tau) = 0$ for all τ, we still have $x(t) = \mathrm{e}^{3t} x(0)$ which goes off to $\pm\infty$ as long as $x(0) \neq 0$. So this system is unstable.

We could instead consider the model

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = -3x(t) + 2u(t) \tag{15}$$

which has solution

$$x(t) = \mathrm{e}^{-3t} x(0) + 2\int_0^t \mathrm{e}^{-3(t-\tau)} u(\tau)\,\mathrm{d}\tau\,. \tag{16}$$

It can be shown that no matter what $x(0)$ is, and with bounded u, that x is bounded, and therefore stable.

Now consider the following example in discrete-time:

$$\vec{x}[i+1] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}[i] \qquad \text{with} \qquad \vec{x}_0 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{17}$$

One can show that A has one distinct eigenvalue – $\lambda_1 = 1$ – and that this eigenvalue has a one-dimensional eigenspace – $\mathrm{Null}(A - \lambda_1 I) = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. This matrix is not diagonalizable, and so we cannot use our marginal stability theorems.

This example is important because all eigenvalues of A have magnitude ≤ 1, but with this initial condition the state trajectory is unbounded. Indeed, one can show that

$$A^i \vec{x}_0 = \begin{bmatrix} i+1 \\ 1 \end{bmatrix} \tag{18}$$

whose norm goes to ∞ in the limit $i \to \infty$. Thus the model is not BIBO stable. So our marginal stability theorem would break if we tried using it.

A corresponding example in the continuous-time case is

$$A := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \vec{x}_0 := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{19}$$

from which one can show that $x_2(t)$ is a nonzero constant and $x_1(t)$ is linear, hence unbounded. Thus the model is not BIBO stable.

3 Feedback Control

We usually want a given LTI model to be stable. Sometimes, our system identification process, or nature itself, gives us an unstable model. In this case we use *state feedback control*.

Definition 11 (Feedback Control)
Feedback control is when the input at a given time is a function of the state at that time:

$$\vec{u}[i] = \vec{f}(\vec{x}[i]) \qquad \text{or} \qquad \vec{u}(t) = \vec{f}(\vec{x}(t)) \tag{20}$$

for some function $\vec{f} \colon \mathbb{R}^n \to \mathbb{R}^m$.

Feedback control is also called *closed loop* control, because the input is a function of the state, which itself is a linear function of the previous input, and so on.

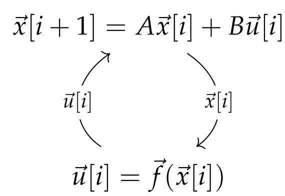

Figure 5: Closed-loop system in discrete-time.

This is in opposition to *open loop* control, which is when the inputs are not a function of the state.

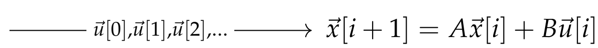

Figure 6: Open-loop system in discrete-time.

3.1 Discrete-Time Feedback Control

Theorem 12 (Discrete-Time Feedback Control)
Suppose in the Discrete-Time LTI Difference Equation Model we apply the closed-loop control:

$$\vec{u}[i] := \vec{u}_{\text{CL}}[i] := F\vec{x}[i] + \vec{u}_{\text{OL}}[i] \tag{21}$$

for some matrix $F \in \mathbb{R}^{m \times n}$ and some open-loop sequence of inputs $\vec{u}_{\text{OL}} : \mathbb{N} \to \mathbb{R}^m$. Then the model becomes

$$\vec{x}[i+1] = A_{\text{CL}}\vec{x}[i] + B\vec{u}_{\text{OL}}[i] \tag{22}$$

$$\vec{x}[0] = \vec{x}_0 \tag{23}$$

where $A_{\text{CL}} := A + BF$.

Proof.

$$\begin{aligned}
\vec{x}[i+1] &= A\vec{x}[i] + B\vec{u}[i] && (24)\\
&= A\vec{x}[i] + B\vec{u}_{\text{CL}}[i] && (25)\\
&= A\vec{x}[i] + B(F\vec{x}[i] + \vec{u}_{\text{OL}}[i]) && (26)\\
&= A\vec{x}[i] + BF\vec{x}[i] + B\vec{u}_{\text{OL}}[i] && (27)\\
&= (A + BF)\vec{x}[i] + B\vec{u}_{\text{OL}}[i] && (28)\\
&= A_{\text{CL}}\vec{x}[i] + B\vec{u}_{\text{OL}}[i]. && (29)
\end{aligned}$$

□

NOTE: We usually take $\vec{u}_{\text{OL}}[i] = 0$ for all i, so that $\vec{u}[i] = F\vec{x}[i]$ for all i. There are some cases, however, where we would like the open-loop input to be nonzero.

NOTE: By Theorem 6, *if all eigenvalues λ of A_{CL} have $|\lambda| < 1$, this system is stable.*

3.2 Continuous-Time Feedback Control

The analysis for the continuous-time feedback control is not much different from the discrete-time feedback control.

Theorem 13 (Continuous-Time Feedback Control)
Suppose in the Continuous-Time LTI Differential Equation Model we apply the closed-loop control:

$$\vec{u}(t) := \vec{u}_{\text{CL}}(t) := F\vec{x}(t) + \vec{u}_{\text{OL}}(t) \tag{30}$$

for some matrix $F \in \mathbb{R}^{m \times n}$ and some open-loop input function $\vec{u}_{\text{OL}} : \mathbb{R}_+ \to \mathbb{R}^m$. Then the model becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A_{\text{CL}}\vec{x}(t) + B\vec{u}_{\text{OL}}(t) \tag{31}$$

$$\vec{x}(0) = \vec{x}_0 \tag{32}$$

where $A_{\text{CL}} := A + BF$.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{33}$$
$$= A\vec{x}(t) + B\vec{u}_{\mathrm{CL}}(t) \tag{34}$$
$$= A\vec{x}(t) + B(F\vec{x}(t) + \vec{u}_{\mathrm{OL}}(t)) \tag{35}$$
$$= A\vec{x}(t) + BF\vec{x}(t) + B\vec{u}_{\mathrm{OL}}(t) \tag{36}$$
$$= (A + BF)\vec{x}(t) + B\vec{u}_{\mathrm{OL}}(t) \tag{37}$$
$$= A_{\mathrm{CL}}\vec{x}(t) + B\vec{u}_{\mathrm{OL}}(t). \tag{38}$$

□

NOTE: Again, we usually take $\vec{u}_{\mathrm{OL}}(t) = 0$ for all t.

NOTE: By Theorem 8, *if all eigenvalues λ of A_{CL} have $\mathrm{Re}\{\lambda\} < 0$, then the system is stable.*

3.3 Stabilizing a Model

Since in both cases the A matrix for the closed-loop system is $A_{\mathrm{CL}} = A + BF$, we are able to use the following procedure to see if a model is stabilizable by linear closed-loop control.

Algorithm 14 Stabilizing Models

Input: Original $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ matrices

Output: Whether or not closed-loop control can be used to make the model stable

1: Write symbolically $F := \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix}$

2: Compute eigenvalues of $A_{\mathrm{CL}} := A + BF$ in terms of $f_{11}, \ldots, f_{mn}$.

3: **if** there is a way to set $f_{11}, \ldots, f_{mn}$ so that system is stable **then**

4: **return** STABILIZABLE, and such a $(f_{11}, \ldots, f_{mn})$

5: **else**

6: **return** NOT STABILIZABLE

7: **end if**

An example of this algorithm will be included in the next subsection.

3.4 Example

We do a discrete-time feedback control example; the continuous-time case works similarly, with the caveat that the conditions for stability as described by Theorem 6 are different from those described by Theorem 8, so it is important to be careful in determining which one applies.

Imagine our original model is:

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u[i]. \tag{39}$$

If we apply the feedback

$$u[i] = F\vec{x}[i] \qquad \text{where} \qquad F := \begin{bmatrix} f_1 & f_2 \end{bmatrix} \tag{40}$$

then we get

$$A_{\text{CL}} := A + BF = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 + f_1 & 3 + f_2 \end{bmatrix}. \tag{41}$$

The eigenvalues of this are determined by the characteristic polynomial

$$\begin{aligned} p_{A_{\text{CL}}}(\lambda) &:= \det(A_{\text{CL}} - \lambda I_2) & (42) \\ &= \det\left(\begin{bmatrix} -\lambda & 1 \\ 2 + f_1 & 3 + f_2 - \lambda \end{bmatrix}\right) & (43) \\ &= (-\lambda)(3 + f_2 - \lambda) - 1 \cdot (2 + f_1) & (44) \\ &= \lambda^2 - (3 + f_2)\lambda - (2 + f_1). & (45) \end{aligned}$$

We can find the roots, and thus the eigenvalues using the following method. Imagine that $p_{A_{\text{CL}}}$ has two roots λ_1 and λ_2. Since the leading coefficient of $p_{A_{\text{CL}}}(\lambda)$ is 1, we know that it has the form

$$\begin{aligned} p_{A_{\text{CL}}}(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2) & (46) \\ &= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2. & (47) \end{aligned}$$

Thus, matching coefficients of λ, we have the system of equations

$$\begin{aligned} 3 + f_2 &= \lambda_1 + \lambda_2 & (48) \\ -2 - f_1 &= \lambda_1\lambda_2. & (49) \end{aligned}$$

Thus

$$\begin{aligned} f_1 &= -2 - \lambda_1\lambda_2 & (50) \\ f_2 &= \lambda_1 + \lambda_2 - 3. & (51) \end{aligned}$$

If we want to set particular eigenvalues λ_1 and λ_2 for A_{CL}, we use the above expression to determine the feedback control.

4 Final Comments

Overall, we have discussed the limiting behavior of the LTI models we usually work with. We have also discussed when and how we can change this behavior by applying the correct desired sequence of inputs.

This is the first time that we explicitly discuss the effect inputs may have on a control system. In this note, the analysis is somewhat qualitative and centers around limiting behavior. In future notes, we will be more quantitative about how to use control inputs to reach a particular state in a certain amount of steps (Note 12) and minimal energy consumption (Note 16).

A Proofs for Section 2

Our stability proofs in Section 2 assume everything is actually in the complex domain, i.e., all numbers in the matrices and vectors may be complex numbers. This is so that we don't have any issues with complex eigenvalues breaking our theorems when we construct vector solutions from scalar solutions.

We will need the following very useful result.

Theorem 15 (Triangle Inequality)

We have

$$\begin{aligned} |x+y| &\leq |x|+|y| && \text{for all } x,y \in \mathbb{R} && \text{(Absolute value.)} \\ |x+y| &\leq |x|+|y| && \text{for all } x,y \in \mathbb{C} && \text{(Complex magnitude.)} \\ \|\vec{x}+\vec{y}\| &\leq \|\vec{x}\|+\|\vec{y}\| && \text{for all } \vec{x},\vec{y} \in \mathbb{R}^n && \text{(Norm.)} \end{aligned} \tag{52}$$

Interpreting integrals as weighted sums, we also have

$$\begin{aligned} \left|\int_a^b f(x)\,\mathrm{d}x\right| &\leq \int_a^b |f(x)|\,\mathrm{d}x && \text{for } f\colon \mathbb{R} \to \mathbb{R} && \text{(Absolute value.)} \\ \left|\int_a^b f(x)\,\mathrm{d}x\right| &\leq \int_a^b |f(x)|\,\mathrm{d}x && \text{for } f\colon \mathbb{C} \to \mathbb{C} && \text{(Complex magnitude.)} \\ \left\|\int_a^b \vec{f}(x)\,\mathrm{d}x\right\| &\leq \int_a^b \left\|\vec{f}(x)\right\|\mathrm{d}x && \text{for } f\colon \mathbb{R} \to \mathbb{R}^n && \text{(Norm.)} \end{aligned} \tag{53}$$

A.1 Proofs for Section 2.2

A.1.1 Proofs for Section 2.2 in Scalar Case

We will first prove the conditions for asymptotic stability in the scalar case. More formally, suppose we have the models

$$x[i+1] = ax[i] + bu[i] \tag{54}$$

$$x[0] = x_0 \tag{55}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t) + bu(t) \tag{56}$$

$$x(0) = x_0 \tag{57}$$

where $a, b \in \mathbb{C}$. We assume that the coefficients are complex numbers here, because we want to use the scalar case as a sub-problem when dealing with the vector case (where we might get complex eigenvalues).

Proof of Theorem 6 in Scalar Case. Recall from Note 9 that the state trajectory is

$$x[i] = a^i x_0 + \sum_{k=0}^{i-1} a^{i-k-1} bu[k] \tag{58}$$

$$= a^i x_0 + \sum_{k=0}^{i-1} a^k bu[i-k-1]. \tag{59}$$

Here the changing of indices in the summation is nothing but changing the order of summation.

(i) Suppose $|a| < 1$. Let $x_0 \in \mathbb{C}$ be any complex number, and u be an input sequence bounded by R_u (so that $|u[i]| < R_u$ for all i). Then we use the triangle inequality and geometric sum formula to get

$$\begin{aligned}
|x[i]| &= \left|a^i x_0 + \sum_{k=0}^{i-1} a^k b u[i-k-1]\right| && (60)\\
&\leq \left|a^i x_0\right| + \sum_{k=0}^{i-1} \left|a^k b u[i-k-1]\right| && (61)\\
&= \underbrace{|a|^i}_{\leq 1} |x_0| + \sum_{k=0}^{i-1} |a|^k |b| \underbrace{|u[i-k-1]|}_{\leq R_u} && (62)\\
&\leq |x_0| + |b| R_u \sum_{k=0}^{i-1} \underbrace{|a|^k}_{\geq 0 \text{ for all } k} && (63)\\
&\leq |x_0| + |b| R_u \underbrace{\sum_{k=0}^{\infty} |a|^k}_{=\frac{1}{1-|a|}} && (64)\\
&= |x_0| + R_u \frac{|b|}{1-|a|}. && (65)
\end{aligned}$$

Thus if $R_x := |x_0| + R_u \frac{|b|}{1-|a|}$ then $|x[i]| \leq R_x$ for all i, so x is bounded and the model is BIBO stable.

(ii) Suppose $|a| > 1$. Let $x_0 \neq 0$. Further let $u[k] := 0$ for all k. Then we have

$$\begin{aligned}
|x[i]| &= \left|a^i x[0] + \sum_{k=0}^{i-1} a^k b \underbrace{u[i-k-1]}_{=0}\right| && (66)\\
&= \left|a^i x[0]\right| && (67)\\
&= |a|^i |x[0]| && (68)\\
\lim_{i\to\infty} |x[i]| &= |x[0]| \lim_{i\to\infty} |a|^i && (69)\\
&= \infty. && (70)
\end{aligned}$$

Thus we have a bounded input and initial condition which makes the state "blow up", and thus the model is BIBO unstable.

□

Proof of Theorem 8 in Scalar Case. Recall from Note 9 that the state trajectory is

$$x(t) = e^{at} x_0 + \int_0^t e^{a(t-\tau)} b u(\tau) \, d\tau. \tag{71}$$

(i) Suppose $\mathrm{Re}\{a\} < 0$. Let $x_0 \in \mathbb{C}$ be any complex number, and let u be an input function bounded by R_u (so that $|u(t)| < R_u$ for all t). Then we use the triangle inequality to get

$$\begin{aligned}
|x(t)| &= \left|e^{at} x_0 + \int_0^t e^{a(t-\tau)} b u(\tau) \, d\tau\right| && (72)\\
&\leq \left|e^{at} x_0\right| + \left|\int_0^t e^{a(t-\tau)} b u(\tau) \, d\tau\right| && (73)
\end{aligned}$$

$$\leq \left|e^{at}x_0\right| + \int_0^t \left|e^{a(t-\tau)}bu(\tau)\right| d\tau \tag{74}$$

$$= \underbrace{\left|e^{at}\right|}_{e^{\mathrm{Re}\{a\}t}} |x_0| + \int_0^t \underbrace{\left|e^{a(t-\tau)}\right|}_{=e^{\mathrm{Re}\{a\}(t-\tau)}} |b| \underbrace{|u(\tau)|}_{\leq R_u} d\tau \tag{75}$$

$$= e^{\mathrm{Re}\{a\}t}|x_0| + |b|R_u \int_0^t e^{\mathrm{Re}\{a\}(t-\tau)} d\tau \tag{76}$$

$$= \underbrace{e^{\mathrm{Re}\{a\}t}}_{\leq 1}|x_0| + |b|R_u \cdot \underbrace{\frac{e^{\mathrm{Re}\{a\}t} - 1}{\mathrm{Re}\{a\}}}_{\leq -\frac{1}{\mathrm{Re}\{a\}}} \tag{77}$$

$$\leq |x_0| - R_u \frac{|b|}{\mathrm{Re}\{a\}}. \tag{78}$$

Thus if $R_x := |x_0| + R_u \frac{|b|}{\mathrm{Re}\{a\}}$ then $|x(t)| \leq R_x$ for all t, so x is bounded and the model is BIBO stable.

(ii) Suppose $\mathrm{Re}\{a\} > 0$. Let $x_0 \neq 0$. Further let $u(\tau) := 0$ for all τ. Then we have

$$|x(t)| = \left| e^{at}x_0 + \int_0^t e^{a(t-\tau)} b \underbrace{u(\tau)}_{=0} d\tau \right| \tag{79}$$

$$= \left|e^{at}x_0\right| \tag{80}$$

$$= e^{\mathrm{Re}\{a\}t}|x_0| \tag{81}$$

$$\lim_{t\to\infty} |x(t)| = |x_0| \lim_{t\to\infty} e^{\mathrm{Re}\{a\}t} \tag{82}$$

$$= \infty. \tag{83}$$

Thus we have a bounded input and initial condition which makes the state "blow up", and thus the model is BIBO unstable.

□

Key Idea 16

Notice the difference between the two styles of proof:

- To show a model is *stable*, we have to show, for *every* initial condition and choice of bounded input, that the state trajectory is bounded.
- To show a model is *unstable*, we have to show, for *one* initial condition and choice of bounded input, that the state trajectory goes to ∞.

This is a result of how we define stability; it is an important distinction to keep in mind.

A.1.2 Proofs for Section 2.2 in Diagonalizable Case

We will now prove the conditions for asymptotic stability in the diagonalizable case. More formally, suppose we have the models

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] \tag{84}$$

$$\vec{x}[0] = \vec{x}_0 \tag{85}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{86}$$
$$\vec{x}(0) = \vec{x}_0 \tag{87}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. Now suppose A is diagonalizable, i.e., $A = V\Lambda V^{-1}$ where V is the matrix of eigenvectors of A and Λ is the diagonal matrix of corresponding eigenvalues.

In our work, we will need the following lemma.

Lemma 17 (Boundedness of Vector Sequences and Functions)
Let $\vec{u}\colon \mathbb{N} \to \mathbb{R}^m$ be a sequence of vectors.

(i) $\vec{u}$ is bounded if and only if $u_k\colon \mathbb{N} \to \mathbb{R}$ is bounded for all $k \in \{1, \ldots, m\}$.

(ii) $\vec{u}$ is bounded if and only if for any matrix $C \in \mathbb{R}^{n\times m}$ we have that $C\vec{u}\colon \mathbb{N} \to \mathbb{R}^n$ is bounded.

Let $\vec{u}\colon \mathbb{R}_+ \to \mathbb{R}^m$ be a vector-valued function.

(i) $\vec{u}$ is bounded if and only if $u_k\colon \mathbb{R}_+ \to \mathbb{R}$ is bounded for all $k \in \{1, \ldots, m\}$.

(ii) $\vec{u}$ is bounded if and only if for any matrix $C \in \mathbb{R}^{n\times m}$ we have that $C\vec{u}\colon \mathbb{R}_+ \to \mathbb{R}^n$ is bounded.

Concept Check: Use Theorem 15 to prove Lemma 17.

Proof of Theorem 6 in Diagonalizable Case. We define the $\vec{z}$ coordinates by $\vec{x}[i] = V\vec{z}[i]$. Then we have

$$\vec{z}[i+1] = V^{-1}\vec{x}[i+1] \tag{88}$$
$$= V^{-1}(A\vec{x}[i] + B\vec{u}[i]) \tag{89}$$
$$= V^{-1}A\vec{x}[i] + V^{-1}B\vec{u}[i] \tag{90}$$
$$= V^{-1}AV\vec{z}[i] + V^{-1}B\vec{u}[i] \tag{91}$$
$$= \Lambda\vec{z}[i] + V^{-1}B\vec{u}[i]. \tag{92}$$

The initial condition is

$$\vec{z}[0] = V^{-1}\vec{x}[0] = V^{-1}\vec{x}_0. \tag{93}$$

Each row of this vector model is the different scalar model

$$z_k[i+1] = \lambda_k z_k[i] + (V^{-1}B\vec{u}[i])_k \qquad k \in \{1, \ldots, n\} \tag{94}$$
$$z_k[0] = (V^{-1}\vec{x}_0)_k. \tag{95}$$

(i) Suppose $|\lambda_i| < 1$ for all i. By the scalar case, the model for z_k is BIBO stable.

Let $\vec{x}_0$ be any initial condition. Suppose $\vec{u}$ is bounded. Then by Lemma 17, $V^{-1}B\vec{u}$ is bounded, and again by the same lemma, $(V^{-1}B\vec{u})_k$ is bounded for every k. Thus by the scalar case, z_k is bounded for every k. By Lemma 17, $\vec{z}$ is bounded, and again by the same lemma, $\vec{x} = V\vec{z}$ is bounded. Thus the model for $\vec{x}$ is BIBO stable.

(ii) Suppose there is a λ_j with $|\lambda_j| > 1$. Let $\vec{x}_0 = \vec{v}_j$ be any corresponding normalized eigenvector corresponding to λ_j; then

$$\vec{z}[0] = V^{-1}\vec{x}_0 = V^{-1}\vec{v}_j = \vec{e}_j. \tag{96}$$

Further let $\vec{u}[k] = \vec{0}_m$ for all k. Then from Note 9, the state trajectory is given by

$$z_j[i] = \lambda_j^i z_j[0] + \sum_{k=0}^{i-1} \lambda_j^k (V^{-1}B\vec{u}[i-1-k])_j. \tag{97}$$

Then

$$\begin{aligned}
|z_j[i]| &= \left|\lambda_j^i z_j[0] + \sum_{k=0}^{i-1} \lambda_j^k (V^{-1}B\underbrace{\vec{u}[i-1-k]}_{=\vec{0}_m})_j\right| && (98)\\
&= \left|\lambda_j^i z_j[0]\right| && (99)\\
&= \left|\lambda_j^i (\vec{e}_j)_j\right| && (100)\\
&= \left|\lambda_j^i\right| && (101)\\
&= |\lambda_j|^i && (102)\\
\lim_{i\to\infty} |\vec{z}_j[i]| &= \lim_{i\to\infty} |\lambda_j|^i && (103)\\
&= \infty. && (104)
\end{aligned}$$

Thus we have a bounded input and initial condition which makes z_j"blow up", and by Lemma 17 $\vec{z}$ also "blows up". Again by Lemma 17, $\vec{x} = V\vec{z}$ also "blows up", and thus the model is BIBO unstable.

□

Proof of Theorem 8 in Diagonalizable Case. We define the $\vec{z}$ coordinates by $\vec{x}(t) = V\vec{z}(t)$. Then we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(V^{-1}\vec{x}(t)\right) && (105)\\
&= V^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t)\right) && (106)\\
&= V^{-1}(A\vec{x}(t) + B\vec{u}(t)) && (107)\\
&= V^{-1}A\vec{x}(t) + V^{-1}B\vec{u}(t) && (108)\\
&= V^{-1}AV\vec{z}(t) + V^{-1}B\vec{u}(t) && (109)\\
&= \Lambda\vec{z}(t) + V^{-1}B\vec{u}(t). && (110)
\end{aligned}$$

The initial condition is

$$\vec{z}(0) = V^{-1}\vec{x}(0) = V^{-1}\vec{x}_0. \tag{111}$$

Each row of this vector model is the different scalar model

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}z_k(t) &= \lambda_k z_k(t) + (V^{-1}B\vec{u}(t))_k \qquad k \in \{1, \ldots, n\} && (112)\\
z_k(0) &= (V^{-1}\vec{x}_0)_k. && (113)
\end{aligned}$$

(i) Suppose $\text{Re}\{\lambda_i\} < 0$ for all i. By the scalar case, the model for z_k is BIBO stable.

Let $\vec{x}_0$ be any initial condition. Suppose $\vec{u}$ is bounded. Then by Lemma 17, $V^{-1}B\vec{u}$ is bounded, and again by the same lemma, $(V^{-1}B\vec{u})_k$ is bounded for every k. Thus by the scalar case, z_k is bounded for every k. By Lemma 17, $\vec{z}$ is bounded, and again by the same lemma, $\vec{x} = V\vec{z}$ is bounded. Thus the model for $\vec{x}$ is BIBO stable.

(ii) Suppose there is a λ_j with $\text{Re}\{\lambda_j\} > 0$. Let $\vec{x}_0 = \vec{v}_j$ be any corresponding normalized eigenvector corresponding to λ_j; then

$$\vec{z}(0) = V^{-1}\vec{x}_0 = V^{-1}\vec{v}_j = \vec{e}_j. \tag{114}$$

Further let $\vec{u}(\tau) = \vec{0}_m$ for all τ. Then from Note 9, the state trajectory is given by

$$z_j(t) = \mathrm{e}^{\lambda_j t} z_j(0) + \int_0^t (V^{-1}B\vec{u}(\tau))_k \,\mathrm{d}\tau\,. \tag{115}$$

Then

$$\begin{aligned}
\left|z_j(t)\right| &= \left|\mathrm{e}^{\lambda_j t} z_j(0) + \int_0^t (V^{-1}B\vec{u}(\tau))_k \,\mathrm{d}\tau\right| && (116)\\
&\leq \left|\mathrm{e}^{\lambda_j t} z_j(0)\right| + \int_0^t \left|(V^{-1}B\underbrace{\vec{u}(\tau)}_{=\vec{0}_m})_k\right| \mathrm{d}\tau && (117)\\
&= \left|\mathrm{e}^{\lambda_j t} z_j(0)\right| && (118)\\
&= \mathrm{e}^{\text{Re}\{\lambda_j\}t}\left|z_j(0)\right| && (119)\\
&= \mathrm{e}^{\text{Re}\{\lambda_j\}t}\left|(\vec{e}_j)_j\right| && (120)\\
&= \mathrm{e}^{\text{Re}\{\lambda_j\}t} && (121)\\
\lim_{t\to\infty}\left|z_j(t)\right| &= \lim_{t\to\infty} \mathrm{e}^{\text{Re}\{\lambda_j\}t} && (122)\\
&= \infty. && (123)
\end{aligned}$$

Thus we have a bounded input and initial condition which makes z_j"blow up", and by Lemma 17 $\vec{z}$ also "blows up". Again by Lemma 17, $\vec{x} = V\vec{z}$ also "blows up", and thus the model is BIBO unstable.

□

A.1.3 Complete Proofs for Section 2.2

Even when the A matrix is not diagonalizable, Section 2.2 is true. To show this, we invoke a tool that we develop in Note 15, namely *upper triangularization* or *Schur decomposition*.

This decomposition, applicable to every square matrix $A \in \mathbb{C}^{n\times n}$, gives the existence of a matrix $U \in \mathbb{C}^{n\times n}$ and an *upper-triangular* matrix $T \in \mathbb{C}^{n\times n}$ (i.e., $t_{ij} = 0$ for $i > j$) which has the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A on the diagonal, such that

$$A = UTU^{-1} = U \begin{bmatrix} \lambda_1 & t_{12} & \cdots & t_{1n} \\ 0 & \lambda_2 & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} U^{-1}. \tag{124}$$

This special structure allows us to prove Theorem 6 and Theorem 8 in full generality.

Proof of Theorem 6. We define the $\vec{z}$ coordinates by $\vec{x}[i] = U\vec{z}[i]$. Then we have

$$\begin{aligned}\vec{z}[i+1] &= U^{-1}\vec{x}[i+1] && (125)\\ &= U^{-1}(A\vec{x}[i] + B\vec{u}[i]) && (126)\\ &= U^{-1}A\vec{x}[i] + U^{-1}B\vec{u}[i] && (127)\\ &= U^{-1}AU\vec{z}[i] + U^{-1}B\vec{u}[i] && (128)\\ &= T\vec{z}[i] + U^{-1}B\vec{u}[i]. && (129)\end{aligned}$$

The initial condition is

$$\vec{z}[0] = U^{-1}\vec{x}[0] = U^{-1}\vec{x}_0. \tag{130}$$

Each row of this vector model is the different scalar model

$$\begin{aligned} z_k[i+1] &= \lambda_k z_k[i] + (U^{-1}B\vec{u}[i])_k + \sum_{j=k+1}^{n} t_{kj} z_j[i] \qquad k \in \{1, \dots, n\} && (131)\\ z_k[0] &= (U^{-1}\vec{x}_0)_k. && (132)\end{aligned}$$

(i) Suppose $|\lambda_i| < 1$ for all i. Let $\vec{x}_0$ be any initial condition and $\vec{u}$ be a bounded input. We show that each z_k is bounded, by recursion.

By Lemma 17, we know that $U^{-1}B\vec{u}$ is bounded, and by the same lemma that $(U^{-1}B\vec{u})_k$ is bounded for all k.

The recursive base case is $k = n$, where the model is

$$\begin{aligned} z_n[i+1] &= \lambda_n z_n[i] + (U^{-1}B\vec{u}[i])_n && (133)\\ z_k[0] &= (U^{-1}\vec{x}_0)_n. && (134)\end{aligned}$$

Define

$$\widetilde{u}_n[i] := (U^{-1}B\vec{u}[i])_n. \tag{135}$$

We know that $\widetilde{u}_n$ is bounded. Our model becomes

$$\begin{aligned} z_n[i+1] &= \lambda_n z_n[i] + \widetilde{u}_n[i] && (136)\\ z_k[0] &= (U^{-1}\vec{x}_0)_n. && (137)\end{aligned}$$

By appealing to the scalar case, z_n is bounded.

Now in the general recursive case, we have a model

$$\begin{aligned} z_k[i+1] &= \lambda_k z_k[i] + (U^{-1}B\vec{u}[i])_k + \sum_{j=k+1}^{n} t_{kj} z_j[i] \qquad k \in \{1, \dots, n\} && (138)\\ z_k[0] &= (U^{-1}\vec{x}_0)_k. && (139)\end{aligned}$$

From recursion we know that $z_{k+1}, \dots, z_n$ are bounded. Thus if we write

$$\widetilde{u}_k[i] := (U^{-1}B\vec{u}[i])_k + \sum_{j=k+1}^{n} t_{kj} z_j[i] \tag{140}$$

then we know that $\widetilde{u}_k$ is bounded. And our model becomes

$$z_k[i+1] = \lambda_k z_k[i] + \widetilde{u}_k[i] \qquad k \in \{1, \dots, n\} \tag{141}$$

$$z_k[0] = (U^{-1}\vec{x}_0)_k. \tag{142}$$

Thus by appealing to the scalar case, z_k is bounded.

At the end of this recursion, we know that z_k is bounded for all k, so by Lemma 17 we know that $\vec{z}$ is bounded. Thus by the same lemma, $\vec{x} = U\vec{z}$ is bounded.

(ii) Suppose there is a λ_j with $|\lambda_j| > 1$. Let $\vec{x}_0 = \vec{v}_j$ be any corresponding normalized eigenvector corresponding to λ_j. Further let $\vec{u}[k] = \vec{0}_m$ for all k. Then the update rule becomes

$$\vec{x}[i+1] = A\vec{x}[i] = A^i\vec{x}[0] = A^i\vec{x}_0 = A^i\vec{v}_j = \lambda_j^i\vec{v}_j. \tag{143}$$

Then

$$\begin{aligned}
\|\vec{x}[i]\| &= \left\|\lambda_j^i\vec{v}_j\right\| && (144)\\
&= |\lambda_j|^i\|\vec{x}_0\| && (145)\\
&= |\lambda_j|^i && (146)\\
\lim_{i\to\infty}\|\vec{x}[i]\| &= \lim_{i\to\infty}|\lambda_j|^i && (147)\\
&= \infty. && (148)
\end{aligned}$$

Thus we have a bounded input and initial condition which makes $\vec{x}$ "blow up", and thus the model is BIBO unstable.

NOTE: There is nothing stopping us from using this same method to generate a stability counterexample in the scalar or diagonalizable cases, but it is more idiomatic to develop counterexamples based on the specific problem structure, so we present both ways. (The corresponding counterexample which uses upper triangularization structure is very messy to prove, so we omit it and give this general argument instead.)

□

Proof of Theorem 8. We define the $\vec{z}$ coordinates by $\vec{x}(t) = U\vec{z}(t)$. Then we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(U^{-1}\vec{x}(t)\right) && (149)\\
&= U^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t)\right) && (150)\\
&= U^{-1}(A\vec{x}(t) + B\vec{u}(t)) && (151)\\
&= U^{-1}A\vec{x}(t) + U^{-1}B\vec{u}(t) && (152)\\
&= U^{-1}AU\vec{z}(t) + U^{-1}B\vec{u}(t) && (153)\\
&= T\vec{z}(t) + U^{-1}B\vec{u}(t). && (154)
\end{aligned}$$

The initial condition is

$$\vec{z}(0) = U^{-1}\vec{x}(0) = U^{-1}\vec{x}_0. \tag{155}$$

Each row of this vector model is the different scalar model

$$\frac{\mathrm{d}}{\mathrm{d}t}z_k(t) = \lambda_k z_k(t) + (U^{-1}B\vec{u}(t))_k + \sum_{j=k+1}^{n} t_{kj}z_j(t) \qquad k \in \{1,\ldots,n\} \tag{156}$$

$$z_k(0) = (U^{-1}\vec{x}_0)_k. \tag{157}$$

(i) Suppose $\text{Re}\{\lambda_i\} < 0$ for all i. Let $\vec{x}_0$ be any initial condition and $\vec{u}$ be a bounded input. We show that each z_k is bounded, by recursion.

By Lemma 17, we know that $U^{-1}B\vec{u}$ is bounded, and by the same lemma that $(U^{-1}B\vec{u})_k$ is bounded for all k.

The recursive base case is $k = n$, where the model is

$$\frac{\mathrm{d}}{\mathrm{d}t}z_n(t) = \lambda_n z_n(t) + (U^{-1}B\vec{u}(t))_n \tag{158}$$

$$z_k(0) = (U^{-1}\vec{x}_0)_n. \tag{159}$$

Define

$$\widetilde{u}_n(t) := (U^{-1}B\vec{u}(t))_n. \tag{160}$$

We know that $\widetilde{u}_n$ is bounded. Our model becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}z_n(t) = \lambda_n z_n(t) + \widetilde{u}_n(t) \tag{161}$$

$$z_k(0) = (U^{-1}\vec{x}_0)_n. \tag{162}$$

By appealing to the scalar case, z_n is bounded.

Now in the general recursive case, we have a model

$$\frac{\mathrm{d}}{\mathrm{d}t}z_k(t) = \lambda_k z_k(t) + (U^{-1}B\vec{u}(t))_k + \sum_{j=k+1}^{n} t_{kj}z_j(t) \qquad k \in \{1, \ldots, n\} \tag{163}$$

$$z_k(0) = (U^{-1}\vec{x}_0)_k. \tag{164}$$

From recursion we know that $z_{k+1}, \ldots, z_n$ are bounded. Thus if we write

$$\widetilde{u}_k(t) := (U^{-1}B\vec{u}(t))_k + \sum_{j=k+1}^{n} t_{kj}z_j(t) \tag{165}$$

then we know that $\widetilde{u}_k$ is bounded. And our model becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}z_k(t) = \lambda_k z_k(t) + \widetilde{u}_k(t) \qquad k \in \{1, \ldots, n\} \tag{166}$$

$$z_k(0) = (U^{-1}\vec{x}_0)_k. \tag{167}$$

Thus by appealing to the scalar case, z_k is bounded.

At the end of this recursion, we know that z_k is bounded for all k, so by Lemma 17 we know that $\vec{z}$ is bounded. Thus by the same lemma, $\vec{x} = U\vec{z}$ is bounded.

(ii) Suppose there is a λ_j with $|\lambda_j| > 1$. Let $\vec{x}_0 = \vec{v}_j$ be any corresponding normalized eigenvector corresponding to λ_j. Further let $\vec{u}[k] = \vec{0}_m$ for all k. Then the update rule becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t). \tag{168}$$

Moreover, if $\vec{x}_0 = \vec{v}_j$ then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(0) = A\vec{x}(0) = A\vec{x}_0 = A\vec{v}_j = \lambda_j\vec{v}_j = \lambda_j\vec{x}_0 = \lambda_j\vec{x}(0). \tag{169}$$

Since $\vec{x}(0) = \vec{v}_j$ and $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) \propto \vec{x}(t)$, it follows that $\vec{x}(t) \propto \vec{v}_j$ for all t. In particular, we have

$$\vec{x}(t) = \mathrm{e}^{\lambda_j t}\vec{x}_0 = \mathrm{e}^{\lambda_j t}\vec{v}_j. \tag{170}$$

Then

$$\begin{aligned}
\|\vec{x}(t)\| &= \left\|\mathrm{e}^{\lambda_j t}\vec{v}_j\right\| && (171)\\
&= \left|\mathrm{e}^{\lambda_j t}\right|\|\vec{v}_j\| && (172)\\
&= \mathrm{e}^{\mathrm{Re}\{\lambda_j\}t}\underbrace{\|\vec{v}_j\|}_{=1} && (173)\\
&= \mathrm{e}^{\mathrm{Re}\{\lambda_j\}t} && (174)\\
\lim_{t\to\infty}\|\vec{x}(t)\| &= \lim_{t\to\infty}\mathrm{e}^{\mathrm{Re}\{\lambda_j\}t} && (175)\\
&= \infty. && (176)
\end{aligned}$$

Thus we have a bounded input and initial condition which makes $\vec{x}$ "blow up", and thus the model is BIBO unstable.

NOTE: There is nothing stopping us from using this same method to generate a stability counterexample in the scalar or diagonalizable cases, but it is more idiomatic to develop counterexamples based on the specific problem structure, so we present both ways. (The corresponding counterexample which uses upper triangularization structure is very messy to prove, so we omit it and give this general argument instead.)

□

A.2 Proofs for Section 2.3

Proof of Theorem 9. We use the same proof as Theorem 6 in the diagonalizable case. When in the $\vec{z}$ coordinates, we have

$$\begin{aligned}
z_k[i+1] &= \lambda_k z_k[i] + (V^{-1}B\vec{u}[i])_k && (177)\\
&= \lambda_k z_k[i] + (V^{-1}B)_{:,k}\vec{u}[i]. && (178)
\end{aligned}$$

for each k, and in particular for each k such that $|\lambda_k| = 1$.

(i) If for all k such that $|\lambda_k| = 1$, we have $(V^{-1}B)_{:,k} = \vec{0}_m^\top$, then for these k the model becomes

$$z_k[i+1] = \lambda_k z_k[i] \tag{179}$$

which implies that

$$z_k[i] = \lambda_k^i z_k[0] = \lambda_k^i (V^{-1}\vec{x}_0)_k. \tag{180}$$

Then

$$|z_k[i]| = \left|\lambda_k^i (V^{-1}\vec{x}_0)_k\right| = \underbrace{|\lambda_k|^i}_{=1}\left|(V^{-1}\vec{x}_0)_k\right| = \left|(V^{-1}\vec{x}_0)_k\right| \tag{181}$$

which is bounded by the constant $R_x := \left|(V^{-1}x_0)_k\right|$ for all timesteps i. Applying this to each k such that $|\lambda_k| = 1$, and using the scalar asymptotic stability case for all k such that $|\lambda_k| < 1$, yields that $\vec{z}$ is bounded given a bounded input $\vec{u}$. Thus $\vec{x}$ is bounded given a bounded input $\vec{u}$, and the model is stable.

(ii) If j is such that $|\lambda_j| > 1$ then by the scalar asymptotic stability case, we have that z_j is unbounded for a bounded input $\vec{u}$. Thus $\vec{z}$ is unbounded and so $\vec{x}$ is unbounded, given a bounded input $\vec{u}$. Thus the model is not BIBO stable.

Now suppose that all $|\lambda_k| \leq 1$, but there exists j such that the j^{th} row of $V^{-1}B$ is nonzero. For this j, the model becomes

$$\begin{aligned} z_j[i+1] &= \lambda_j z_j[i] + (V^{-1}B\vec{u}[i])_j && (182) \\ &= \lambda_j z_j[i] + (V^{-1}B)_{:,j}\vec{u}[i]. && (183) \end{aligned}$$

The state trajectory is thus

$$z_j[i] = \lambda_j^i (V^{-1}\vec{x}_0)_j + \sum_{k=0}^{i-1} \lambda_j^{i-1-k}(V^{-1}B)_{:,j}\vec{u}[k]. \tag{184}$$

Picking

$$\vec{x}_0 := \vec{0}_n \qquad \vec{u}[k] := \lambda_j^k (V^{-1}B)_{:,j}^\top \tag{185}$$

ensures that $\vec{u}$ is bounded (by $R_u := \left\|(V^{-1}B)_{:,j}^\top\right\|$) and also that

$$\begin{aligned} z_j[i] &= \lambda_j^i (V^{-1}\vec{x}_0)_j + \sum_{k=0}^{i-1} \lambda_j^{i-1-k}(V^{-1}B)_{:,j}\vec{u}[i] && (186) \\ &= \lambda_j^i (V^{-1}\underbrace{\vec{x}_0}_{=\vec{0}_n})_j + \sum_{k=0}^{i-1} \lambda_j^{i-1-k}(V^{-1}B)_{:,j}\underbrace{\vec{u}[i]}_{=\lambda_j^k (V^{-1}B)_{:,j}^\top} && (187) \\ &= \sum_{k=0}^{i-1} \lambda_j^{i-1}\left\|(V^{-1}B)_{:,j}^\top\right\|^2 && (188) \\ &= i \cdot \lambda_j^{i-1} \cdot \left\|(V^{-1}B)_{:,j}^\top\right\|^2 && (189) \\ |z_j[i]| &= \left| i \cdot \lambda_j^{i-1} \cdot \left\|(V^{-1}B)_{:,j}^\top\right\|^2 \right| && (190) \\ &= i \underbrace{|\lambda_j|^{i-1}}_{=1} \left\|(V^{-1}B)_{:,j}^\top\right\|^2 && (191) \\ &= i\left\|(V^{-1}B)_{:,j}^\top\right\|^2 && (192) \\ \lim_{i\to\infty} |z_j[i]| &= \lim_{i\to\infty} i\left\|(V^{-1}B)_{:,j}^\top\right\|^2 && (193) \\ &= \infty. && (194) \end{aligned}$$

Thus, given a bounded input $\vec{u}$, we know that z_j is unbounded. Then $\vec{z}$ is unbounded and so $\vec{x}$ is unbounded. Thus the model is not BIBO stable.

□

Proof of Theorem 10. We use the same proof as Theorem 8 in the diagonalizable case. When in the $\vec{z}$ coordinates, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} z_k(t) = \lambda_k z_k(t) + (V^{-1}B\vec{u}(t))_k \tag{195}$$

$$= \lambda_k z_k(t) + (V^{-1}B)_{:,k}\vec{u}(t). \tag{196}$$

for each k, and in particular for each k such that $\text{Re}\{\lambda_k\} = 0$.

(i) If for all k such that $\text{Re}\{\lambda_k\} = 0$, we have $(V^{-1}B)_{:,k} = \vec{0}_m^\top$, then for these k the model becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}z_k(t) = \lambda_k z_k(t) \tag{197}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}z_k(t) = \mathrm{e}^{\lambda_k t}z_k(0) = \mathrm{e}^{\lambda_k t}(V^{-1}\vec{x}_0)_k. \tag{198}$$

Then

$$|z_k(t)| = \left|\mathrm{e}^{\lambda_k t}(V^{-1}\vec{x}_0)_k\right| = \left|\mathrm{e}^{\lambda_k t}\right|\left|(V^{-1}\vec{x}_0)_k\right| = \underbrace{\mathrm{e}^{\text{Re}\{\lambda_k\}t}}_{=1}\left|(V^{-1}\vec{x}_0)_k\right| = \left|(V^{-1}\vec{x}_0)_k\right|. \tag{199}$$

which is bounded by the constant $R_x := \left|(V^{-1}x_0)_k\right|$ for all times t. Applying this to each k such that $\text{Re}\{\lambda_k\} = 0$, and using the scalar asymptotic stability case for all k such that $\text{Re}\{\lambda_k\} < 0$, yields that $\vec{z}$ is bounded given a bounded input $\vec{u}$. Thus $\vec{x}$ is bounded given a bounded input $\vec{u}$, and the model is stable.

(ii) If j is such that $\text{Re}\{\lambda_j\} > 0$ then by the scalar asymptotic stability case, we have that z_j is unbounded for a bounded input $\vec{u}$. Thus $\vec{z}$ is unbounded and so $\vec{x}$ is unbounded, given a bounded input $\vec{u}$. Thus the model is not BIBO stable.

Now suppose that all $\text{Re}\{\lambda_k\} \leq 0$, but there exists j such that the j^{th} row of $V^{-1}B$ is nonzero. For this j, the model becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}z_j(t) = \lambda_j z_j(t) + (V^{-1}B\vec{u}(t))_j \tag{200}$$

$$= \lambda_j z_j(t) + (V^{-1}B)_{:,j}\vec{u}(t). \tag{201}$$

The state trajectory is thus

$$z_j(t) = \mathrm{e}^{\lambda_j t}(V^{-1}\vec{x}_0)_j + \int_0^t \mathrm{e}^{\lambda_j(t-\tau)}(V^{-1}B)_{:,j}\vec{u}(\tau)\,\mathrm{d}\tau\,. \tag{202}$$

Picking

$$\vec{x}_0 := \vec{0}_n \qquad \vec{u}(\tau) := \mathrm{e}^{\lambda_j\tau}(V^{-1}B)_{:,j}^\top \tag{203}$$

ensures that $\vec{u}$ is bounded (by $R_u := \left\|(V^{-1}B)_{:,j}^\top\right\|$) and also that

$$z_j(t) = \mathrm{e}^{\lambda_j t}(V^{-1}\underbrace{\vec{x}_0}_{=\vec{0}_n})_j + \int_0^t \mathrm{e}^{\lambda_j(t-\tau)}(V^{-1}B)_{:,j}\underbrace{\vec{u}(\tau)}_{=\mathrm{e}^{\lambda_j\tau}(V^{-1}B)_{:,j}^\top}\mathrm{d}\tau \tag{204}$$

$$= \int_0^t \mathrm{e}^{\lambda_j t}\left\|(V^{-1}B)_{:,j}^\top\right\|^2\mathrm{d}\tau \tag{205}$$

$$= t\mathrm{e}^{\lambda_j t}\left\|(V^{-1}B)_{:,j}^\top\right\|^2 \tag{206}$$

$$|z_j(t)| = \left|t\mathrm{e}^{\lambda_j t}\left\|(V^{-1}B)_{:,j}^\top\right\|^2\right| \tag{207}$$

$$= t\underbrace{\left|\mathrm{e}^{\lambda_j t}\right|}_{=\mathrm{e}^{\text{Re}\{\lambda_j t\}}=1}\left\|(V^{-1}B)_{:,j}^\top\right\|^2 \tag{208}$$

$$= t\left\|(V^{-1}B)_{:,j}^{\top}\right\|^2 \tag{209}$$

$$\lim_{t\to\infty} \left|z_j(t)\right| = \lim_{t\to\infty} t\left\|(V^{-1}B)_{:,j}^{\top}\right\|^2 \tag{210}$$

$$= \infty. \tag{211}$$

Thus, given a bounded input $\vec{u}$, we know that z_j is unbounded. Then $\vec{z}$ is unbounded and so $\vec{x}$ is unbounded. Thus the model is not BIBO stable.

□

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