EECS 16B   Designing Information Devices and Systems II
Fall 2021   Midterm Exam Solutions   Midterm 1

Section 0: Pre-exam questions (4 points)

1. Honor Code: Please copy the following statement in the space provided below and sign your name.

As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

2. What's your favorite thing about this semester? (2 pts)

3. Who is a movie, TV, or fiction character that inspires you? (2 pts)

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4. Complex Numbers (2 points)

(a) (1 pt) Express the complex number j in polar form $j = Ae^{j\theta}$.

What is A? What is θ? Your θ should be between $-\pi$rad and $+\pi$rad.

Solution: $j = 0 + j \cdot 1 = \sqrt{0^2 + 1^2}e^{j\,\text{atan2}(1,0)} = 1e^{j\frac{\pi}{2}}$

(b) (1 pt) Express the complex number $e^{j\frac{\pi}{3}}$ in rectangular form $e^{j\frac{\pi}{3}} = a + jb$.
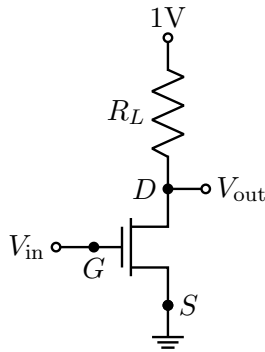
What is a? What is b?

The following table may be useful:

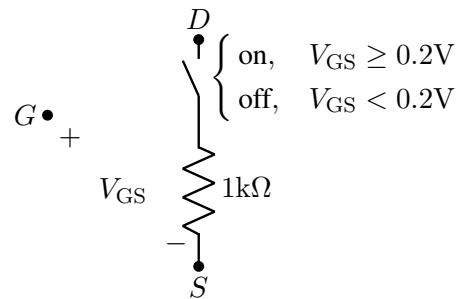| Angle θ (rad)	$\sin(\theta)$	$\cos(\theta)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

Solution: By Euler's identity: $e^{j\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) + j\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + j\frac{\sqrt{3}}{2}$.

5. NMOS Transistor Inverter (4 points)

Consider the following schematic and NMOS model.
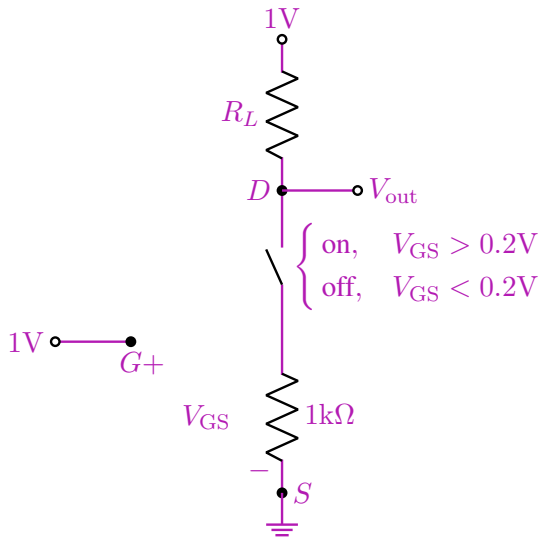

(a) An NMOS Transistor circuit

**(b)** Resistor and switch model for NMOS transistor.

Figure 1: NMOS figures.

What value of R_L is required to produce $V_{\text{out}} = 0.1$V when $V_{\text{in}} = 1$V? Show your work.

Solution: First substitute in the transistor model, and examine V_{GS} to see if we expect the NMOS to be on or off. The circuit with the transistor model substituted is visible in figure 2b. Since $V_{\text{GS}} = 1$V > 0.2V, the switch is on and we have the following circuit in figure **??**.

(a) NMOS circuit with transistor model substituted in.
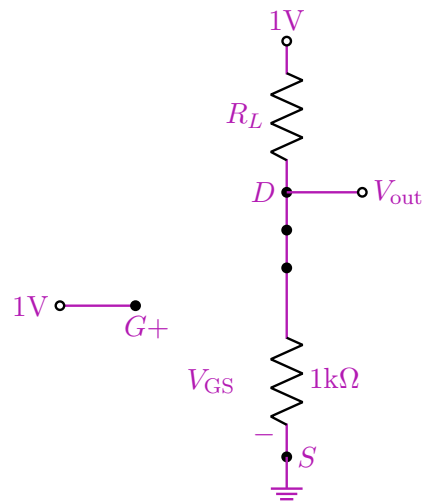
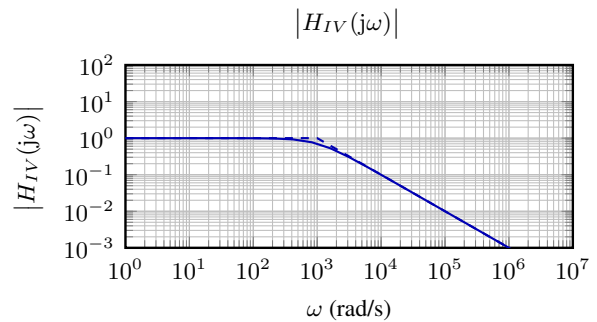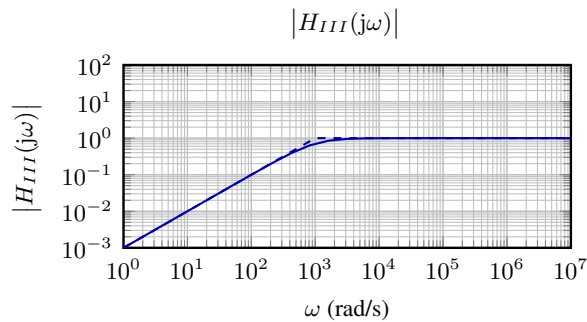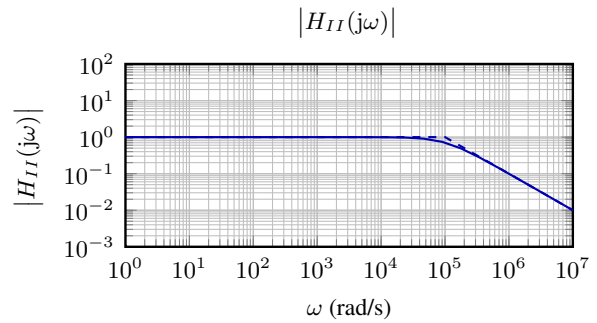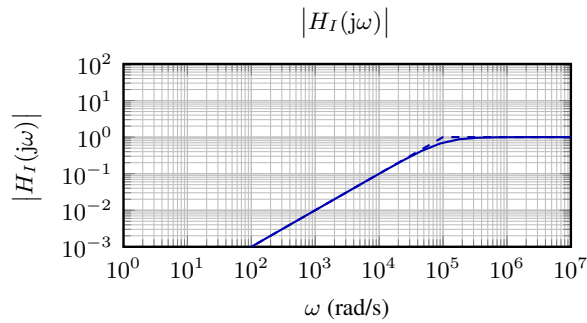**(b)** NMOS circuit with transistor model substituted in.

We have a voltage divider. $V_{\text{out}} = \frac{1\text{k}\Omega}{1\text{k}\Omega + R_L} 1\text{V} = 0.1\text{V}$.

$$\frac{1\text{k}\Omega}{1\text{k}\Omega + R_L} = \frac{1}{10} \implies R_L = 9\text{k}\Omega$$

6. Transfer Function Matching (8 points)

Below, you have filter circuits A,B,C,D, each with specific component values. **Fill in the bubbles to match each filter to its corresponding magnitude Transfer Function Plot out of choices I, II, III, IV.**

Note that each plot may be assigned to filters once, more than once, or not at all. Each filter has exactly one corresponding plot. *A table of SI Prefixes and some info. about common filters is on the next page (scratch).*

(A) Filter A.

(B) Filter B.

(C) Filter C.

(D) Filter D.

$$\left|H_I(j\omega)\right|$$

(I) Plot I.

$$\left|H_{II}(j\omega)\right|$$

(II) Plot II.

$$\left|H_{III}(j\omega)\right|$$

(III) Plot III.

$$\left|H_{IV}(j\omega)\right|$$
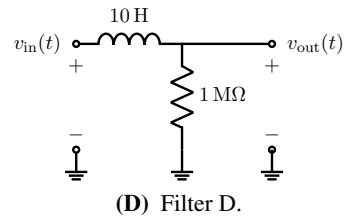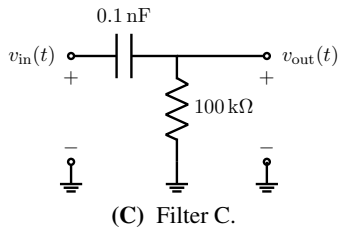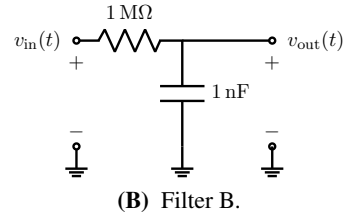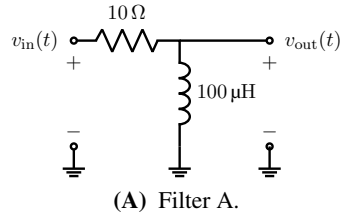
(IV) Plot IV.

Filter Letter	Plot I	Plot II	Plot III	Plot IV
A	◯	◯	◯	◯
B	◯	◯	◯	◯
C	◯	◯	◯	◯
D	◯	◯	◯	◯

Solution:

Filter Letter	Plot I	Plot II	Plot III	Plot IV
A	●	○	○	○
B	○	○	○	●
C	●	○	○	○
D	○	●	○	○

First, we can categorize each plot by whether it's a first-order low-pass filter (LPF) or first-order high-pass filter (HPF). We can then inspect each plot's cutoff frequency (as the "knee" of the exact transfer function plot, and the frequency where the Bode Plot linear approximations intersect.)

- Plot I: HPF, cutoff at $\omega_c = 1 \times 10^5 \frac{\text{rad}}{\text{s}}$.

- Plot II: LPF, cutoff at $\omega_c = 1 \times 10^5 \frac{\text{rad}}{\text{s}}$.

- Plot III: HPF, cutoff at $\omega_c = 1 \times 10^3 \frac{\text{rad}}{\text{s}}$.

- Plot IV: LPF, cutoff at $\omega_c = 1 \times 10^3 \frac{\text{rad}}{\text{s}}$.

As for the filters, we have the following cutoff frequencies and transfer functions:

Filter Letter	ω_c	$H(\mathrm{j}\omega)$
A	$\frac{R}{L} = \frac{10\Omega}{100\mu\text{H}} = 10^5 \frac{\text{rad}}{\text{s}}$	$H(\mathrm{j}\omega) = \frac{\mathrm{j}\omega L}{R + \mathrm{j}\omega L} = \frac{\frac{\mathrm{j}\omega}{\omega_c}}{1 + \frac{\mathrm{j}\omega}{\omega_c}}$ (HPF)
B	$\frac{1}{RC} = \frac{1}{1\text{M}\Omega \cdot 1\text{nF}} = 10^3 \frac{\text{rad}}{\text{s}}$	$H(\mathrm{j}\omega) = \frac{\frac{1}{\mathrm{j}\omega C}}{\frac{1}{\mathrm{j}\omega C} + R} = \frac{1}{1 + \frac{\mathrm{j}\omega}{\omega_c}}$ (LPF)
C	$\frac{1}{RC} = \frac{1}{100\text{k}\Omega \cdot 0.1\text{nF}} = 10^5 \frac{\text{rad}}{\text{s}}$	$H(\mathrm{j}\omega) = \frac{R}{\frac{1}{\mathrm{j}\omega C} + R} = \frac{\frac{\mathrm{j}\omega}{\omega_c}}{1 + \frac{\mathrm{j}\omega}{\omega_c}}$ (HPF)
D	$\frac{R}{L} = \frac{1\text{M}\Omega}{10\text{H}} = 10^5 \frac{\text{rad}}{\text{s}}$	$H(\mathrm{j}\omega) = \frac{R}{R + \mathrm{j}\omega L} = \frac{1}{1 + \frac{\mathrm{j}\omega}{\omega_c}}$ (LPF)

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[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

SI Prefix	Exponent Definition
nano (n)	10^{-9}
micro (μ)	10^{-6}
milli (m)	10^{-3}
kilo (k)	10^{3}
mega (M)	10^{6}
giga (G)	10^{9}

- RC low-pass filter: $\quad H(j\omega) = \frac{1}{1+j\frac{\omega}{\omega_c}} \qquad \omega_c = \frac{1}{RC}$

- RC high-pass filter: $\quad H(j\omega) = \frac{j\frac{\omega}{\omega_c}}{1+j\frac{\omega}{\omega_c}} \qquad \omega_c = \frac{1}{RC}$

- RL low-pass filter: $\quad H(j\omega) = \frac{1}{1+j\frac{\omega}{\omega_c}} \qquad \omega_c = \frac{R}{L}$

- RL high-pass filter: $\quad H(j\omega) = \frac{j\frac{\omega}{\omega_c}}{1+j\frac{\omega}{\omega_c}} \qquad \omega_c = \frac{R}{L}$

7. **Magnitude, Phase, and Cascades (6 points)**

Suppose you have the transfer function $H(j\omega)$ for a system as given in eq. (1) below:

$$H(j\omega) = \frac{10}{1 + j\frac{\omega}{10^6}} \tag{1}$$

Answer the following questions.

(a) (2 pts) **What is the transfer function's approximate magnitude $\left|H(j\omega)\right|$ at $\omega = 1 \times 10^4 \frac{\text{rad}}{\text{s}}$? Select the closest option from the list below by filling in a bubble.**

   ○ 100   ○ $\sqrt{2}$

   ○ $10\sqrt{2}$ ○ 1

   ○ 10   ○ 0.1

   ○ $\frac{10}{\sqrt{2}}$ ○ 0.01

   ○ 2   ○ 0.001

Solution: This is a low-pass filter with gain 10, and the given frequency is much smaller than cutoff. So, the magnitude is about 10. Precisely, we can say $\left|H(j\omega)\right| = \left|\frac{10}{1+j\frac{10^4}{10^6}}\right| = \frac{10}{\sqrt{1^2 + \left(\frac{10^4}{10^6}\right)^2}} = \frac{10}{1.00005} = 9.99950004 \approx 10$.

(b) (2 pts) **What is the transfer function's approximate phase $\angle H(j\omega)$ at $\omega = 1 \times 10^9 \frac{\text{rad}}{\text{s}}$? Select the closest option from the list below by filling in a bubble.**

   ○ $90°, \frac{\pi}{2}\text{rad}$ ○ $-6°$

   ○ $84°$ ○ $-45°, -\frac{\pi}{4}\text{rad}$

   ○ $45°, \frac{\pi}{4}\text{rad}$ ○ $-84°$

   ○ $6°$

   ○ $0°, 0\,\text{rad}$ ○ $-90°, -\frac{\pi}{2}\text{rad}$

Solution: Since this is a low-pass filter and the given frequency is much larger than cutoff frequency, we have $-90°$. To compute the phase we calculate: $\angle H(j1 \times 10^9 \frac{\text{rad}}{\text{s}}) = \angle\left(\frac{10}{1+j\frac{10^9}{10^6}}\right) = \angle(10) - \angle(1 + j10^3) \approx 0 - \angle(j10^3) = -\frac{\pi}{2}\text{rad}$ or $-90°$.

(c) (2 pts) You cascade three of the systems as defined by the transfer function in eq. (1). You place one after the other, with unity-gain buffers in between. **Write the overall transfer function $H_{\text{ov}}(j\omega)$ in terms of the given transfer function $H(j\omega)$.** *You do not have to simplify your answer.*

Solution: Cascading filters leads to the product of their transfer functions as the output of one filter will become the input of the next. Thus we have: $H_{\text{ov}}(j\omega) = H(j\omega)^3 = \left(\frac{10}{1+j\frac{\omega}{10^6}}\right)^3$

8. **Recurrence Stability (6 points)**

Just as you have seen second-order vector differential equations, we can define similar second-order recurrence relations in discrete-time. Consider:

$$x[i+1] = x[i] - 3x[i-1] + w[i] \tag{2}$$

where $x[0] = 0, x[1] = 0$ and $|w[i]| \leq \varepsilon$.

This can be written in vector form for $i \geq 1$ as:

$$\begin{bmatrix} x[i] \\ x[i+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix}}_{A_d} \begin{bmatrix} x[i-1] \\ x[i] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w[i], \tag{3}$$

with initial condition $\begin{bmatrix} x[0] \\ x[1] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Is the discrete-time system above stable or unstable? Fill in the appropriate bubble and then explain below.

○ Stable

○ Unstable

The following information about the matrix A_d may be useful. You may use as much or as little of the following information as you find helpful. It is here to simplify calculations you may want to perform.

Eigenvalues of A_d :	$\lambda_1 = \frac{1}{2}\left(1 + j\sqrt{11}\right)$	$\lambda_2 = \frac{1}{2}\left(1 - j\sqrt{11}\right)$
Eigenvalue magnitudes:	$|\lambda_1| \approx 1.73$	$|\lambda_2| \approx 1.73$
Eigenvectors for associated λ_i :	$\vec{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$	$\vec{v}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$
Matrix inverse:	$A_d^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$	

If stable, explain why. If unstable, just give a sequence of bounded scalar disturbances $w[i]$ that would cause the states $x[i]$ to grow unboundedly. You don't need to justify your choice of $w[i]$ sequence here.

Solution: This system is unstable; to see so from the given information, we notice that the magnitude of the eigenvalues is larger than 1, which is the condition to check a discrete-time system's stability. We only needed one of these eigenvalues' magnitudes to exceed 1.

If we use the input sequence $w[i] = +\varepsilon, 0, 0, \cdots$ we see that $x[2] = +\varepsilon, x[3] = +\varepsilon, x[4] = -2\varepsilon, x[5] = -5\varepsilon, x[6] = \varepsilon, x[7] = 16\varepsilon, \cdots$ and this continues to grow in magnitude.

9. **Nonhomogeneous Solution (4 points)**

Solve the following differential equation for $t \geq 0$:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -3x(t) + 6 \tag{4}$$

with initial condition $x(0) = -2$. **What is $x(t)$?**

You do not have to show any work for this problem.

Solution: From the nonhomogenous differential equation solution (which you could either apply directly or derive):

$$x(t) = x_0 e^{\lambda t} + \int_0^t e^{\lambda(t-\theta)} u(\theta)\,\mathrm{d}\theta \tag{5}$$

$$x(t) = -2e^{-3t} + \int_0^t e^{-3(t-\theta)} 6\,\mathrm{d}\theta \tag{6}$$

$$x(t) = -2e^{-3t} + 6\int_0^t e^{-3(t-\theta)}\,\mathrm{d}\theta \tag{7}$$

$$x(t) = -2e^{-3t} + 6e^{-3t}\int_0^t e^{3\theta}\,\mathrm{d}\theta \tag{8}$$

$$x(t) = -2e^{-3t} + 6e^{-3t}\left(\frac{1}{3}\left(e^{3t} - 1\right)\right) \tag{9}$$

$$x(t) = -2e^{-3t} + 2e^{-3t}\left(e^{3t} - 1\right) \tag{10}$$

$$x(t) = -2e^{-3t} + 2\left(1 - e^{-3t}\right) \tag{11}$$

$$x(t) = -2e^{-3t} + 2 - 2e^{-3t} \tag{12}$$

$$x(t) = 2 - 4e^{-3t}. \tag{13}$$

Alternatively, we could use our substitution technique.

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -3(x(t) - 2), x(0) = -2 \tag{14}$$

$$y(t) := x(t) - 2 \tag{15}$$

$$y(0) = -4 \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(y(t) - 2\right) = -3y(t) \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = -3y(t) \tag{18}$$

$$y(t) = -4e^{-3t} \tag{19}$$

$$x(t) = 2 - 4e^{-3t} \tag{20}$$

There is yet another method; that of "guess-and-solve". From experience with this category of differential equation, we know that the value will approach a constant after a long time t, and we know also that a term of the form ce^{-3t} will be present (capturing the exponential behavior in the meantime). Solving for the long-term condition, we get a constant of 2. Then, solving for the initial condition yields -4 as the value of c in the guess.

Since you have seen this sort of equation many times in the course, it is also fine to directly write down the solution by inspection.

10. Get Real! (8 points)

(a) (2 pts) Suppose you have the input signal in eq. (21):

$$v_{\text{in}}(t) = 4\cos\left(10^4 t + \frac{\pi}{4}\right) \tag{21}$$

For the sinusoidal input voltage signal given in eq. (21), what is the angular frequency ω?

Solution: The angular frequency is the coefficient of t in the argument of the sinusoid. since we have $10^4 t$ in the argument, our angular frequency is $\omega = 1 \times 10^4 \frac{\text{rad}}{\text{s}}$.

(b) (4 pts) You now have a circuit given in fig. 5. We want to add an element Z_E in series to the R and C such that the equivalent impedance of the series combination is purely real. Notice the same input signal eq. (21) is also labeled in the diagram fig. 5.
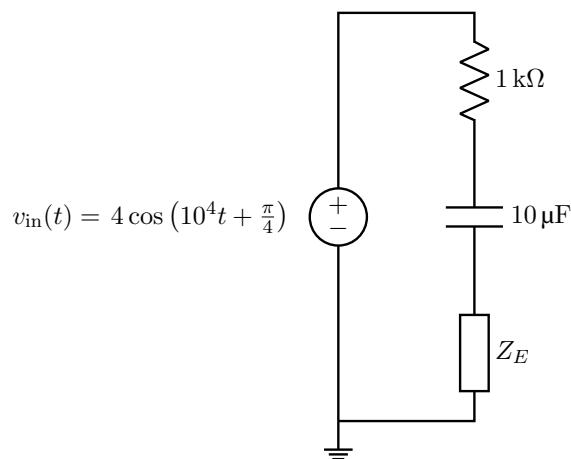

Figure 5: A given circuit for which we want to make the equivalent series impedance purely real.

In order to make the equivalent impedance of the series interconnection of R, C and Z_E purely real at the input angular frequency ω you identified above, **what should the impedance $Z_E(\text{j}\omega)$ be? Show your work.** Give a specific numerical value, and do not worry about either units or the dependence on ω.

Solution: The total series equivalent impedance is given by: $Z_{\text{eq}} = Z_R + Z_C + Z_E$.

$$Z_{\text{eq}} = R + \frac{1}{\text{j}\omega C} + Z_E$$
$$= 1\text{k}\Omega - \text{j}\frac{1}{10^4 \frac{\text{rad}}{\text{s}} 10\mu\text{F}} + Z_E$$
$$= 1\text{k}\Omega - \text{j}10\Omega + Z_E$$

To have Z_{eq} be purely real, we must have the capacitive impedance $-\text{j}10\Omega$ and the unknown Z_E cancel out. So $Z_E = \text{j}10\Omega$.

Since this impedance is positive, we know to use an inductor, and as we were given a frequency of operation $\omega = 1 \times 10^4 \frac{\text{rad}}{\text{s}}$, we can actually solve for the exact inductance needed to satisfy the condition of real equivalent impedance. Setting $\text{j}\omega L = \text{j}10$, we solve: $L = \frac{10}{\omega} = \frac{10}{1 \times 10^4} = 1 \times 10^{-3}$ H $= 1\,\text{mH}$. This last step was not required by the problem as stated.

(c) (2 pts) **What single circuit element could you use to get the above Z_E? Select one of the options below by filling in a bubble.**

○ Resistor

○ Capacitor

○ Inductor

Solution: To achieve a positive imaginary impedance, we can use an inductor, which will have $Z_L(j\omega) = j\omega L$.

11. System ID (8 points)

Consider a scalar system we want to model by:

$$x[i+1] = a_1 x[i] + a_2 x[i-1] + bu[i] + w[i], \tag{22}$$

where $w[i]$ is a hopefully small disturbance.

We collect a trace of values for $x[i]$ and $u[i]$ from time $i = 0, 1, \ldots, \ell$. Setup a least squares problem for the unknown parameters a_1, a_2, b, in the form of solving an approximate system of equations $D\vec{p} \approx \vec{s}$. **What are the matrix D, the vector of unknowns $\vec{p}$, and the vector $\vec{s}$?** **Solution:** We can write the following equations:

$$x[2] \approx a_1 x[1] + a_2 x[0] + bu[1]$$
$$x[3] \approx a_1 x[2] + a_2 x[1] + bu[2]$$
$$\vdots \quad \approx \quad \vdots$$
$$x[\ell-1] \approx a_1 x[\ell-2] + a_2 x[\ell-3] + bu[\ell-2]$$
$$x[\ell] \approx a_1 x[\ell-1] + a_2 x[\ell-2] + bu[\ell-1]$$

Grouping up terms on the left and the right as vectors, and factoring out our parameters we get the following equation.

$$\underbrace{\begin{bmatrix} x[2] \\ x[3] \\ \vdots \\ x[\ell-1] \\ x[\ell] \end{bmatrix}}_{\vec{s}} \approx \begin{bmatrix} a_1 x[1] + a_2 x[0] + bu[1] \\ a_1 x[2] + a_2 x[1] + bu[2] \\ \vdots \\ a_1 x[\ell-2] + a_2 x[\ell-3] + bu[\ell-2] \\ a_1 x[\ell-1] + a_2 x[\ell-2] + bu[\ell-3] \end{bmatrix} = \underbrace{\begin{bmatrix} x[1] & x[0] & u[1] \\ x[2] & x[1] & u[1] \\ \vdots & \vdots & \vdots \\ x[\ell-2] & x[\ell-3] & u[\ell-2] \\ x[\ell-1] & x[\ell-2] & u[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ b \end{bmatrix}}_{\vec{p}}$$

12. s-Impedance Derivation (8 points)

(a) (4 pts) You are given a component with the I-V relationship $V(t) = p_0 \frac{d^2}{dt^2} I(t) - p_1 \frac{d}{dt} I(t)$. Suppose you apply $I(t) = e^{st}$. **What is $V(t)$?**

Solution: Consider $I(t) = e^{st}$.

$$V(t) = p_0 \frac{d^2}{dt^2}\left(e^{st}\right) - p_1 \frac{d}{dt}\left(e^{st}\right)$$
$$= p_0 s^2 e^{st} - p_1 s e^{st}$$
$$= (p_0 s^2 - p_1 s)e^{st}$$

(b) (4 pts) In (a) you should have noticed the answer is proportional to e^{st}. The s-impedance is defined as the ratio $\frac{V(t)}{I(t)}$ when $I(t) = e^{st}$. **What is the s-impedance for this component?**
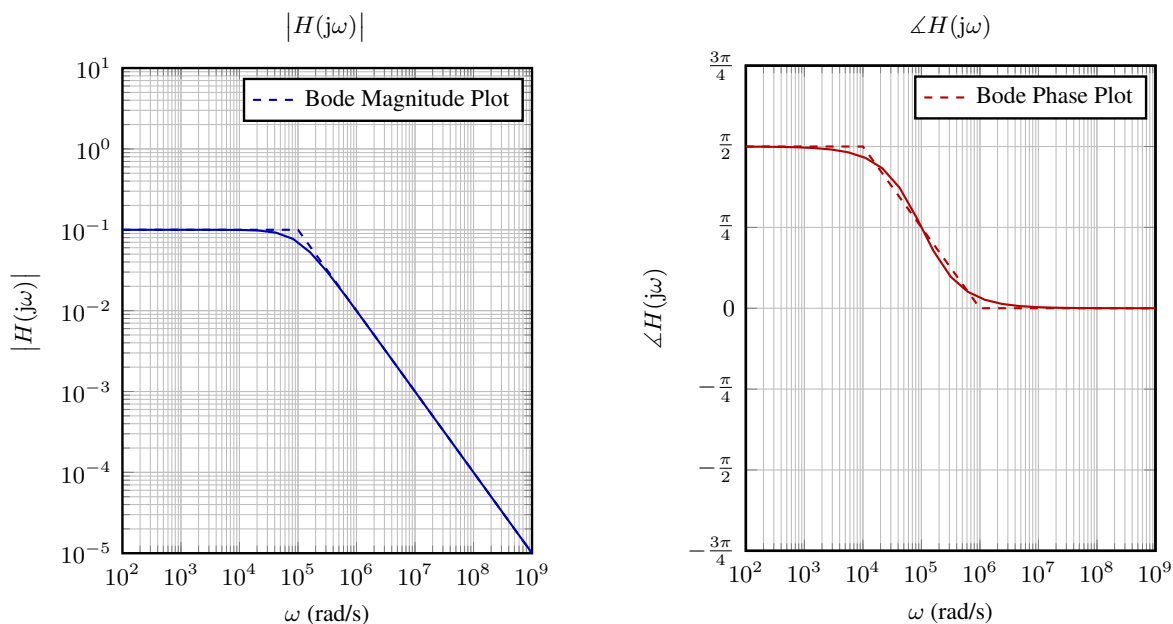
Solution: We have current $I(t) = e^{st}$ and voltage $V(t) = (p_0 s^2 - p_1 s)e^{st}$. The s-impedance is the ratio of the two which gives us:

$$Z(s) = \frac{V(t)}{I(t)} = p_0 s^2 - p_1 s$$

13. Conceptual Transfer Functions and Bode Plot Analysis (10 points)

In fig. 6 are magnitude and phase Bode plots for some filter with transfer function $H(j\omega)$.

$\boxed{5\,\text{min} - 8\,\text{min}}$

(i) Part a) Magnitude Plot

(ii) Part a) Phase Plot

Figure 6: Part a) Magnitude and Phase Bode Plots for a transfer function $H(j\omega)$.

(a) (4 pts) Suppose $v_{\text{in},1}(t)$ below is an input voltage signal to the above filter with transfer function $H(j\omega)$.

$$v_{\text{in},1}(t) = 3\sin\left(10^3 t + \frac{\pi}{3}\right) \tag{23}$$

Label the input angular frequency of $v_{\text{in},1}(t)$ on both plots above, in fig. 6, using a vertical line.
Next, compute the output voltage $v_{\text{out},1}(t)$ after this input signal passes through the filter defined by $H(j\omega)$.
Your answer for the output voltage should have the form $A_1 \sin(\omega_1 t + \phi_1)$. What are $A_1, \omega_1,$ and ϕ_1?

Solution: The first step here is to identify the input angular frequency ω; we know that this is the coefficient of t in the $\sin$ term. Here, this is $\omega = 1 \times 10^3 \frac{\text{rad}}{\text{s}}$, and we would label this frequency with vertical lines in the plots.

Next, we consider what a transfer function of a filter represents for magnitude; the magnitude of the transfer function at a given frequency indicates how much gain (in our case, attenuation) the filter applies to the magnitude of the input signal. Here, we find that the A_1 terms of the output is 3 (input magnitude) multiplied by the magnitude of $H(j\omega)$ at $\omega = 1 \times 10^3 \frac{\text{rad}}{\text{s}}$, which is 0.1. That is, $A_1 = 3 \cdot 0.1 = 0.3$.

Similarly, the phase of the transfer function at a given frequency indicates how much phase the filter adds to the phase of the input signal. From the plot, we see that this phase addition is $\frac{\pi}{2}$ rad. So, $\phi_1 = \frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6}$ rad.

Finally, our filter will not change the input frequency, so $\omega_1 = \omega_{\text{in},1} = 1 \times 10^3 \, \frac{\text{rad}}{\text{s}}$.

Putting this all together, we find that $v_{\text{out},1}(t) = 0.3 \sin\left(10^3 t + \frac{5\pi}{6}\right)$. Equivalently, $v_{\text{out},1}(t) = -0.3 \sin\left(10^3 t - \frac{\pi}{6}\right)$

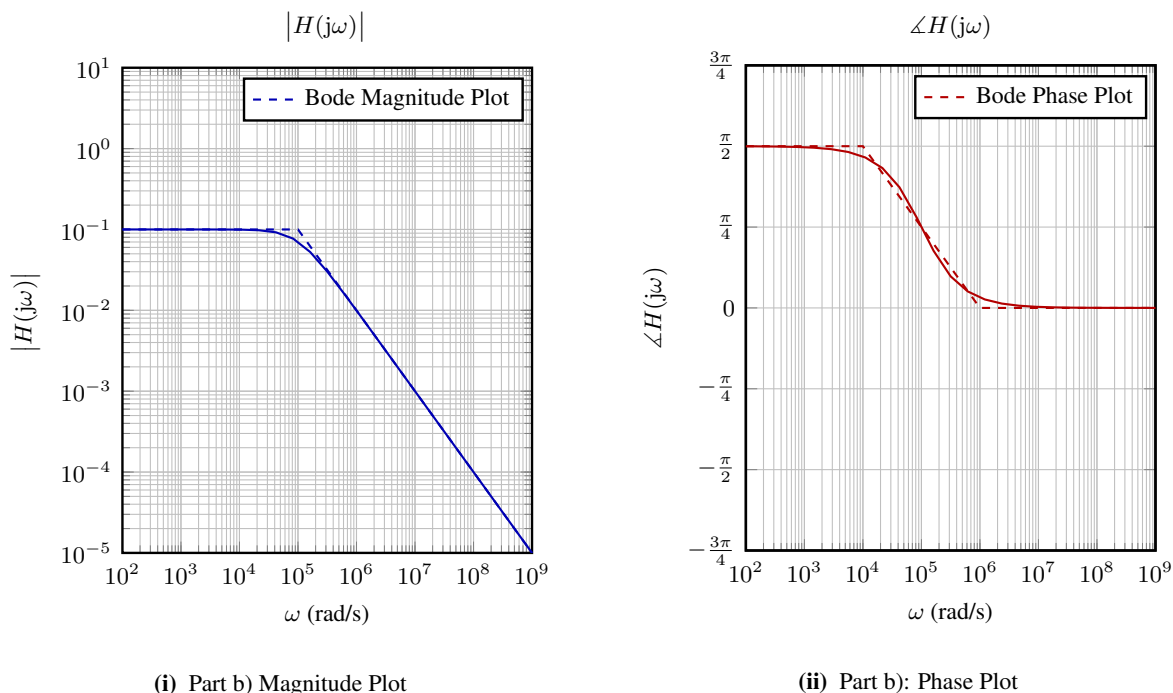(b) (4 pts) For convenience, we have copied the same plots from part (a) below.

(i) Part b) Magnitude Plot

(ii) Part b): Phase Plot

Figure 7: Part b): Magnitude and Phase Bode Plots for a transfer function $H(j\omega)$.

Suppose $v_{\text{in},2}(t)$ below is a second input voltage signal to the filter with transfer function $H(j\omega)$ on the previous page.

$$v_{\text{in},2}(t) = 6\cos\left(10^8 t - \frac{\pi}{6}\right) \tag{24}$$

Label the input angular frequency of $v_{\text{in},2}(t)$ on both plots in fig. 7, using a vertical line.

Next, compute the output voltage $v_{\text{out},2}(t)$ after this input signal passes through the filter defined by $H(j\omega)$.

Your answer for the output voltage should have the form $A_2\cos(\omega_2 t + \phi_2)$. What are A_2, ω_2, and ϕ_2?

Solution: As for the previous subpart, we identify ω as the coefficient of t in the cos term. Here, this is $\omega = 1 \times 10^8 \frac{\text{rad}}{\text{s}}$, and we would label this frequency with vertical lines in the plots.

A_2 is equal to the input magnitude multiplied by the magnitude of $H(j\omega)$ at $\omega = 1 \times 10^8 \frac{\text{rad}}{\text{s}}$. That is, $A_2 = 6 \cdot 1 \times 10^{-4} = 6 \times 10^{-4} = 0.0006$.

From the plot, the phase addition is 0, so $\phi_2 = -\frac{\pi}{6}\text{rad}$.

Finally, our filter will not change the input frequency, so $\omega_2 = \omega_{\text{in},2} = 1 \times 10^8 \frac{\text{rad}}{\text{s}}$.

Putting this all together, we find that $v_{\text{out},2}(t) = 0.0006\cos\left(10^8 t - \frac{\pi}{6}\right)$.

(c) (2 pts) **Based on your answers in parts (a) and (b), if $v_{\text{in}}(t) = v_{\text{in},1}(t) + v_{\text{in},2}(t)$, what is the corresponding $v_{\text{out}}(t)$?**

You can leave your answer in terms of the A_i, ω_i, ϕ_i from the previous two parts. You don't have to substitute in values.

Solution: The output is the sum of the two output signal components; symbolically, this i:

$$v_{\text{out}}(t) = v_{\text{out},1}(t) + v_{\text{out},2}(t) \tag{25}$$

$$v_{\text{out}}(t) = A_1 \sin\left(\omega_1 t + \phi_1\right) + A_2 \cos\left(\omega_2 t + \phi_2\right) \tag{26}$$

Leaving it in symbolic form as above sufficiently answers the question. Substituting in the numbers, we have:

$$v_{\text{out}}(t) = 0.3 \sin\left(10^3 t + \frac{5\pi}{6}\right) + 0.0006 \cos\left(10^8 t - \frac{\pi}{6}\right). \tag{27}$$

14. **Being able to control with feedback (22 points)**

    Consider the discrete-time dynamic system:

    $$\vec{x}[i + 1] = A\vec{x}[i] + \vec{b}u[i] + \vec{w}[i], \tag{28}$$

    where the purely real matrix $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

    (a) (3 pts) **Show that if $\beta = 0$, then $\vec{b}$ is an eigenvector of A.**

       (HINT: What does it mean to be an eigenvector?)

       Solution:

       We just write stuff out:

       $$A\vec{b} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{29}$$

       $$= \begin{bmatrix} \beta \\ \delta \end{bmatrix}. \tag{30}$$

       If $\beta = 0$, then $A\vec{b} = \delta\vec{b}$ so $\vec{b}$ is an eigenvector.

    (b) (5 pts) Let η_1 and η_2 be a pair of complex numbers that are complex conjugates. (i.e. $\eta_1 = \overline{\eta_2}$.)

       Show that the polynomial $(\lambda - \eta_1)(\lambda - \eta_2) = \lambda^2 + c_1\lambda + c_0$ has purely real coefficients c_1 and c_0.

       Solution:

       It is convenient to write $\eta_1 = a + bj$ and $\eta_2 = a - bj$ to capture their complex conjugate relationship. We just multiply out the monomial terms:

       $$(\lambda - \eta_1)(\lambda - \eta_2) = \lambda^2 - (\eta_1 + \eta_2)\lambda + \eta_1\eta_2 \tag{31}$$

       and then match coefficients to see that $c_1 = -(\eta_1 + \eta_2) = -(\overline{\eta_2} + \eta_2) = -2\operatorname{Re}(\eta_2) = -2a$ and $c_0 = \eta_1\eta_2 = \overline{\eta_2}\eta_2 = (a + bj)(a - bj) = a^2 + b^2 = |\eta_2|^2$. We know that the magnitude squared of a complex number is purely real as is the sum of a complex number with its own conjugate since the imaginary parts cancel. This shows that c_1 and c_0 are both purely real numbers.

(c) (14 pts) Suppose we knew that $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is **not** an eigenvector of the real matrix $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$.

Choose real feedback gains f_1, f_2 so that $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i]$ places the eigenvalues of the closed-loop matrix $A_{cl} = (A + \vec{b} \begin{bmatrix} f_1 & f_2 \end{bmatrix})$ at the complex conjugate pair η_1 and η_2 from the previous part. Show your work.

It is fine if you leave your answer in terms of c_1 and c_0 from the previous part.

(HINT: The final gains f_1 and f_2 are going to depend on the $\alpha, \beta, \gamma, \delta$ that define A as well as η_1 and η_2 through c_1 and c_0 from part (b).

At some point in the derivation, your life might become easier if you solve for f_2 first before trying to solve for f_1.)

(Potentially heavier algebra warning for this part. Don't get bogged down for too long.)

Solution:

Looking at the closed-loop dynamics, we get:

$$A + \vec{b} \begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \tag{32}$$

$$= \begin{bmatrix} \alpha & \beta \\ \gamma + f_1 & \delta + f_2 \end{bmatrix}. \tag{33}$$

Evaluating the characteristic polynomial:

$$\det \left(\lambda I - (A + \vec{b} \begin{bmatrix} f_1 & f_2 \end{bmatrix}) \right) = \det \begin{bmatrix} \lambda - \alpha & -\beta \\ -\gamma - f_1 & \lambda - \delta - f_2 \end{bmatrix} \tag{34}$$

$$= (\lambda - \alpha)(\lambda - \delta - f_2) - (-\beta(-\gamma - f_1)) \tag{35}$$

$$= \lambda^2 - (\alpha + \delta + f_2)\lambda + (\alpha\delta + \alpha f_2 - \beta\gamma - \beta f_1). \tag{36}$$

To get the eigenvalues to be at η_1 and η_2, it is necessary and sufficient to have the characteristic polynomial be $\lambda^2 + c_1 \lambda + c_0$ for the c_1 and c_0 computed in the previous part.

Matching terms for the coefficient c_1 of λ in the characteristic polynomial we see that we must set f_2 to satisfy:

$$-2\operatorname{Re}(\eta_2) = -(\alpha + \delta + f_2) \tag{37}$$

which we can solve immediately for

$$f_2 = 2\operatorname{Re}(\eta_2) - \alpha - \delta. \tag{38}$$

Matching terms for the constant term c_0 in the characteristic polynomial we see that we must set f_1 to satisfy:

$$|\eta_2|^2 = \alpha\delta + \alpha f_2 - \beta\gamma - \beta f_1 \tag{39}$$

$$= \alpha\delta + \alpha(2\operatorname{Re}(\eta_2) - \alpha - \delta) - \beta\gamma - \beta f_1 \tag{40}$$

$$= \alpha(2\operatorname{Re}(\eta_2) - \alpha) - \beta\gamma - \beta f_1, \tag{41}$$

where we used eq. (38) to substitute in the known value for f_2. Given that we know that $\beta \neq 0$, we can immediately solve for

$$f_1 = \frac{\alpha(2\operatorname{Re}(\eta_2) - \alpha) - \beta\gamma - |\eta_2|^2}{\beta} \tag{42}$$

Because $\alpha, \beta, \gamma, \delta, \operatorname{Re}(\eta_2), |\eta_2|$ are all real quantities, f_1 and f_2 are also real. Since the above choice of f_1 and f_2 sets the characteristic polynomial to have roots η_1 and η_2, we have placed the eigenvalues of $A + \vec{b}[f_1 \ f_2]$ at η_1 and η_2 by this choice of feedback control.

15. Op-Amp Model Analysis (16 points)

(a) (10 pts) You are given the op-amp circuit model in fig. 8i. This circuit has a phasor-domain repre-
sentation as given in fig. 8ii. **Find the transfer function from v_{in} to v_{out}. i.e. Find $H(j\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$
corresponding to fig. 8ii. Show your work.**

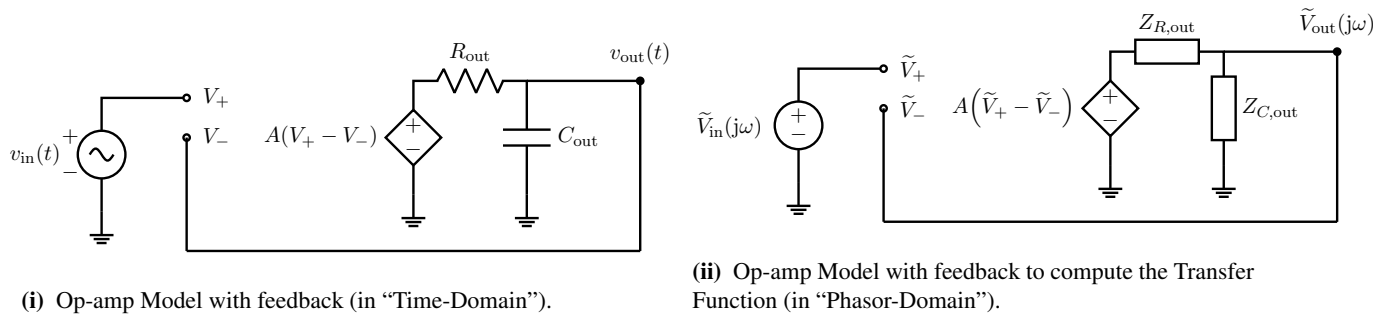You don't have to fully simplify your answer but it has to be a valid transfer function.

(i) Op-amp Model with feedback (in "Time-Domain").

(ii) Op-amp Model with feedback to compute the Transfer
Function (in "Phasor-Domain").

Figure 8: Op-amp Models in "Time-Domain" and "Phasor-Domain".

Solution:

$$\widetilde{V}_- = \widetilde{V}_{\text{out}}$$

$$\widetilde{V}_+ = \widetilde{V}_{\text{in}}$$

$$\widetilde{V}_{\text{out}} = \frac{Z_C}{Z_C + Z_R} A(\widetilde{V}_+ - \widetilde{V}_-)$$

$$\widetilde{V}_{\text{out}} = \frac{\frac{1}{j\omega C_{\text{out}}}}{\frac{1}{j\omega C_{\text{out}}} + R_{\text{out}}} A(\widetilde{V}_+ - \widetilde{V}_-)$$

$$\frac{1 + j\omega R_{\text{out}} C_{\text{out}}}{A} \widetilde{V}_{\text{out}} = \widetilde{V}_+ - \widetilde{V}_-$$

$$\frac{1 + j\omega R_{\text{out}} C_{\text{out}}}{A} \widetilde{V}_{\text{out}} = \widetilde{V}_{\text{in}} - \widetilde{V}_{\text{out}}$$

$$\left(\frac{1 + j\omega R_{\text{out}} C_{\text{out}}}{A} + 1\right) \widetilde{V}_{\text{out}} = \widetilde{V}_{\text{in}}$$

So we can conclude that $H(j\omega) = \frac{1}{1 + \frac{1}{A} + \frac{j\omega}{\omega_0 A}}$, where $\omega_0 = \frac{1}{R_{\text{out}} C_{\text{out}}}$.

(b) (6 pts) Now suppose that your A is large so that you can approximate your transfer function $H(j\omega)$ as $\widehat{H}(j\omega)$ as in eq. (43).

$$\widehat{H}(j\omega) = \frac{1}{1 + \frac{j\omega}{\omega_0 A}} \tag{43}$$

For the transfer function $\widehat{H}(j\omega)$, sketch a Bode Plot (straight-line approximations are fine) for the magnitude on fig. 9 with $w_0 = 10^2 \frac{\text{rad}}{\text{s}}$ and $A = 10^5$.
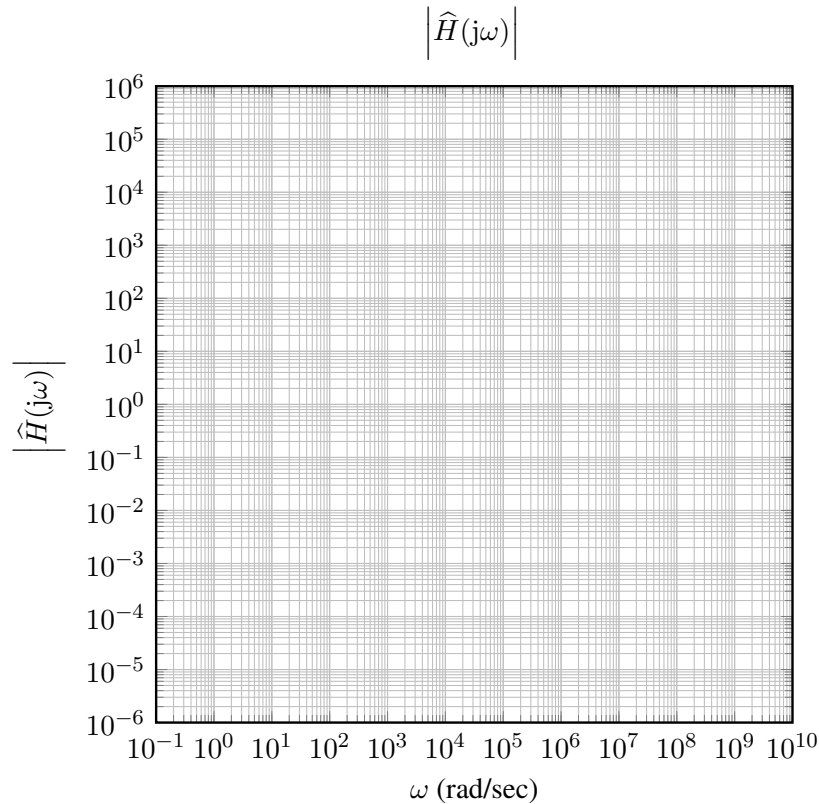
$$\left|\widehat{H}(j\omega)\right|$$

Figure 9: Magnitude Bode Plots: Template for part (b).

Solution: To generate a Bode Plot, we have to identify a few key features of the transfer function. From the form, we know it's a first-order filter, either low-pass or high-pass. To identify which kind, we can either pattern-match to the known forms for each filter and inspect the type, or substitute in low and high frequencies to get a sense of the behavior of the transfer function at various frequencies. This is a low-pass filter, with magnitude of 1 in the pass-band.

Next, we identify the cutoff frequency. The general form of a first-order low-pass filter is as follows:

$$H_{\text{LP}}(j\omega) = \frac{1}{1 + \frac{j\omega}{\omega_c}} \tag{44}$$

Therefore, the cutoff frequency here is $\omega_c = \omega_0 A$. For the specific values given ($w_0 = 10^2 \frac{\text{rad}}{\text{s}}$ and $A = 10^5$), we find that $\omega_c = 1 \times 10^7 \frac{\text{rad}}{\text{s}}$. Now, we can sketch the plot.
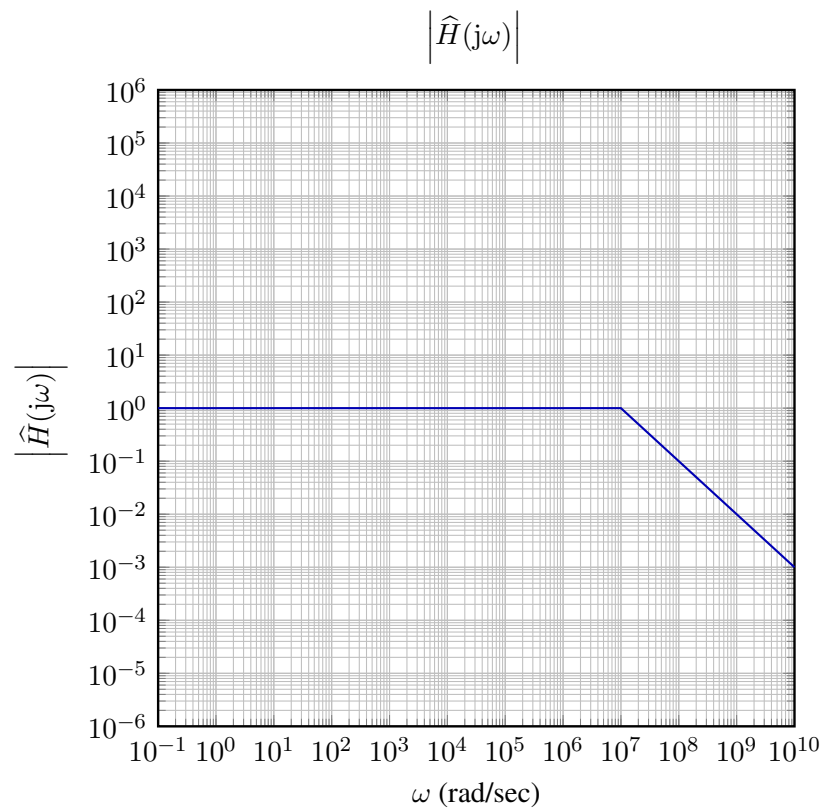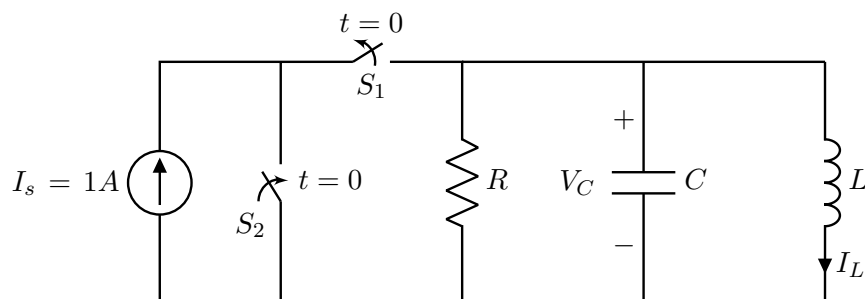
Figure 10: Magnitude Bode Plot Solution for part (b).

16. Parallel RLC with Current Source (28 points)

Consider the following circuit:

(a) (10 pts) At $t = 0$, switch S_1 became open and switch S_2 became closed. We first need to construct our state space system for $t \geq 0$. Our natural state variables are the current through the inductor $x_1(t) = I_L(t)$ and the voltage across the capacitor $x_2(t) = V_C(t)$ since these are the quantities whose derivatives show up in the system of equations governing our circuit.

Find the system of differential equations in terms of our state variables that describes this circuit for $t \geq 0$. Leave the system symbolic in terms of $I_s, R, L,$ and C. Write the system of differential equations in vector/matrix form with the vector state variable:

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix} \tag{45}$$

This should be in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.
Show your work.

Solution:
Let's first find the relationship between $\frac{\mathrm{d}I_L(t)}{\mathrm{d}t}$ and $V_C(t)$:

$$V_C(t) = V_L(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} \rightarrow \frac{\mathrm{d}I_L(t)}{\mathrm{d}t} = \frac{1}{L}V_C(t) \tag{46}$$

Using KCL, we can also find the relationship between $\frac{\mathrm{d}V_C(t)}{\mathrm{d}t}$ and $I_L(t)$:

$$I_C(t) + I_L(t) + I_R(t) = 0, \quad I_C(t) = C\frac{\mathrm{d}V_C(t)}{\mathrm{d}t}, \quad I_R = \frac{V_C(t)}{R} \tag{47}$$

$$C\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} + I_L(t) + \frac{V_C(t)}{R} = 0 \rightarrow \frac{\mathrm{d}V_C(t)}{\mathrm{d}t} = -\frac{1}{C}I_L(t) - \frac{1}{RC}V_C(t) \tag{48}$$

Note that $I_C(t) + I_L(t) + I_R(t) = 0$ because the current source (I_s) is disconnected from the RLC network at $t = 0$.
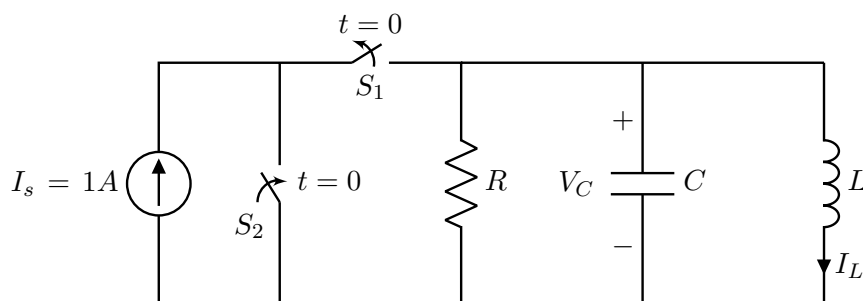In summary, we have the following differential equations:

$$\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} = \frac{1}{L}V_C(t) \tag{49}$$

$$\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} = -\frac{1}{C}I_L(t) - \frac{1}{RC}V_C(t). \tag{50}$$

Finally, we can represent the above differential equations in the matrix/vector multiplication form as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix}}_{A} \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix}. \tag{51}$$

(b) (3 pts) For a long time in the past $t < 0$, assume that the switch S_1 had remained closed and S_2 had remained open. **What is $I_L(0)$ and $V_C(0)$? Give a brief justification as well.**

Solution:

If something has been happening for a long time, we expect to see that things are no longer changing. For the current in the inductor to no longer be changing, the voltage across it must be zero. Since the current is connected in parallel with the capacitor, it must be that $V_C(0) = 0V$. With a zero voltage across it, the resistor must have zero current flowing through it. For the voltage across the capacitor to not be changing, there must be zero current flowing through it as well. To satisfy, KCL, this means all the current from the current source must be flowing through the inductor, setting $I_L(0) = 1A$.

The above is a fuller solution than we expected. A shorter one could simply be: if a system has been left as it is for a long time, it should be in steady state. In steady-state, an inductor is a short and a capacitor is an open circuit. This means that all the current goes through the short, making $I_L(0) = 1A$ and since it is a short, the voltage across it and hence the capacitor is $0V$.

Writing the above in math, we'd get: The given RLC network reaches steady state ($\frac{dV_C(t)}{dt} = 0$ and $\frac{dI_L(t)}{dt} = 0$) after we wait for long enough. This can be interpreted as $I_C = 0$ and $V_L = 0$ since $I_C = C\frac{dV_C(t)}{dt}$ and $V_L = L\frac{dI_L(t)}{dt}$. Therefore the capacitor and the inductor behave as an open and a short circuit, respectively. Thus, all of the input current flows through the inductor ($I_L = I_s = 1A$).

A significantly longer answer would be to realize that the nonhomogeneous differential equation that corresponds to the configuration for $t < 0$ is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix}}_{A} \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix} I_s. \tag{52}$$

This is stable because $A's$ eigenvalues all have real part less than 0 as long as $R > 0$ (as confirmed in part (c)) and hence all transients will have long since decayed away. The only (because A is invertible) steady-state solution (with the derivatives being zero since nothing is changing in steady-state) this has for $I_s = 1A$ is $I_L = 1A$ and $V_C = 0V$.

(c) (3 pts) For the rest of this problem, assume $R = 1\,\mathrm{M\Omega}, L = 25\,\mathrm{\mu H}, C = 10\,\mathrm{nF}$. With these values, we get the following eigenvalues and eigenvectors for A in the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$.

$$\lambda_1 = -Z_0 + \mathrm{j}\omega_0 \qquad \lambda_2 = -Z_0 - \mathrm{j}\omega_0, \tag{53}$$

$$\text{where } Z_0 = \sqrt{\frac{L}{C}} = 50 \text{ and } \omega_0 = \frac{1}{\sqrt{LC}} = 2 \times 10^6$$

$$V = \begin{bmatrix} \vec{v}_{\lambda_1} & \vec{v}_{\lambda_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ a & \bar{a} \end{bmatrix} \tag{54}$$

$$V^{-1} = \begin{bmatrix} b & -\text{j}0.01 \\ \bar{b} & \text{j}0.01 \end{bmatrix} \tag{55}$$

Consider a nice coordinate system for which we can write $\vec{x}(t) = V\widetilde{\vec{x}}(t)$.

What is the $\widetilde{A}$ so that $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\vec{x}}(t) = \widetilde{A}\widetilde{\vec{x}}(t)$? You can leave the answer symbolic in terms of λ_1, λ_2.
Note that you **do not** need to find/evaluate a, b. You can still answer all the questions below without knowing these values.

Solution:
A calculation is not required as the effect of multiplying A appropriately by V and V^{-1} yields a matrix with the eigenvalues on its diagonal, in the order corresponding to the eigenvectors in the V matrix.

$$\widetilde{A} = V^{-1}AV = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -Z_0 + \text{j}\omega_0 & 0 \\ 0 & -Z_0 - \text{j}\omega_0 \end{bmatrix} \tag{56}$$

However, a quick way to verify is the following calculation:

$$\widetilde{A} = V^{-1}AV \tag{57}$$

$$= V^{-1} \begin{bmatrix} | & & | \\ A\vec{v}_{\lambda_1} & \cdots & A\vec{v}_{\lambda_n} \\ | & & | \end{bmatrix} \tag{58}$$

$$= V^{-1} \begin{bmatrix} | & & | \\ \lambda_1\vec{v}_{\lambda_1} & \cdots & \lambda_n\vec{v}_{\lambda_n} \\ | & & | \end{bmatrix} \tag{59}$$

$$= \begin{bmatrix} | & & | \\ \lambda_1 V^{-1}\vec{v}_{\lambda_1} & \cdots & \lambda_n V^{-1}\vec{v}_{\lambda_n} \\ | & & | \end{bmatrix} \tag{60}$$

$$= \begin{bmatrix} | & & | \\ \lambda_1\vec{e}_1 & \cdots & \lambda_n\vec{e}_n \\ | & & | \end{bmatrix} \tag{61}$$

$$= \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \tag{62}$$

Note that $\vec{e}_i$ is the i-th column of the identity matrix I - this comes from setting the columns of $V^{-1}V = I$ equal to each other after distributing V^{-1} on the columns of V.

Followings are for the verification of eigenvalues and eigenvectors:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & \frac{1}{L} \\ \frac{-1}{C} & \frac{-1}{RC} - \lambda \end{vmatrix} = \lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC} = 0 \tag{63}$$

$$\lambda = -\frac{1}{2RC} \pm \frac{1}{2}\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}} = -50 \pm j\left(2 \times 10^6\right) \tag{64}$$

Assuming the eigenvectors have the following form: $\vec{v} = \begin{bmatrix} 1 \\ y \end{bmatrix}$, then we have:

$$A\vec{v} = \lambda\vec{v} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix}\begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} \lambda_i \\ y \times \lambda_i \end{bmatrix} \tag{65}$$

From the above, we can easily find that $y = L\lambda_i$. Based on this, we can calculate the eigenvectors:

$$v_1 = \begin{bmatrix} 1 \\ L\lambda_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.00125 + j50 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ L\lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.00125 - j50 \end{bmatrix} \tag{66}$$

(d) (12 pts) Now, suppose that our initial conditions for $I_L(0)$ and $V_C(0)$ have changed to the following:

$$\begin{bmatrix} I_L(0) \\ V_C(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{67}$$

Using the information from (c), **find $I_L(t)$ for $t \geq 0$ and write the answer in a form involving real exponentials and sinusoids. Does $I_L(t)$ converge as $t \to \infty$? If so, what does it converge to? Show your work.**

Solution:

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} b & -j0.01 \\ \bar{b} & j0.01 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -j0.01 \\ j0.01 \end{bmatrix} \tag{68}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \widetilde{x}_1(t) \\ \widetilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \widetilde{x}_1(t) \\ \widetilde{x}_2(t) \end{bmatrix} \tag{69}$$

$$\widetilde{x}_1(t) = K_1 e^{\lambda_1 t} \quad \text{and} \quad \widetilde{x}_1(0) = -j0.01 \tag{70}$$

$$\widetilde{x}_2(t) = K_2 e^{\lambda_2 t} \quad \text{and} \quad \widetilde{x}_1(0) = j0.01 \tag{71}$$

Solving above yields follows:

$$\widetilde{x}_1(t) = (-j0.01)\, e^{\lambda_1 t} \tag{72}$$

$$\widetilde{x}_2(t) = (j0.01)\, e^{\lambda_2 t} \tag{73}$$

Finally, we can calculate $x(t)$ by computing $V\vec{\tilde{x}}(t)$:

$$\begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix} = \vec{x}(t) = V\vec{\tilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ a & \bar{a} \end{bmatrix} \begin{bmatrix} (-j0.01)\, e^{\lambda_1 t} \\ (j0.01)\, e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ a & \bar{a} \end{bmatrix} \begin{bmatrix} (-j0.01)\, e^{(-Z_0 + j\omega_0)t} \\ (j0.01)\, e^{(-Z_0 - j\omega_0)t} \end{bmatrix} \tag{74}$$

$$I_L(t) = -j0.01 e^{-Z_0 t} \left(e^{j\omega_0 t} - e^{-j\omega_0 t} \right) = 2j \times -j0.01 e^{-Z_0 t} \frac{\left(e^{j\omega_0 t} - e^{-j\omega_0 t} \right)}{2j} \tag{75}$$

$$\therefore I_L(t) = 0.02 e^{-Z_0 t} \sin(\omega_0 t) \tag{76}$$

Regarding convergence, we have for the current $\displaystyle\lim_{t \to \infty} I_L(t) = 0$ since the $e^{-Z_0 t}$ term goes to 0 as $t \to \infty$.

Verification:

$$\begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix} = V\vec{\tilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ -0.00125 + j50 & -0.00125 - j50 \end{bmatrix} \begin{bmatrix} (-j0.01)\, e^{(-Z_0 + j\omega_0)t} \\ (j0.01)\, e^{(-Z_0 - j\omega_0)t} \end{bmatrix} \tag{77}$$

$$I_L(t) = 0.02 e^{-Z_0 t} \sin(\omega_0 t) \tag{78}$$

$$V_C(t) = e^{-Z_0 t} \cos(\omega_0 t) - (25 \times 10^{-6}) e^{-Z_0 t} \sin(\omega_0 t) \tag{79}$$

Let us check if above satisfies $V_C(t) = V_L(t) = L\frac{dI_L(t)}{dt}$.

$$L\frac{dI_L(t)}{dt} = (25 \times 10^{-6})(0.02)e^{-Z_0 t}\left(\omega_0 \cos(\omega_0 t) - Z_0 \sin(\omega_0 t)\right) \tag{80}$$

$$= e^{-Z_0 t}\cos(\omega_0 t) - (25 \times 10^{-6})e^{-Z_0 t}\sin(\omega_0 t) \tag{81}$$

Therefore, the pair $(I_L(t)$ and $V_C(t))$ satisfies the differential equation.

[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.

If needed, you can also use this space to work on problems. But if you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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