EECS 16B Designing Information Devices and Systems II
Spring 2019 UC Berkeley

Final Exam

Exam location: GPB 100

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ________________________________, ____________________
(first and last) (signature)

PRINT your discussion section and GSI(s) (the one(s) you attend): ________________________________

Row Number (front row is 1): __________ Seat Number (left most is 1): __________

Name and SID of the person to your left: ________________________________

Name and SID of the person to your right: ________________________________

Name and SID of the person in front of you: ________________________________

Name and SID of the person behind you: ________________________________

Section 0: Pre-exam questions (3 points)

1. **What has been your favorite course at UC Berkeley so far? (1 pt)**

2. **What would you do on your perfect vacation? Describe how you would feel. (2 pts)**

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

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3. PCA (14 pts)

In this problem, we are going to think of our data points as being given in **columns**. You can imagine that the data points are recordings from a microphone. We take many such recordings. Our goal is to identify the principal components so that we could, in the future, project fresh recordings from the microphone onto those principal components to help us better understand what was being said.

(a) **(2 pts)** Suppose for this part, that you have four observed data vectors (say corresponding to the same spoken word, being repeated four times) and all of them just happened to be multiples of the following 6-dimensional vector $\vec{v} = \begin{bmatrix} 3 \\ -4 \\ 5 \\ -5 \\ 4 \\ -3 \end{bmatrix}$. (For your convenience, note that $\|\vec{v}\| = 10$.)

You arrange the data vectors as the columns of a matrix A given by:

$$A = \begin{bmatrix} | & | & | & | \\ -\vec{v} & -2\vec{v} & \vec{v} & 2\vec{v} \\ | & | & | & | \end{bmatrix} \tag{1}$$

You want to perform PCA to better understand your data. **Find the first principal component vector of A to explain the nature of your data points.**

(HINT: You don't need to compute any covariance matrices or compute any eigenvalue/eigenvectors in this simple case. Also, be sure to think about what size vector you want as the answer. Don't forget to normalize!)

(b) **(6 pts)** Suppose that now, we have two more data points (say, corresponding to a *different* word being spoken twice) that are multiples of a different vector $\vec{p}$ where:

$$\vec{p} = \begin{bmatrix} \frac{36}{\sqrt{134}} \\ -\frac{8}{\sqrt{134}} \\ -\frac{28}{\sqrt{134}} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$ (For your convenience, note that $\|\vec{p}\| = 4$ and that $\vec{p}^T \vec{v} = 0$.)

We augment our data matrix with these two new data points to get:

$$A = \begin{bmatrix} | & | & | & | & | & | \\ -\vec{v} & -2\vec{v} & \vec{v} & 2\vec{v} & -4\vec{p} & 4\vec{p} \\ | & | & | & | & | & | \end{bmatrix} \tag{2}$$

Find the principal components corresponding to the nonzero singular values of A. **What is the first principal component vector? What is the second principal component vector? Justify your answer.**

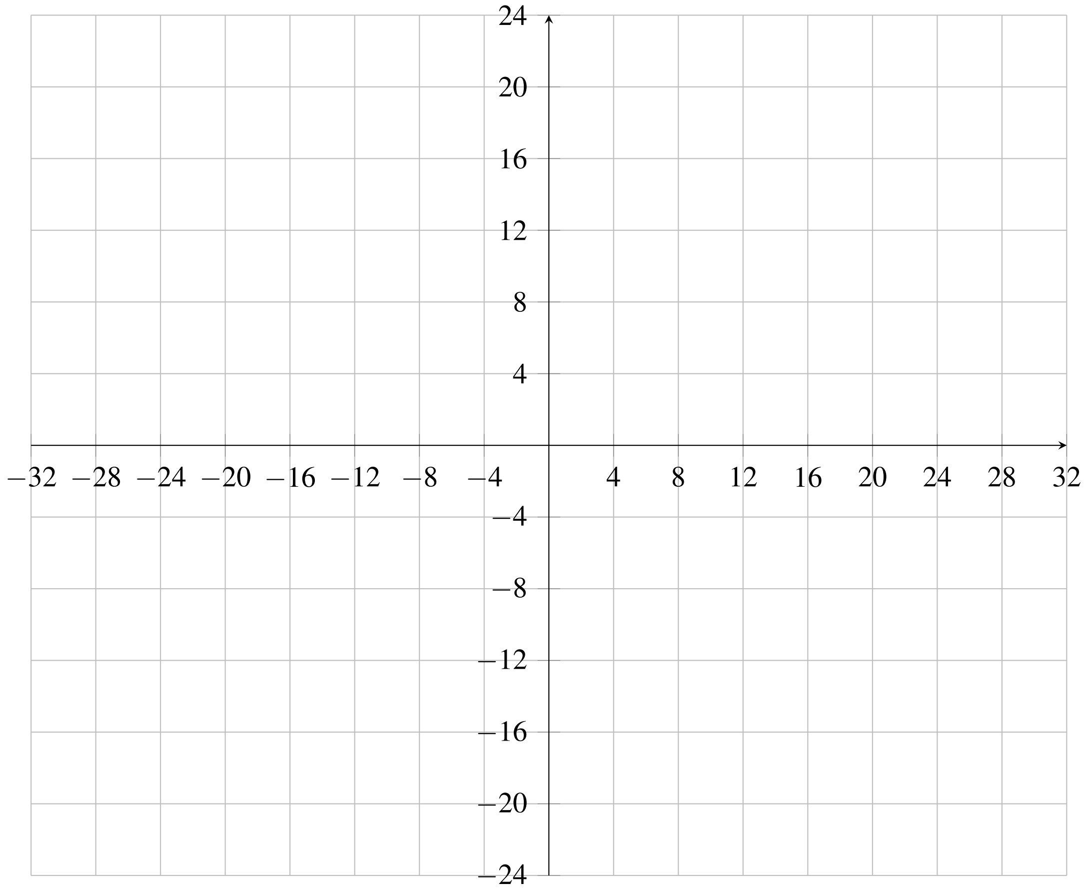
(Hint: Think about the inner product of $\vec{v}$ and $\vec{p}$ and what that implies for being able to appropriately decompose A. Again, very little computation is required here.)

(c) **(6 pts)** In the previous part, you had

$$A = \begin{bmatrix} | & | & | & | & | & | \\ -\vec{v} & -2\vec{v} & \vec{v} & 2\vec{v} & -4\vec{p} & 4\vec{p} \\ | & | & | & | & | & | \end{bmatrix}$$

with $\|\vec{v}\| = 10$ and $\|\vec{p}\| = 4$, satisfying $\vec{p}^T\vec{v} = 0$.

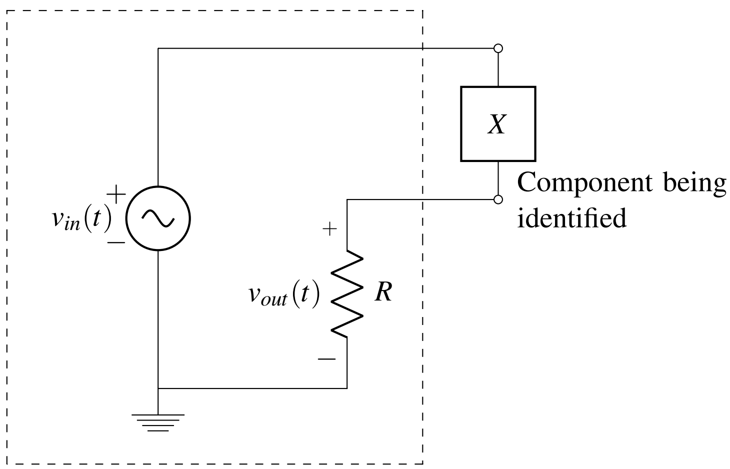
If we use $\vec{a}_i$ to denote the i-th column of A, **plot the data points $\vec{a}_i$ projected onto the first and second principal component vectors.** The coordinate along the first principal component should be represented by horizontal axis and the coordinate along the second principal component should be the vertical axis. **Label each point**.



4. Identifying an Unknown Circuit Component (24 pts)

Suppose we have an unknown circuit component, which we'll denote as X and represent with the symbol $\boxed{X}$. X could either be a resistor, a capacitor, or an inductor, but we don't know which one it is, nor do we know what its *component value* (that is, its resistance, capacitance, or inductance) could be. If you needed to identify X, that is figure out what kind of component X is and figure out its value, you would use a tool called an *RLC meter*. **In this problem, you will examine how an RLC meter can identify unknown circuit components with the help of transfer functions and the DFT**.

In circuit form, an RLC meter looks like this:



RLC meter

Here, $v_{in}(t) = A_{in}\cos(2\pi f_0 t + \theta_{in})$ is a known sinusoidal test input of known frequency f_0, known amplitude A_{in}, and known phase θ_{in}; while R is also a known resistance. Under this setup, we know that $v_{out}(t)$ will also be a sinusoid, which we'll denote as $v_{out}(t) = A_{out}\cos(2\pi f_0 t + \theta_{out})$.

When X is connected to the RLC meter, an on-board microcontroller takes samples from $v_{in}(t)$ and $v_{out}(t)$ and uses these samples to compute $Z_X|_{f_0}$, the impedance of the unknown component at frequency f_0. From the value of $Z_X|_{f_0}$, it can figure out whether X is a resistor, a capacitor, or an inductor, as well as what resistance, capacitance, or inductance it has.

(a) **(4 pts) Find the transfer function $H(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}}$ when the unknown component is connected to the RLC meter**. Here, $\widetilde{V}_{out}$ and $\widetilde{V}_{in}$ denote the phasor representations of $v_{out}(t)$ and $v_{in}(t)$, respectively. Answer in terms of R, the known resistance, and $Z_X(\omega)$, the unknown impedance.

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(b) **(4 pts)** Suppose that we know $H(\omega_0)$, that is the (possibly complex) numerical value of $H(\omega)$ at the angular frequency $\omega_0 = 2\pi f_0$. **Show how to use the value of $H(\omega_0)$ to calculate $Z_X|_{f_0}$, the impedance of the unknown component at the frequency** f_0. Your result should be an equation for $Z_X|_{f_0}$ in terms of quantities whose values we know.

(c) **(6 pts)** Suppose that we knew $Z_X|_{f_0}$. **Describe how to use $Z_X|_{f_0}$ to determine both what kind of component X is and the corresponding component value?** *(HINT: Physical resistances, capacitances, and inductances are always positive. And $\frac{1}{j} = -j$ for $j = \sqrt{-1}$.)*

(d) **(10 pts)** Now we just need to calculate $H(\omega_0)$ using the samples of $v_{in}(t) = A_{in}\cos(2\pi f_0 t + \theta_{in})$ and $v_{out}(t) = A_{out}\cos(2\pi f_0 t + \theta_{out})$ that the on-board microcontroller collected. Suppose we have collected N samples from $v_{in}(t)$ and $v_{out}(t)$ with sampling interval $\Delta = \frac{1}{f_0 N}$, and we have arranged these samples into N-dimensional vectors $\vec{v}_{in}$ and $\vec{v}_{out}$. These sample vectors have the form

$$\vec{v}_{in} = \begin{bmatrix} A_{in}\cos(\theta_{in}) \\ A_{in}\cos\big(2\pi f_0(\Delta) + \theta_{in}\big) \\ A_{in}\cos\big(2\pi f_0(2\Delta) + \theta_{in}\big) \\ \vdots \\ A_{in}\cos\big(2\pi f_0((N-1)\Delta) + \theta_{in}\big) \end{bmatrix}, \quad \vec{v}_{out} = \begin{bmatrix} A_{out}\cos(\theta_{out}) \\ A_{out}\cos\big(2\pi f_0(\Delta) + \theta_{out}\big) \\ A_{out}\cos\big(2\pi f_0(2\Delta) + \theta_{out}\big) \\ \vdots \\ A_{out}\cos\big(2\pi f_0((N-1)\Delta) + \theta_{out}\big) \end{bmatrix}. \tag{3}$$

Since $\widetilde{V}_{out} = \frac{1}{2}A_{out}e^{j\theta_{out}}$ and $\widetilde{V}_{in} = \frac{1}{2}A_{in}e^{j\theta_{in}}$, these samples contain information about the currently unknown phasors $\widetilde{V}_{out}$ and $\widetilde{V}_{in}$ at the frequency f_0. These phasors need to be recovered from the samples using the DFT. **Express the DFTs $\vec{V}_{in} = F_N\vec{v}_{in}$ and $\vec{V}_{out} = F_N\vec{v}_{out}$ symbolically in terms of the phasors $\widetilde{V}_{out}$ and $\widetilde{V}_{in}$, the standard basis vectors $\vec{e}_i$, and the number of samples N.**

Here, the DFT transformation matrix $F_N = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}1} & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}3} & \cdots & e^{-j\frac{2\pi}{N}(N-1)1} \\ 1 & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}4} & e^{-j\frac{2\pi}{N}6} & \cdots & e^{-j\frac{2\pi}{N}(N-1)2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi}{N}(N-1)} & e^{-j\frac{2\pi}{N}2(N-1)} & e^{-j\frac{2\pi}{N}3(N-1)} & \cdots & e^{-j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}$

and the inverse of this matrix is just $\frac{1}{N}F_N^*$.

5. DFT and Circuit Filters (30 pts)

You have been introduced to low-pass and high-pass filter circuits that pass some range of input signal frequencies while attenuating other ranges of signal frequency. You have also seen how we can break signals down and view the frequency components of sampled signals using the DFT. In this problem, we will see how we can combine these two bases of knowledge. Throughout this problem, if we have an N-dimensional vector $\vec{x}$, its DFT coefficients are given by the vector $\vec{X} = F_N\vec{x}$ where the DFT transformation matrix is

$$F_N = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}1} & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}3} & \cdots & e^{-j\frac{2\pi}{N}(N-1)1} \\ 1 & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}4} & e^{-j\frac{2\pi}{N}6} & \cdots & e^{-j\frac{2\pi}{N}(N-1)2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi}{N}(N-1)} & e^{-j\frac{2\pi}{N}2(N-1)} & e^{-j\frac{2\pi}{N}3(N-1)} & \cdots & e^{-j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}$$

and the inverse is $F_N^{-1} = \frac{1}{N}F_N^*$.

(a) **(6 pts)** If you sample every Δ seconds and you take N samples, the 0^{th} DFT coefficient $\vec{X}[0]$ corresponds to the DC (or constant) term. The 1^{st} DFT coefficient $\vec{X}[1]$ corresponds to the fundamental frequency $f_0 = \frac{1}{N\Delta}$.

Say you have a signal $v_{in}(t) = \cos\left(\frac{2\pi}{3}t\right) + \cos\left(\frac{2\pi}{9}t\right)$. You take $N = 9$ samples of the function every $\Delta = 1$ second; i.e. at $t = \{0, 1, 2, \ldots, 8\}$, forming a 9 element vector of samples $\vec{v}_{in}$. **What are the DFT coefficients $\vec{V}_{in}$ of the sampled signal $\vec{v}_{in}$?**

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(b) **(12 pts)** You are given the circuit below.

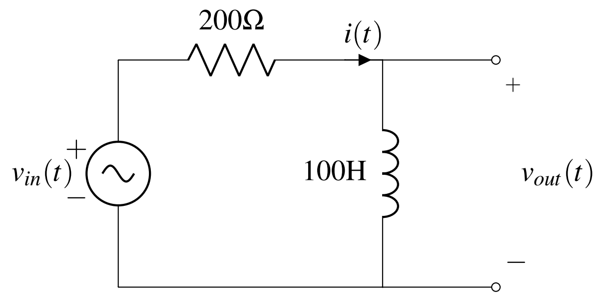
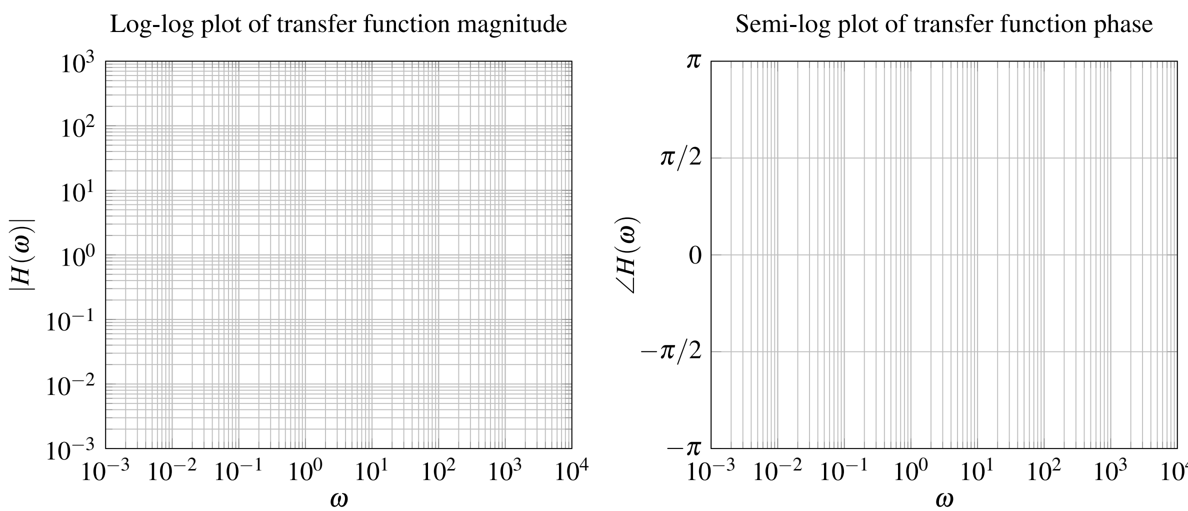


Figure 1: Filter circuit

Is this a high-pass or low-pass filter? What is its cutoff angular frequency, ω_c? Sketch the piecewise-linear approximations of the magnitude and phase Bode plots of the transfer function $H(\omega) = \frac{\widetilde{V}_{out}(\omega)}{\widetilde{V}_{in}(\omega)}$ below.



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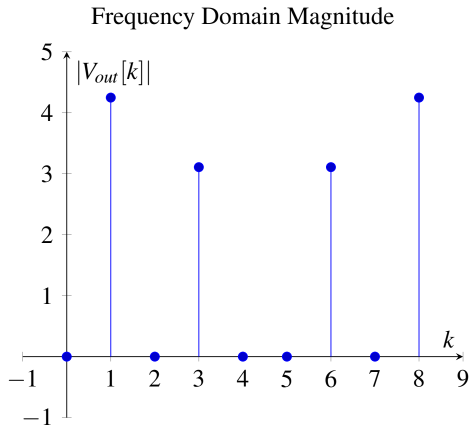
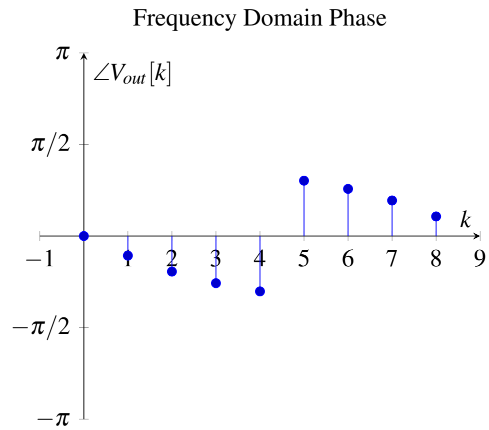
(c) **(12 pts)** The signal $v_{in}(t) = \cos\left(\frac{2\pi}{3}t\right) + \cos\left(\frac{2\pi}{9}t\right)$ is input into the circuit in Figure 1, giving output signal $v_{out}(t)$. You take $N = 9$ samples of the function $v_{out}(t)$ every $\Delta = 1$ seconds; i.e. at $t = \{0, 1, 2, \ldots, 8\}$, forming a 9 element vector of samples $\vec{v}_{out}$. We have given you several possible plots below that may represent the DFT coefficients $\vec{V}_{out}$ of the sampled signal $\vec{v}_{out}$. **For each of the four candidate solutions, circle the statement which is true. Provide a one-sentence explanation for your choice in the box provided.** *Reminder:* $\omega = 2\pi f$.

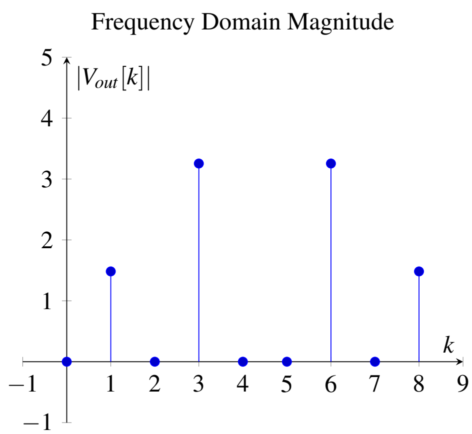
(HINT: Exactly one of the candidate solutions below is correct. Consequently, no precise numerical calculations are required to get full credit.)

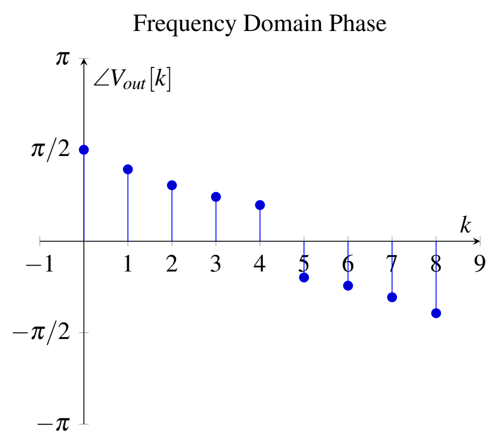




Circle one: Correct $\vec{V}_{out}$ Incorrect $\vec{V}_{out}$

Reason:





Circle one: Correct $\vec{V}_{out}$ Incorrect $\vec{V}_{out}$

Reason:

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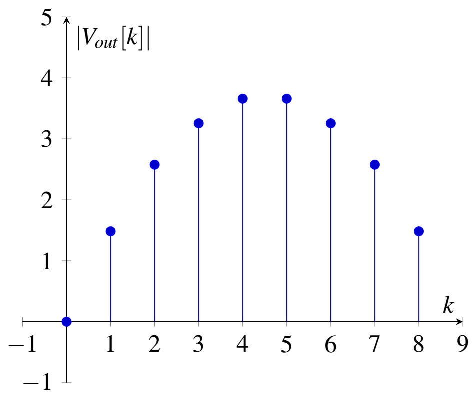
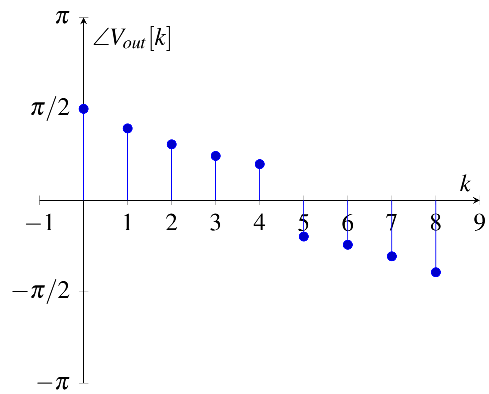
Frequency Domain Magnitude



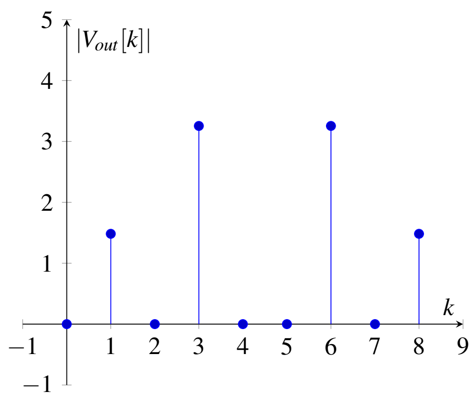
Frequency Domain Phase



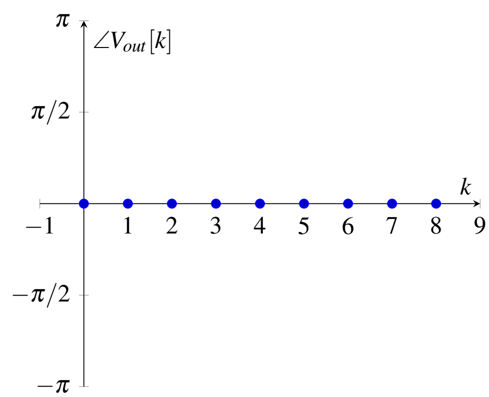
Circle one: Correct $\vec{V}_{out}$ Incorrect $\vec{V}_{out}$

Reason:

Frequency Domain Magnitude



Frequency Domain Phase



Circle one: Correct $\vec{V}_{out}$ Incorrect $\vec{V}_{out}$

Reason:

6. Lagrange Polynomials (24 pts)

In this question, we consider the interpolation of a function $f(x)$, at N points $x_0, \ldots, x_{N-1}$. The samples are collected in vector $\vec{f}_N = [f(x_0), f(x_1), \ldots, f(x_{N-1})]^\top$. The k-th component of $\vec{f}_N$ is denoted by $f_N[k]$.

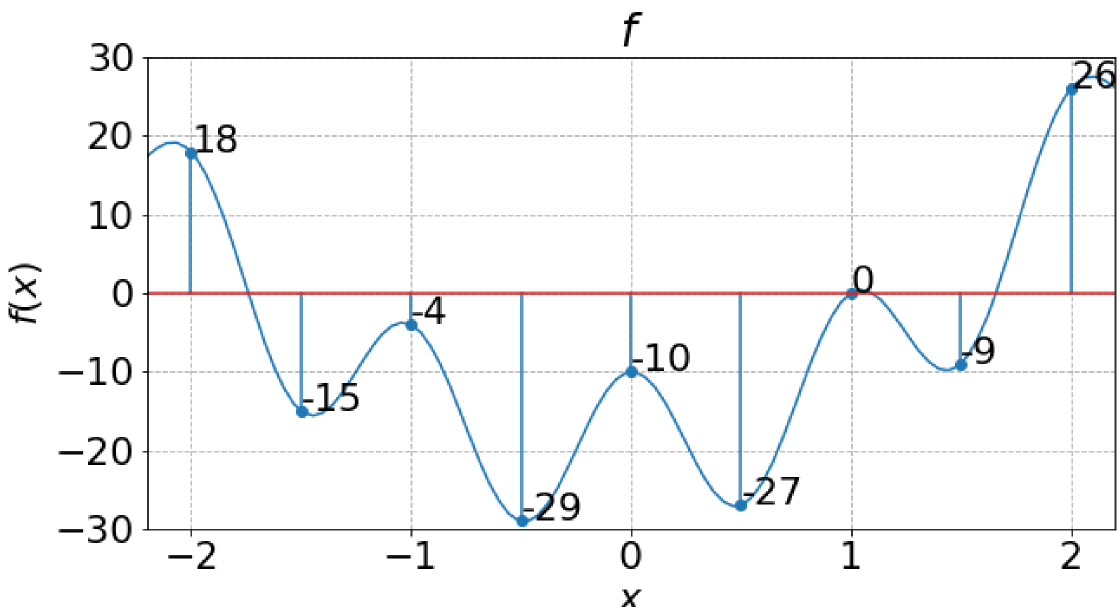


Figure 2: Plot of an example function $f(x)$

(a) **(2 pts)** For function $f(x)$ given in Figure 2, **give the vector $\vec{f}_8$ of samples for** $x_k = -2 + \frac{k}{2}$**,** $k = 0, \ldots, 7$**.**

(b) **(6 pts)** Recall the family of Lagrange polynomials $\{L_i\}$ of degree at most $N-1$ from discussion and homework. For all $i = 0, \ldots, N-1$, the polynomial L_i is of degree at most $N-1$ and is given by:

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{N-1} \left(\frac{x - x_j}{x_i - x_j} \right) \tag{4}$$

Explicitly write out the vector $\vec{\ell}_i = [L_i(x_0), L_i(x_1), \ldots, L_i(x_{N-1})]^\top$ of samples for the i-th Lagrange polynomial L_i sampled at $x_0, x_1, \ldots, x_i, \ldots, x_{N-1}$, and argue why this family of vectors $\{\vec{\ell}_i\}$ is orthonormal.

(c) **(4 pts)** For a sample vector $\vec{f}_N$, the polynomial H that interpolates it can be written $H(x) = \sum_{i=0}^{N-1} b[i] L_i(x)$. **Write $b[i]$ in terms of $\vec{f}_N$**, using the special properties of the Lagrange polynomials that you found in the previous part.

(Hint: Interpolation means that $H(x_j) = f_N[j]$ for $j = 0, \ldots, N-1$.)

(d) **(8 pts)** The same polynomial H can also be written as follows: For all x, $H(x) = \sum_{j=0}^{N-1} a[j]x^j$. For $j = 0, \ldots, N-1$, **write $a[j]$ in terms of $\vec{f}_N$ using the matrix form $\vec{a} = P\vec{f}_N$. What is the matrix P here?**

It is fine to leave your result for P in terms of other matrices and matrix operations, as long as it is explicit and unambiguous.

(e) **(4 pts)** If the samples are taken at $x_k = \omega_N^k = e^{j\frac{2\pi}{N}k}$, **do you recognize the P matrix that gives us the coefficients of the interpolating polynomial? What is it?**

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

7. Observability Lost (10 pts)

In this problem, we will be considering discrete-time systems with outputs, that is systems of the form

$$\vec{x}_d(t+1) = A\vec{x}_d(t) + B\vec{u}_d(t) \tag{5}$$
$$y_d(t) = C\vec{x}_d(t). \tag{6}$$

As you know, the overwhelming majority of such systems are observable. Nevertheless, there are some instances where systems that are not observable *do* arise.

Suppose that at least one of the eigenvectors of A is in the nullspace of C. That is, there exists at least one vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$ and $C\vec{v} = \vec{0}$. **Prove that, under these conditions, the system cannot be observable**.

Here, feel free to assume that $C = c^T$ is a row vector and that $y_d(t)$ is a scalar valued function of time.

8. Real Eigenvalues (15 pts)

Suppose S is a complex matrix that can be written in the form $S = B^*B$, for some other complex matrix B. **Show that the eigenvalues of S are all real and non-negative.**

(Hint: Remember that $\vec{v}^*\vec{v} = \|\vec{v}\|^2 \geq 0$ for all $\vec{v}$ and that the norm $\|\cdot\|$ is always real valued, even for complex vectors.)

9. Weighted minimum norm (25 pts)

You saw in lecture in the context of open-loop control, how we consider problems in which we have a wide matrix A and solve $A\vec{x} = \vec{y}$ such that $\vec{x}$ is a minimum norm solution:

$$||\vec{x}|| \leq ||\vec{z}||$$

for all $\vec{z}$ such that $A\vec{z} = \vec{y}$. You then saw this idea again in the homeworks in the context of MIMO communication and also worked out how to compute the appropriate "pseudo-inverse" for such wide matrices.

But what if you weren't interested in just the norm of $\vec{x}$? What if you instead cared about minimizing the norm of a linear transformation $C\vec{x}$? For example, suppose that controls were more or less costly at different times.

The problem can be written out mathematically as:

Given a wide matrix A and a matrix C find $\vec{x}$ such that $A\vec{x} = \vec{y}$ and $||C\vec{x}|| \leq ||C\vec{z}||$ for all $\vec{z}$ such that $A\vec{z} = \vec{y}$.

(a) **(10 pts)** Let's start with the case of C being invertible. **Solve this problem (i.e. find the optimal $\vec{x}$ with the minimum $||C\vec{x}||$) for the specific matrices and $\vec{y}$ given below. Show your work.**
It is fine to leave your answer as an explicit product of matrices and vectors.
(HINT: You might want to change variables to solve this problem. Don't forget to change back!)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For convenience, $C^{-1} = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0 \end{bmatrix}$ and you are also given some SVDs on the following page.

$$A = (U_A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})(\Sigma_A = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix})(V_A^T = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}) \tag{7}$$

$$C = (U_C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix})(\Sigma_C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix})(V_C^T = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}) \tag{8}$$

$$AC = (U_{AC} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (\Sigma_{AC} = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix})(V_{AC}^T = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix}) \tag{9}$$

$$AC^{-1} = (U_{AC^{-1}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})(\Sigma_{AC^{-1}} = \begin{bmatrix} \frac{\sqrt{5}}{2} & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix})(V_{AC^{-1}}^T = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{bmatrix}) \tag{10}$$

(b) **(15 pts)** What if C were a tall matrix with linearly independent columns? **Explicitly describe how you would solve this problem in that case, step by step.**

For convenience, we have copied the problem statement again here: Given a wide matrix A and a matrix C **find** $\vec{x}$ such that $A\vec{x} = \vec{y}$ and $\|C\vec{x}\| \leq \|C\vec{z}\|$ for all $\vec{z}$ such that $A\vec{z} = \vec{y}$.

Here, you can assume that the wide matrix A has linearly-independent rows but is otherwise generic. Similarly, $\vec{y}$ is a generic vector.

(HINT: Does C have a nullspace? Does $C^T C$ have a nullspace? Does the SVD of C suggest any (invertible) change of coordinates from $\vec{x}$ to $\vec{\tilde{x}}$ such that $\|\vec{\tilde{x}}\| = \|C\vec{x}\|$?)

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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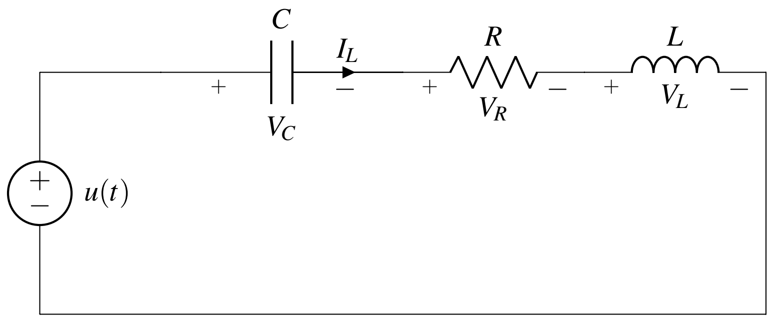
10. Circuit Discretization (18 pts)

Let's consider the following RLC circuit that you have encountered before.



(a) **(6 pts)** Find the matrix differential equation for the above system using the state-vector $\vec{x} = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}$ as

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t).$$

What is A? What is $\vec{b}$?

Your answers should be in terms of R, L, C.

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[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

(b) **(12 pts)** Now, assume for some specific component values we get the following differential equation:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t). \tag{11}$$

Unfortunately, we are unable to measure our state vector continuously. Suppose that we sample the system with some sampling interval Δ. Let us discretize the above system. Assume that we use piecewise constant voltage inputs $u(t) = u_d(k)$ for $t \in [k\Delta, (k+1)\Delta)$.

Recall from the homework that for a hypothetical scalar differential equation $\frac{d}{dt}x(t) = \lambda x(t) + bu(t)$, we can discretize it as long as $\lambda \neq 0$ as follows:

$$x_d(k+1) = e^{\lambda\Delta} x_d(k) + \frac{e^{\lambda\Delta} - 1}{\lambda} b u_d(k). \tag{12}$$

Here $x_d(k) = x(k\Delta)$.

Using equation (12), calculate the discrete-time system for Equation (11)'s continuous-time vector system in the form:

$$\vec{x}_d(k+1) = A_d \vec{x}_d(k) + \vec{b}_d u_d(k).$$

More concretely, find A_d and $\vec{b}_d$.

You do not need to multiply out any matrices. It is fine if you give your answers as explicit products of matrices/vectors/etc.

Hint: We have provided information regarding the matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ in (11) for your convenience (not all of this is needed) on the opposite page.

i. The determinant of A: $\det(A) = 2$.

ii. The trace of A: $\mathrm{tr}(A) = -3$.

iii. $A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$.

iv. We can diagonalize the matrix as $A = V\Lambda V^{-1}$, where, Λ is a diagonal matrix with the eigenvalues in its diagonal and the columns of V are the eigenvectors of the corresponding eigenvalues

v. The eigenvalues/eigenvectors for A are:

$$\text{For } \lambda_1 = -2 : \ \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{For } \lambda_2 = -1 : \ \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

vi. For $V = [\vec{v}_1, \vec{v}_2]$, we have $V^{-1} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$.

11. Discretization With Piecewise Linear Controls (18 pts)

In most of this course, when discretizing a continuous-time control system, we forced the input to be constant between time steps, *i.e.*, between some $k\Delta$ and $(k+1)\Delta$, (this is alternatively called a zero-order hold) and then changed it instantly and discontinuously to its new value. However, applying such a discontinuous control might be physically impossible for a real-world system. Suppose we decided instead to use something piecewise-linear (see Figure 3) for our continuous-time input.

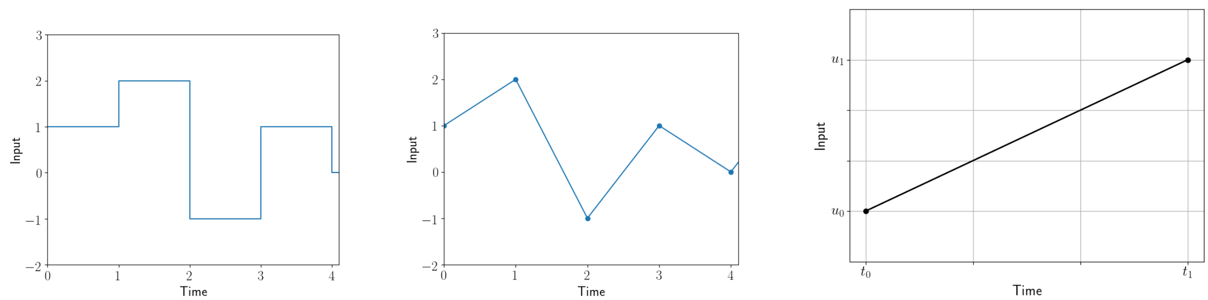

Figure 3: Piecewise constant vs. piecewise linear control inputs and a pure affine control input.

Consider a scalar differential-equation with scalar input $u(t)$:

$$\frac{d}{dt}x(t) = \lambda x(t) + bu(t) \tag{13}$$

with an initial condition $x(t_0)$. If we use a pure affine input (see the third panel in the figure above) for $u(t)$, we get the following continuous-time scalar differential equation:

$$\frac{d}{dt}x(t) = \lambda x(t) + b\left(m(t-t_0) + u_0\right) \tag{14}$$

where $m = \left(\frac{u_1-u_0}{t_1-t_0}\right)$ is the slope of the input $u(t)$ and u_0 is where the input $u(t)$ starts at time t_0, with u_1 being where the input $u(t)$ ends at time $t_1 > t_0$. Assuming $\lambda \neq 0$, solving this differential equation (14) for an arbitrary initial condition $x(t_0)$, we get the following solution for all $t_0 \leq t \leq t_1$:

$$x(t) = x(t_0)e^{\lambda(t-t_0)} - \frac{b}{\lambda}m(t-t_0) + \frac{b}{\lambda}\left(\frac{m}{\lambda} + u_0\right)(e^{\lambda(t-t_0)} - 1). \tag{15}$$

The goal in this problem is to extend (15) to let us discretize the continuous-time differential equation (13) under piecewise-linear inputs. The twist comes from the fact that each linear segment is defined by two numbers — “slope” and “intercept.”

(a) **(6 pts)** The first step in discretizing Equation (13) is to consider each discrete time step (between $t = k\Delta$ and $t = (k+1)\Delta$) as virtually giving us not one, but two discrete-time inputs $\begin{bmatrix} s_d(k) \\ m_d(k) \end{bmatrix}$. Namely: $s_d(k) = u(k\Delta)$, the "intercept" where the input $u(t)$ starts for this interval and $m_d(k) = \frac{u((k+1)\Delta)-u(k\Delta)}{\Delta}$ as the "slope" of the $u(t)$ input in the interval between $t = k\Delta$ and $t = (k+1)\Delta$.

We can write the behavior of the discrete-time state $x_d(k) = x(k\Delta)$ as obeying a scalar discrete-time controlled recurrence relation:

$$x_d(k+1) = \lambda_d x_d(k) + b_{d,m} m_d(k) + b_{d,s} s_d(k). \tag{16}$$

What are λ_d, $b_{d,m}$, $b_{d,s}$ in terms of the given λ, Δ, b?

(Hint: use Equation (15) *as appropriate.)*

(b) **(4 pts)** We want to understand how this system behaves in discrete-time as a function of the sequence of endpoints $u_d(k) = u((k+1)\Delta)$ of the piecewise constant input $u(t)$.

In reality, both the $m_d(k)$ input and the $s_d(k)$ input depend on $u_d(k)$ and $u_d(k-1)$. **Find $b_{d,1}$ and $b_{d,2}$ (in terms of the $\Delta, \lambda_d, b_{d,m}, b_{d,s}$ from above) so that Equation** (16) **can be rewritten as:**

$$x_d(k+1) = \lambda_d x_d(k) + b_{d,1} u_d(k) + b_{d,2} u_d(k-1). \tag{17}$$

(HINT: remember how $m_d(k)$ was defined. And that $u_d(k)$ is the $u(t)$ at the end of the interval and $u_d(k-1)$ is the $u(t)$ at the beginning of the interval from $[k\Delta, (k+1)\Delta]$.)

(c) **(8 pts)** To get this into standard vector/matrix form, we realize that we need to remember $u_d(k-1)$ for the next time step. Everything that needs to be remembered has be a part of the state, and so let's augment our state vector as

$$\widetilde{\vec{x}}_d(k) = \begin{bmatrix} x_d(k) \\ u_d(k-1) \end{bmatrix}.$$

Starting with Equation (17), write a matrix time-evolution equation using $\widetilde{\vec{x}}_d$ as

$$\widetilde{\vec{x}}_d(k+1) = A_d \widetilde{\vec{x}}_d(k) + \vec{b}_d u_d(k).$$

More concretely, find A_d and $\vec{b}_d$, in terms of λ_d, $b_{d,1}$ and $b_{d,2}$.

PRINT your name and student ID: ____________________

[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.]