EECS 16B   Designing Information Devices and Systems II
Spring 2019   UC Berkeley
Midterm 2

Exam location: Dwinelle 145

PRINT your student ID: _____

PRINT AND SIGN your name: _____, _____ _____
                                    (last)   (first)   (signature)

PRINT your discussion section and GSI (the one you attend): _____

Row Number (front row is 1): _____   Seat Number (left most is 1): _____

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Section 0: Pre-exam questions (3 points)

1. What has been your favorite part of 16B so far? (1 pt)

2. What are you most looking forward to about summer? (2 pts)

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

3. System Identification (24 pts)

You are given a discrete-time system as a black-box. You don't know the specifics of the system but you know that it takes one scalar input and has two states that you can observe. You assume that the system is linear and of the form

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t) + \vec{w}(t), \tag{1}$$

where $\vec{w}(t)$ is an external unseen disturbance that you hope is small, $u(t)$ is a scalar input, and

$$A = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix}. \tag{2}$$

You want to identify the system parameters from measured data. You need to find the unknowns: a_0, a_1, a_2, a_3, b_0 and b_1, however, you can only interact with the system via a blackbox model.

(a) **(6 pts)** You observe that the system has state $\vec{x}(t) = [x_0(t), x_1(t)]^T$ at time t. You pass input $u(t)$ into the blackbox and observe the next state of the system: $\vec{x}(t+1) = [x_0(t+1), x_1(t+1)]^T$.

Write scalar equations for the new states, $x_0(t+1)$ and $x_1(t+1)$. Write these equations in terms of the a_i, b_i, the states $x_0(t)$, $x_1(t)$ and the input $u(t)$. Here, assume that $\vec{w}(t) = \vec{0}$ (i.e. the model is perfect).

(b) **(4 pts)** Now we want to identify the system parameters. We observe the system at the start state $\vec{x}(0) = \begin{bmatrix} x_0(0) \\ x_1(0) \end{bmatrix}$. We can then input $u(0)$ and observe the next state $\vec{x}(1) = \begin{bmatrix} x_0(1) \\ x_1(1) \end{bmatrix}$. We can continue this for an m long sequence of inputs.

Let us define an m long trace to be $[x_0(0), x_1(0), u(0), x_0(1), x_1(1), u(1), x_0(2), x_1(2), u(2), \ldots, x_0(m-1), x_1(m-1), u(m-1), x_0(m), x_1(m)]$. **What is the minimum value of m you need to identify the system parameters?**

(c) **(8 pts)** Say we feed in a total of 4 inputs $[u(0), u(1), u(2), u(3)]$ into our blackbox. This allows us to observe the following states $[x_0(0), x_0(1), x_0(2), x_0(3), x_0(4)]$ and $[x_1(0), x_1(1), x_1(2), x_1(3), x_1(4)]$, which we can use to identify the system.

To identify the system we need to set up an approximate (because of potential disturbances) matrix equation

$$D\vec{s} \approx \vec{y}$$

using the observed values above and the unknown parameters we want to find. Suppose you are given the form of D in terms of some of the observed data:

$$D = \begin{bmatrix} x_0(0) & x_1(0) & u(0) & 0 & 0 & 0 \\ x_0(1) & x_1(1) & u(1) & 0 & 0 & 0 \\ x_0(2) & x_1(2) & u(2) & 0 & 0 & 0 \\ x_0(3) & x_1(3) & u(3) & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0(0) & x_1(0) & u(0) \\ 0 & 0 & 0 & x_0(1) & x_1(1) & u(1) \\ 0 & 0 & 0 & x_0(2) & x_1(2) & u(2) \\ 0 & 0 & 0 & x_0(3) & x_1(3) & u(3) \end{bmatrix}. \tag{3}$$

For this D, **what are $\vec{y}$ and the unknowns $\vec{s}$ so that $D\vec{s} \approx \vec{y}$ makes sense?** Tell us what the components of these vectors are, written in vector form.

(Feel free to assume the columns of D are linearly independent if that is causing concern.)

(Hint: Recall what you did in lab.)

(d) **(6 pts)** Now that we have set up $D\vec{s} \approx \vec{y}$, **explain how you would use this approximate equation to estimate the unknown values** a_0, a_1, a_2, a_3, b_0 **and** b_1**.** In particular, give an expression for your estimate $\widehat{\vec{s}}$ for the unknowns in terms of the D and $\vec{y}$.

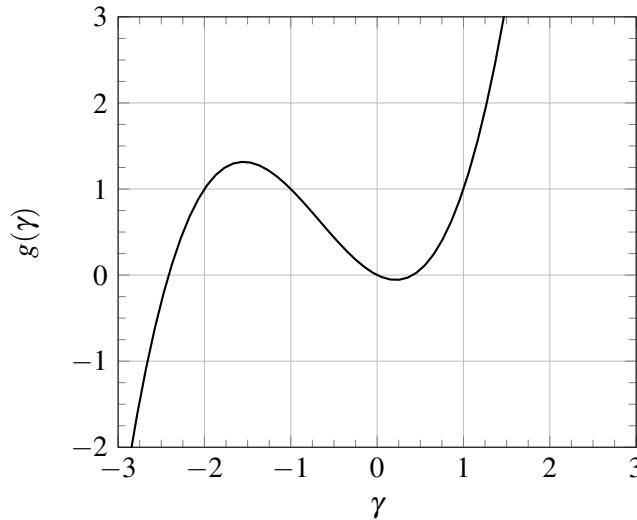*(HINT: Don't forget that D is not a square matrix. It is taller than it is wide.)*

4. Linearizing a Two-state System (14 pts)

We have a two-state nonlinear system defined by the following differential equation:

$$\frac{d}{dt}\begin{bmatrix} \beta(t) \\ \gamma(t) \end{bmatrix} = \frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -2\beta(t) + \gamma(t) \\ g(\gamma(t)) + u(t) \end{bmatrix} = \vec{f}(\vec{x}(t), u(t)) \tag{4}$$

where $\vec{x}(t) = \begin{bmatrix} \beta(t) \\ \gamma(t) \end{bmatrix}$ and $g(\cdot)$ is a nonlinear function with the following graph:

The $g(\cdot)$ is the only nonlinearity in this system. We want to linearize this entire system around a DC operating point.

(a) **(6 pts)** Take the DC operating input to be $u^\star = -1$. For this choice of $u^\star$, this system has three DC operating points. In other words, there are three states, $\vec{x}_1^\star$, $\vec{x}_2^\star$, and $\vec{x}_3^\star$, such that $\vec{f}(\vec{x}_1^\star, u^\star) = \vec{f}(\vec{x}_2^\star, u^\star) = \vec{f}(\vec{x}_3^\star, u^\star) = \vec{0}$. To be unambiguous, order them by increasing γ coordinates.

For this part, **find $\vec{x}_1^\star$, $\vec{x}_2^\star$, and $\vec{x}_3^\star$**. Your argument needn't be algebraic, especially since you don't have an expression for $g(\gamma)$. A graphical argument, using the above plot of $g(\gamma)$, will suffice for the γ parts.

*(HINT: understand what is happening with γ first, then figure out β.)*

(b) **(8 pts)** Now that you have the three DC operating points, **linearize the system about the DC operating point $(\vec{x}_3^\star, u^\star)$ that has the largest value for** γ. Specifically, what we want is as follows. Let $\vec{\delta x}_i(t) = \vec{x}(t) - \vec{x}_i^\star$ for $i = 1, 2, 3$, and $\delta u(t) = u(t) - u^\star$. We can in principle write the *linearized system* for each DC operating point in the following form:

$$\text{(linearization about } (\vec{x}_i^\star, u^\star)) \quad \frac{d}{dt}\vec{\delta x}_i(t) = A_i\vec{\delta x}_i(t) + B_i\delta u(t) + \vec{w}_i(t) \tag{5}$$

where $\vec{w}_i(t)$ is a disturbance that also includes the approximation error due to linearization.

For this part, **find A_3 and B_3.**

We have provided below the function $g(\gamma)$ and its derivative $\frac{\partial g}{\partial \gamma}$.
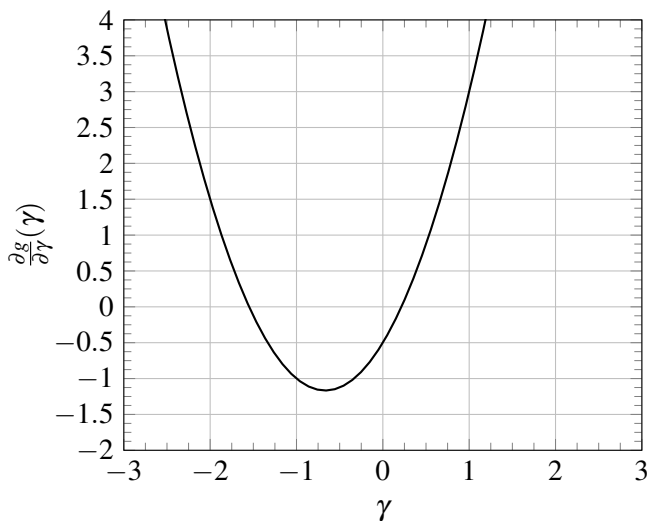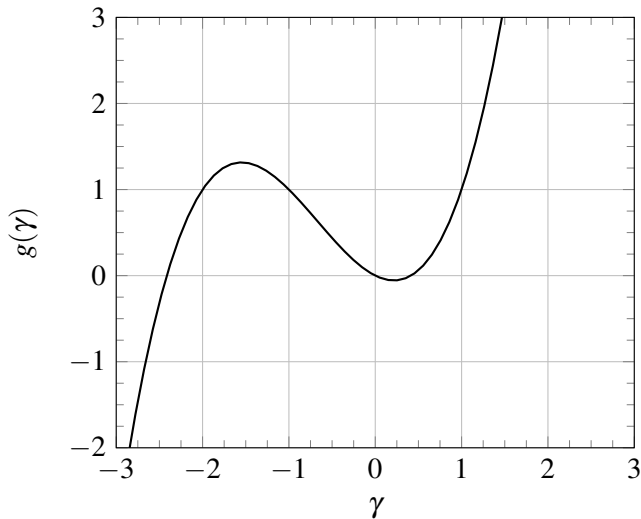
[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

5. Smog Control (28 pts)

Smog is created with the emission of nitric oxide NO interacting with ozone O_3.

Let's examine the regulation of nitric oxide and ozone to see how we might control smog in our cities. We set our state vector to be

$$\vec{x} = \begin{bmatrix} N_{\text{NO}} \\ N_{\text{O}_3} \end{bmatrix},$$

where N_{NO} is the amount of NO molecules and N_{O_3} is the amount of O_3 molecules in the air. Say we have a simplified discrete-time model where the evolution of the molecules every t weeks is given by

$$\vec{x}(t+1) = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{6} \end{bmatrix} \vec{x}(t).$$

(a) **(2 pts) Is the system stable? Why or why not?**

Here, we give you that

$$A = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

and that the characteristic polynomial $\det(\lambda I - A) = \lambda^2 - \frac{13}{6}\lambda + 1$.

(b) **(6 pts)** Suppose that we can use chemical methods to capture NO from the air. We can model the effects of these controls as an input $u(t)$ to our system:

$$\vec{x}(t+1) = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{6} \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t).$$

Is the system controllable?

(c) **(10 pts)** We want to convert the system

$$\vec{x}(t+1) = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{6} \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

to controllable canonical form, which in this case is given by

$$\vec{z}(t+1) = \begin{bmatrix} 0 & 1 \\ -1 & \frac{13}{6} \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

where $\vec{z}(t)$ is an appropriately transformed version of $\vec{x}(t)$; that is, $\vec{z}(t) = T\vec{x}(t)$. **Find the transformation T.**

(d) **(10 pts)** In order to keep smog under control, we want to set the closed-loop eigenvalues of the system

$$\vec{z}(t+1) = \begin{bmatrix} 0 & 1 \\ -1 & \frac{13}{6} \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

to be $\lambda_1 = \frac{2}{3}, \lambda_2 = \frac{1}{6}$ using state feedback

$$u(t) = - \begin{bmatrix} \widetilde{k}_1 & \widetilde{k}_2 \end{bmatrix} \vec{z}(t).$$

What specific numeric values of $\widetilde{k}_1$ and $\widetilde{k}_2$ should we use?

If $\vec{z}(t) = T\vec{x}(t)$, **how can you use T to get what the control gains $\vec{k}$ should be so that $u(t) = -\vec{k}^T \vec{x}(t)$?**
You don't need to compute explicit numeric values for the entries of $\vec{k}$.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

6. SVD (8 pts)

(a) **(8 pts)** Given the matrix

$$A = \frac{1}{\sqrt{50}} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} + \frac{3}{\sqrt{50}} \begin{bmatrix} -4 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix},$$

write out a singular value decomposition of matrix A in the form $U\Sigma V^T$. Note the ordering of the singular values in Σ should be from the largest to smallest.

HINT: You don't have to compute any eigenvalues for this. Some useful observations are that

$$[3,4]\begin{bmatrix} -4 \\ 3 \end{bmatrix} = 0, \quad [1,-1]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0, \quad \left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\| = 5, \quad \left\| \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| = \sqrt{2}.$$

7. Norm (12 pts)

Given a matrix $A \in \mathbb{R}^{m,n}$ of rank r with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0$. We want to show that

$$\max_{\vec{x} \neq \vec{0}} \frac{\|A\vec{x}\|}{\|\vec{x}\|} = \sigma_1$$

where $\vec{x} \in \mathbb{R}^n$ (i.e. that the maximum factor by which that A can grow the Euclidean norm is σ_1).

This problem will have you write out two steps in that proof.

We start by noting that since the matrix $A^T A$ is symmetric, we know $A^T A$ has eigenvalue decomposition $V \Lambda V^T$. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a set of orthonormal eigenvectors of $A^T A$ with associated eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, where $\lambda_i = \sigma_i^2$.

(a) **(8 pts)** For $\vec{x} \in \mathbb{R}^n$, **express** $\|A\vec{x}\|^2$ **in terms of** $\vec{v}_i$, $\vec{x}$**, and** σ_i, for $i \in \{1, 2, ..., n\}$.

(HINT: Remember that $\|\vec{y}\|^2 = \vec{y}^T \vec{y}$. Also, you might want to decompose $\vec{x}$ as a linear combination of the set of orthonormal eigenvectors of $A^T A$.)

(b) **(4 pts) What is** $\|A\vec{v}_1\|$ **in terms of other information given in this problem? Show work and simplify as much as possible.**

8. Using upper-triangularization to solve differential equations (20 pts)

This problem really starts on the next page. This page is background that defines the notation.

You know that for any square matrix A with real eigenvalues, there exists a real matrix V with orthonormal columns and a real upper triangular matrix R so that $A = VRV^T$. In particular, to set notation explicitly:

$$V = \begin{bmatrix} \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \end{bmatrix}$$
$$R^T = \begin{bmatrix} \vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n \end{bmatrix}$$

where the rows of the upper-triangular R look like

$$\vec{r}_1^T = [\lambda_1, r_{1,2}, r_{1,3}, \ldots, r_{1,n}]$$
$$\vec{r}_2^T = [0, \lambda_2, r_{2,3}, r_{2,4}, \ldots, r_{2,n}]$$
$$\vec{r}_i^T = [\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, \lambda_i, r_{i,i+1}, r_{i,i+2}, \ldots, r_{i,n}]$$
$$\vec{r}_n^T = [\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, \lambda_n]$$

where the λ_i are the eigenvalues of A.

Here, we also use bracket notation to index into vectors so

$$r_i[k] = \begin{cases} 0 & \text{if } k < i \\ \lambda_i & \text{if } k = i \\ r_{i,k} & \text{if } k > i \end{cases}$$

Note: we use 1-indexing so the first entry has index 1.

Suppose our goal is to solve the n-dimensional system of differential equations written out in vector/matrix form as:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t),$$
$$\vec{x}(0) = \vec{x}_0,$$

where $\vec{x}_0$ is a specified initial condition and $\vec{u}(t)$ is a given vector of functions of time.

Assume that the V and R have already been computed and are accessible to you using the notation above.

Assume that you have access to a function $ScalarSolve(\lambda, y_0, \check{u})$ that takes a real number λ, a real number y_0, and a real-valued function of time $\check{u}$ as inputs and returns a real-valued function of time that is the solution to the scalar differential equation $\frac{d}{dt}y(t) = \lambda y(t) + \check{u}(t)$ with initial condition $y(0) = y_0$.

Also assume that you can do regular arithmetic using real-valued functions and it will do the right thing. So if u is a real-valued function of time, and g is also a real-valued funcion of time, then $5u + 6g$ will be a real valued function of time that evaluates to $5u(t) + 6g(t)$ at time t.

Use V, R to construct a procedure for solving this differential equation

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t),$$
$$\vec{x}(0) = \vec{x}_0,$$

for $\vec{x}(t)$ by filling in the following template in the spots marked ♣, ◇, ♡, ♠.

(HINT: Think back to the very long RLC question on the homework. You also might want to write out for yourself what the differential equation looks like in V-coordinates.)

1: $\vec{\widetilde{x}}_0 = V^T \vec{x}_0$ ▷ Change the initial condition to be in V-coordinates
2: $\vec{\widetilde{u}} = V^T \vec{u}$ ▷ Change the external input functions to be in V-coordinates
3: **for** $i = n$ down to 1 **do**   ▷ Iterate up from the bottom
4:      $\check{u}_i = ♣ + \sum_{j=i+1}^{n} ♠$ ▷ Make the effective input for this level
5:      $\widetilde{x}_i = ScalarSolve(◇, \widetilde{x}_0[i], \check{u}_i)$ ▷ Solve this level's scalar differential equation
6: **end for**

7: $\vec{x}(t) = ♡ \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \vdots \\ \widetilde{x}_n \end{bmatrix}(t)$ ▷ Change back into original coordinates

(a) **(4 pts) Give the expression for ♡ on line 7 of the algorithm above.** (i.e. How do you get from $\vec{\widetilde{x}}(t)$ to $\vec{x}(t)$?)

(b) **(4 pts) Give the expression for ◇ on line 5 of the algorithm above.** (i.e. What are the λ arguments to *ScalarSolve* for the i-th iteration of the for-loop?)

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Algorithm template copied for your convenience:

1: $\vec{\widetilde{x}}_0 = V^T \vec{x}_0$ ▷ Change the initial condition to be in V-coordinates

2: $\vec{\widetilde{u}} = V^T \vec{u}$ ▷ Change the external input functions to be in V-coordinates

3: **for** $i = n$ down to 1 **do**   ▷ Iterate up from the bottom

4:      $\widecheck{u}_i = \clubsuit + \sum_{j=i+1}^{n} \spadesuit$ ▷ Make the effective input for this level

5:      $\widetilde{x}_i = ScalarSolve(\diamondsuit, \widetilde{x}_0[i], \widecheck{u}_i)$ ▷ Solve this level's scalar differential equation

6: **end for**

7: $\vec{x}(t) = \heartsuit \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \vdots \\ \widetilde{x}_n \end{bmatrix}(t)$ ▷ Change back into original coordinates

(c) **(4 pts) Give the expression for ♣ on line 4 of the algorithm above.**

(d) **(8 pts) Give the expression for ♠ on line 4 of the algorithm above.**

[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.]