

## EE 16B Midterm 2, March 21, 2017

Name: Solutions for HW

SID #: \_\_\_\_\_

Discussion Section and TA: \_\_\_\_\_

Lab Section and TA: \_\_\_\_\_

Name of left neighbor: \_\_\_\_\_

Name of right neighbor: \_\_\_\_\_

### Important Instructions:

- **Show your work.** An answer without explanation is not acceptable and does not guarantee any credit.
- **Only the front pages will be scanned and graded.** You can use the back pages as scratch paper.
- **Do not remove pages,** as this disrupts the scanning. Instead, cross the parts that you don't want us to grade.

Problem	Points
1	10
2	15
3	10
4	20
5	15
6	15
7	15
Total	100

1. (10 points) The thirteenth century Italian mathematician Fibonacci described the growth of a rabbit population by the recurrence relation:

$$y(t+2) = y(t+1) + y(t)$$

where  $y(t)$  denotes the number of rabbits at month  $t$ . A sequence generated by this relation from initial values  $y(0)$ ,  $y(1)$  is known as a Fibonacci sequence.

a) (5 points) Bring the recurrence relation above to the state space form using the variables  $x_1(t) = y(t)$  and  $x_2(t) = y(t+1)$ .

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

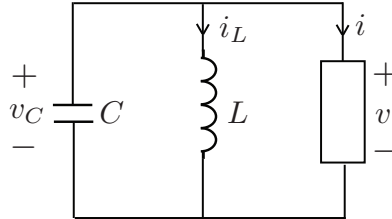
b) (5 points) Determine the stability of this system.

Unstable



2. (15 points) Consider the circuit below that consists of a capacitor, an inductor, and a third element with the nonlinear voltage-current characteristic:

$$i = -v + v^3.$$



a) (5 points) Write a state space model of the form

$$\begin{aligned} \frac{dx_1(t)}{dt} &= f_1(x_1(t), x_2(t)) \\ \frac{dx_2(t)}{dt} &= f_2(x_1(t), x_2(t)) \end{aligned}$$

using the states  $x_1(t) = v_C(t)$  and  $x_2(t) = i_L(t)$ .

$$f_1(x_1, x_2) = \frac{1}{C}(x_2 - x_1^3 - x_2)$$

$$f_2(x_1, x_2) = \frac{1}{L} x_1$$

b) (5 points) Linearize the state model at the equilibrium  $x_1 = x_2 = 0$  and specify the resulting  $A$  matrix.

$$A = \begin{bmatrix} \frac{1}{c} & -\frac{1}{c} \\ \frac{1}{L} & 0 \end{bmatrix}$$

c) (5 points) Determine stability based on the linearization.

Unstable

3. (10 points) Consider the discrete-time system

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

a) (5 points) Determine if the system is controllable.

Not controllable

b) (5 points) Explain whether or not it is possible to move the state vector from  $\vec{x}(0) = 0$  to

$$\vec{x}(T) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

If your answer is yes, specify the smallest possible time  $T$  and an input sequence  $u(0), \dots, u(T-1)$  to accomplish this task.

yes  $T=2$

$$u(0) = 2$$

$$u(1) = 1$$



4. (20 points) Consider the system

$$\vec{x}(t+1) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

where  $\theta$  is a constant.

a) (5 points) For which values of  $\theta$  is the system controllable?

Controllable for all

$$\theta \neq k\pi$$

where  $k$  is an integer

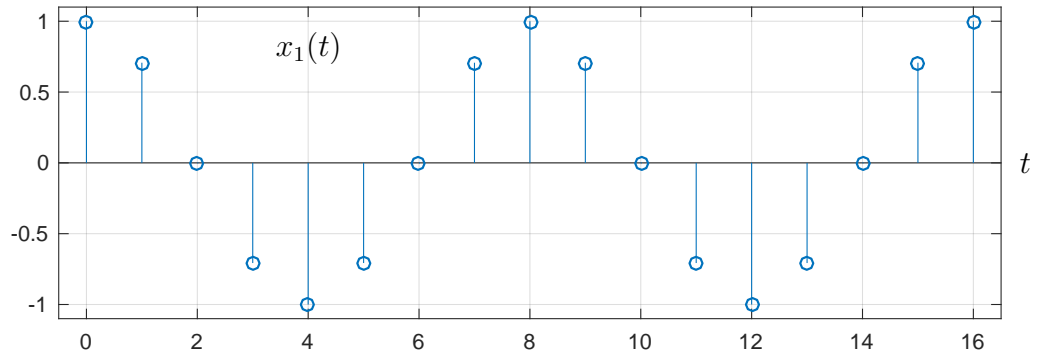
b) (10 points) Select the coefficients  $k_1, k_2$  of the state feedback controller

$$u(t) = k_1 x_1(t) + k_2 x_2(t)$$

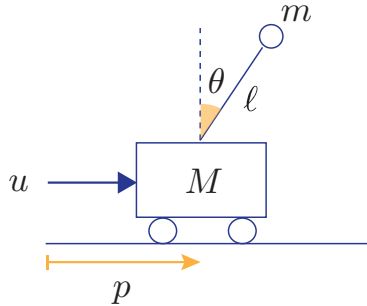
such that the closed-loop eigenvalues are  $\lambda_1 = \lambda_2 = 0$ . Your answer should be symbolic and well-defined for the values of  $\theta$  you specified in part (a).

Additional workspace for Problem 4b.

c) (5 points) Suppose the state variable  $x_1(t)$  evolves as depicted below when no control is applied ( $u = 0$ ). What is the value of  $\theta$ ?



5. (15 points) Consider the inverted pendulum below, where  $p(t)$  is the position of the cart,  $\theta(t)$  is the angle of the pendulum, and  $u(t)$  is the input force.



When linearized about the upright position, the equations of motion are

$$\begin{aligned}\ddot{p}(t) &= -\frac{m}{M}g\theta(t) + \frac{1}{M}u(t) \\ \ddot{\theta}(t) &= \frac{M+m}{M\ell}g\theta(t) - \frac{1}{M\ell}u(t)\end{aligned}\tag{1}$$

where  $M$ ,  $m$ ,  $\ell$ ,  $g$  are positive constants.

a) (5 points) Using (1) write the state model for the vector

$$\vec{x}(t) = [p(t) \quad \dot{p}(t) \quad \theta(t) \quad \dot{\theta}(t)]^T.$$

b) (5 points) Suppose we measure only the position; that is, the output is  $y(t) = x_1(t)$ . Determine if the system is observable with this output.

c) (5 points) Suppose we measure only the angle; that is, the output is  $y(t) = x_3(t)$ . Determine if the system is observable with this output.

6. (15 points) Consider the system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_A \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad y(t) = \underbrace{[0 \quad 1 \quad 0]}_C \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

a) (5 points) Select values for  $\ell_1, \ell_2, \ell_3$  in the observer below such that  $\hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t)$  converge to the true state variables  $\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)$  respectively.

$$\begin{bmatrix} \hat{x}_1(t+1) \\ \hat{x}_2(t+1) \\ \hat{x}_3(t+1) \end{bmatrix} = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix}}_L (\hat{x}_2(t) - y(t)).$$

Additional workspace for Problem 6a.



b) (5 points) Professor Arcak found a solution to part (a) that guarantees convergence of  $\hat{x}_3(t)$  to  $x_3(t)$  in one time step; that is

$$\hat{x}_3(t) = x_3(t) \quad t = 1, 2, 3, \dots$$

for any initial  $\vec{x}(0)$  and  $\hat{x}(0)$ . Determine his  $\ell_3$  value based on this behavior of the observer. Explain your reasoning.

c) (5 points) When Professor Arcak solved part (a), he found the convergence of  $\hat{x}_1(t)$  to  $x_1(t)$  to be rather slow no matter what  $L$  he chose. Explain the reason why no choice of  $L$  can change the convergence rate of  $\hat{x}_1(t)$  to  $x_1(t)$ .

7. (15 points) Consider a system with the symmetric form

$$\frac{d}{dt} \begin{bmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{bmatrix} = \begin{bmatrix} F & H \\ H & F \end{bmatrix} \begin{bmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{bmatrix} + \begin{bmatrix} G \\ G \end{bmatrix} \vec{u}(t), \quad (2)$$

where  $\vec{x}_1$  and  $\vec{x}_2$  have identical dimensions and, therefore,  $F$  and  $H$  are square matrices.

a) (5 points) Define the new variables

$$\vec{z}_1 = \vec{x}_1 + \vec{x}_2 \quad \text{and} \quad \vec{z}_2 = \vec{x}_1 - \vec{x}_2,$$

and write a state model with respect to these variables:

$$\frac{d}{dt} \begin{bmatrix} \vec{z}_1(t) \\ \vec{z}_2(t) \end{bmatrix} = \begin{bmatrix} | & | \\ \hline & \\ \hline | & | \end{bmatrix} \begin{bmatrix} \vec{z}_1(t) \\ \vec{z}_2(t) \end{bmatrix} + \begin{bmatrix} | \\ \hline | \end{bmatrix} u(t).$$

b) (5 points) Show that the system (2) is not controllable.

c) (5 points) Write a state model for the circuit below using the inductor currents as the variables. Show that the model has the symmetric form (2).

