Midterm

1. **Honor Code (0 pts.)**

   Please copy the following statement in the space provided below and sign your name.

   As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

   Note that if you do not copy the honor code and sign your name, you will get a 0 on the exam.

   Solution: Any attempt to copy the honor code and sign should get full points.

2. **What are you planning to do after the midterm? (2 pts.)**

   Solution: Any answer is sufficient.

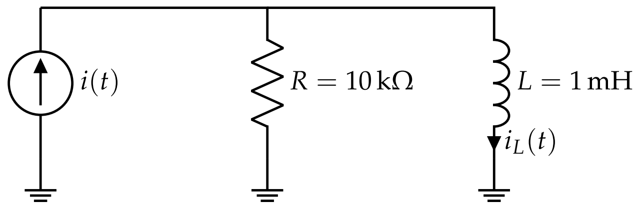
3. **What's your favorite thing to do in Berkeley? (2 pts.)**

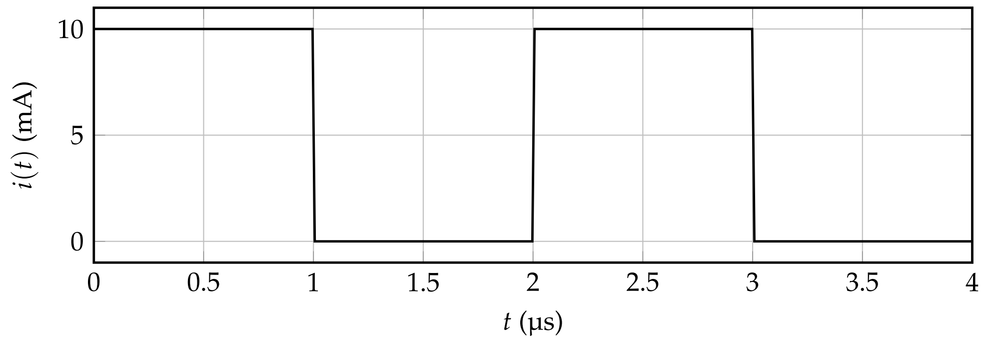
   Solution: Any answer is sufficient.

4. RL Square Wave (16 pts.)

In this problem, we will explore how an RL circuit behaves to a square wave input.

Suppose you have the following circuit, with the plot of input $i(t)$ shown below. Assume $i_L(0) = 0$.





(a) (5 pts.) Find the **differential equation for** $i_L(t)$ in terms of R, L, and $i(t)$.

Solution: To start out, let's write out the KCL equation for the circuit.

$$i(t) = \frac{v_R(t)}{R} + i_L(t) \tag{1}$$

We want to be able to describe $v_R(t)$ in terms of $i_L(t)$. Since $v_R(t) = v_L(t)$, we can use the inductor equation for this.

$$v_R(t) = v_L(t) = L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} \tag{2}$$

Using this, we can find the differential equation for $i_L(t)$.

$$\frac{1}{R}\left(L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t}\right) + i_L(t) = i(t) \tag{3}$$

$$\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} + \frac{R}{L}i_L(t) = \frac{R}{L}i(t) \tag{4}$$

(b) (2 pts.) Find the **time constant** τ that describes this circuit both in terms of R and L, as well as numerically using the element values provided in the circuit diagram.

Solution: Our differential equation can be written in the form

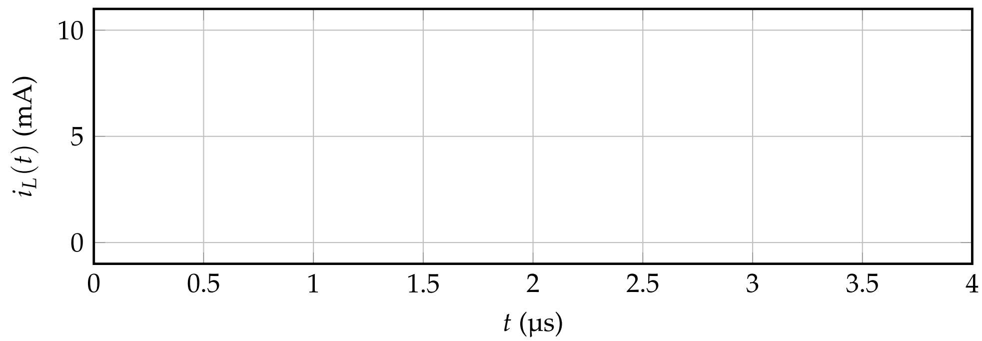
$$\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} + \frac{1}{\tau}i_L(t) = u(t) \tag{5}$$

Thus, we can find that

$$\tau = \frac{L}{R} = \frac{1 \times 10^{-3}}{10 \times 10^{3}} = 1 \times 10^{-7}\,\mathrm{s} \tag{6}$$

Notice that this is the same time constant we found when analyzing an RL circuit with a voltage source.

(c) (2 pts.) **Qualitatively draw $i_L(t)$ for $0\,\mu s < t < 4\,\mu s$** on the provided plot (make sure to look at the units of the plot). Also, remember that $i_L(0) = 0$.



(HINT: Remember how the time constant relates to transitions between voltages.)

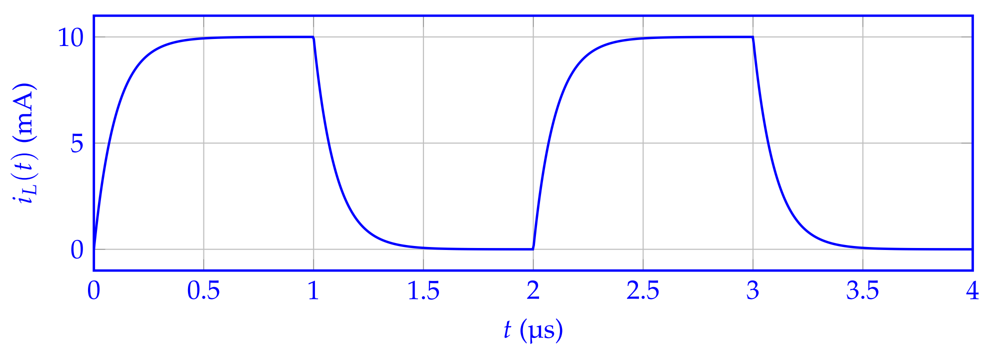
Solution: The structure of our differential equation (as well as the knowledge we have accumulated on RC and RL circuits) imply that the current $i_L(t)$ will change exponentially at a speed related to the time constant τ we determined in the previous part.

Since $\tau = 1 \times 10^{-7}$ s while $i(t)$ transitions every 1×10^{-6} s, the change for $i_L(t)$ will be relatively fast compared to the transitions for $i(t)$ so $i_L(t)$ will be able to approximately reach its asymptotic value every transition.

When $i(t) = 10\,\text{mA}$, the asymptotic value for $i_L(t)$ will be $10\,\text{mA}$ (since this would be the steady state value for $i_L(t)$ since the inductor would act like a short circuit and thus all the current would flow through the inductor).

When $i(t) = 0$, the asymptotic value for $i_L(t)$ will be 0.

The following plot shows these transitions for $i_L(t)$.



For the rest of the problem, we will focus on the time interval $0\,\mu s < t < 1\,\mu s$ (which represents the first transition for the circuit).

(d) (5 pts.) **Solve the differential equation for $i_L(t)$** on the time interval $0\,\mu s < t < 1\,\mu s$.

Solution: On this interval, $i(t) = 10\,\text{mA}$ is constant so we can solve this problem as we would with an RL or RC circuit with constant input.

By inspection, since $i_L(0) = 0$ and $i_L(t)$ approaches $10\,\text{mA}$ with time constant $\tau = 10^{-7}$, we can find the solution

$$i_L(t) = 0.01(1 - e^{-\frac{t}{10^{-7}}}) = 0.01(1 - e^{-10^7 t}) \tag{7}$$

If you use any other method to solve the problem, you should find the same solution.

(e) (2 pts.) Suppose we want to shorten the time period of the input square wave $i(t)$ such that the first transition occurs over the time interval $0 < t < T$ (currently, $T = 1 \times 10^{-6}\,\mathrm{s} = 1\,\mu\mathrm{s}$). However, we want to still ensure that $i_L(t)$ reaches at least 90% of $i_{L,\mathrm{desired}}$, its target/desired value (the value it approaches over the time interval). To find the minimum time T such that this condition is fulfilled, we can solve the following equation:

$$\frac{i_L(T)}{i_{L,\mathrm{desired}}} = 0.9 \tag{8}$$

Using your answer from the previous part, **solve for the minimum time** T. Your answer should be numerical, but you do not need to simplify it.

Solution: From the previous part, we have an equation for $i_L(t)$ over the interval so we can find an expression for $i_L(T)$.

$$i_L(T) = 0.01(1 - \mathrm{e}^{-10^7 T}) \tag{9}$$

The value of $i_{L,\mathrm{desired}} = 0.01$ since this is the steady state value that $i_L(t)$ approaches over the interval (in the ideal case, we would want $i_L(t)$ to transition from 0 to 0.01 instantaneously, but we have learned that this is not possible due to capacitors and inductors as seen in this problem). Thus, we can use the provided equation to solve for T.

$$\frac{0.01(1 - \mathrm{e}^{-10^7 T})}{0.01} = 0.9 \tag{10}$$

$$1 - \mathrm{e}^{-10^7 T} = 0.9 \tag{11}$$

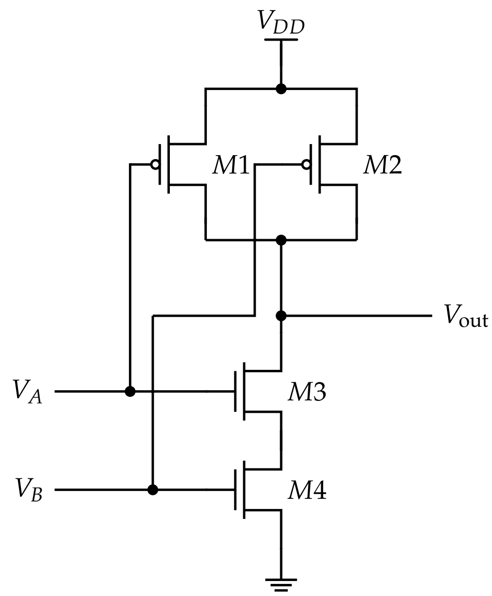
$$\mathrm{e}^{-10^7 T} = 0.1 \tag{12}$$

$$-10^7 T = \ln(0.1) \tag{13}$$

$$T = -10^{-7}\ln(0.1) = 10^{-7}\ln(10) \approx 2.3 \times 10^{-7}\,\mathrm{s} \tag{14}$$

5. Transistor Logic (10 pts.)

For this problem, we will analyze the following transistor logic gate:



Assume that $V_{DD} > V_{Tn}, |V_{Tp}| > 0$ (where V_{Tn} is the threshold voltage for the NMOS transistors and V_{Tp} is the threshold voltage for the PMOS transistors) for all parts of the problem.

(a) (5 pts.) For each row in the following table, for the given input voltages V_A and V_B, fill in the bubble **for each transistor that is active (conducts current) for those input voltages** and fill out one of the bubbles for the **output** V_{out} ($V_{out} = V_{DD}$ or $V_{out} = 0$).

For this part, you may model the transistors as voltage-controlled switches (with no resistance or capacitance). Also, you do not have to choose whether to fill in the bubble or not for transistor M3 for $V_A = V_{DD}$ and $V_B = 0$ (the bubble is not present).

V_A	V_B	M1	M2	M3	M4	$V_{out} = V_{DD}$	$V_{out} = 0$
0	0	◯	◯	◯	◯	◯	◯
0	V_{DD}	◯	◯	◯	◯	◯	◯
V_{DD}	0	◯	◯		◯	◯	◯
V_{DD}	V_{DD}	◯	◯	◯	◯	◯	◯

Solution: Transistors M1 and M2 are PMOS transistors. Transistors M3 and M4 are NMOS transistors. PMOS transistors are active/on (can be modelled as a closed switch) when $V_{SG} > |V_{Tp}|$ and NMOS transistors are active/on (can be modelled as a closed switch) when $V_{GS} > V_{Tn}$. $V_{S,3}$ is the voltage of the source terminal of transistor M3, which is also the drain voltage for transistor M4; when transistor M4 is on, $V_{S,3} = 0$, but when M4 is off, $V_{S,3}$ is a disconnected node that can be between 0 and V_{DD}, which is why the case of $V_A = V_{DD}$ and $V_B = 0$ becomes more complicated (and was not asked on the exam).

For $V_A = V_B = 0$:

- $V_{SG,1} = V_{DD} - 0 = V_{DD} > |V_{Tp}|$ so M1 is on.

- $V_{SG,2} = V_{DD} - 0 = V_{DD} > |V_{Tp}|$ so M2 is on.
- $V_{GS,4} = 0 - 0 = 0 < V_{Tn}$ so M4 is off.
- $V_{GS,3} = 0 - V_{S,3} < V_{Tn}$ so M3 is off. $V_{S,3}$ is unknown, but $0 - V_{S,3} \le 0$ since $0 \le V_{S,3} \le V_{DD}$.

Thus, $V_{\text{out}} = V_{DD}$ in this case since there is a conducting path from V_{DD} to the output through both PMOS transistors (M1 and M2).

For $V_A = 0$ and $V_B = V_{DD}$:

- $V_{SG,1} = V_{DD} - 0 = V_{DD} > |V_{Tp}|$ so M1 is on.
- $V_{SG,2} = V_{DD} - V_{DD} = 0 < |V_{Tp}|$ so M2 is off.
- $V_{GS,4} = V_{DD} - 0 = V_{DD} > V_{Tn}$ so M4 is on. This means that $V_{S,3} = 0$.
- $V_{GS,3} = 0 - V_{S,3} = 0 - 0 < V_{Tn}$ so M3 is off.

Thus, $V_{\text{out}} = V_{DD}$ in this case since there is a conducting path from V_{DD} to the output through PMOS transistor M1.

For $V_A = V_{DD}$ and $V_B = 0$:

- $V_{SG,1} = V_{DD} - V_{DD} = 0 < |V_{Tp}|$ so M1 is off.
- $V_{SG,2} = V_{DD} - 0 = V_{DD} > |V_{Tp}|$ so M2 is on.
- $V_{GS,4} = 0 - 0 = 0 < V_{Tn}$ so M4 is off.
- This was the case where the state of transistor M3 was not asked.

Thus, $V_{\text{out}} = V_{DD}$ in this case since there is a conducting path from V_{DD} to the output through PMOS transistor M2.

The state of transistor M3 does not matter in this case because transistor M4 being off means there cannot be a conductive path to ground and M2 being on means there is a conductive path to V_{DD}. In actuality, M3 would likely remain on until $V_{S,3} = V_{DD} - V_{Tn}$ and would then turn off so in steady state, M3 would be off.

For $V_A = V_B = V_{DD}$:

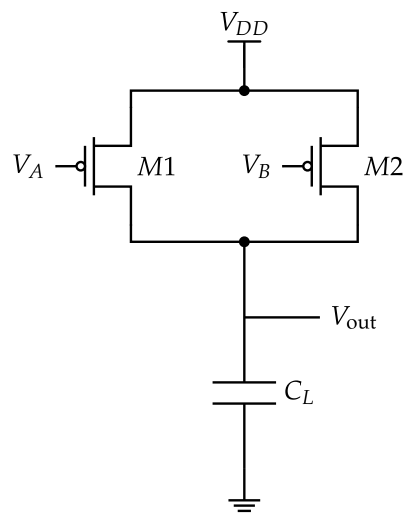
- $V_{SG,1} = V_{DD} - V_{DD} = 0 < |V_{Tp}|$ so M1 is off.
- $V_{SG,2} = V_{DD} - V_{DD} = 0 < |V_{Tp}|$ so M2 is off.
- $V_{GS,4} = V_{DD} - 0 = V_{DD} > V_{Tn}$ so M4 is on. This means that $V_{S,3} = 0$.
- $V_{GS,3} = V_{DD} - V_{D,4} = V_{DD} - 0 = V_{DD} > V_{Tn}$ so M3 is on.

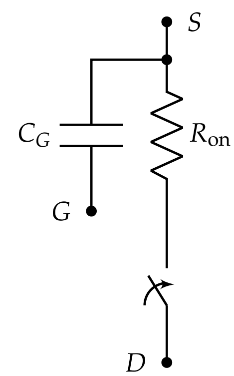
Thus, $V_{\text{out}} = 0$ in this case since there is a conducting path from ground to the output through the combination of both NMOS transistors (M3 and M4).

The completed table would be:

V_A	V_B	M1	M2	M3	M4	$V_{\text{out}} = V_{DD}$	$V_{\text{out}} = 0$
0	0	●	●	○	○	●	○
0	V_{DD}	●	○	○	●	●	○
V_{DD}	0	○	●		○	●	○
V_{DD}	V_{DD}	○	○	●	●	○	●

(b) (5 pts.) Now, consider the following circuit on the left, which can represent part of the transistor gate from the previous part when it drives some load capacitance C_L (which can represent a circuit connected to the output of the transistor gate). The circuit on the right is the model we will use for the two identical transistors (the state of the switch is determined by the relevant threshold voltage condition).



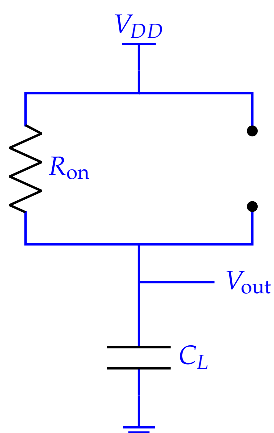


Suppose we have two scenarios:

- Scenario 1: $V_A = 0$, $V_B = V_{DD}$
- Scenario 2: $V_A = V_B = 0$

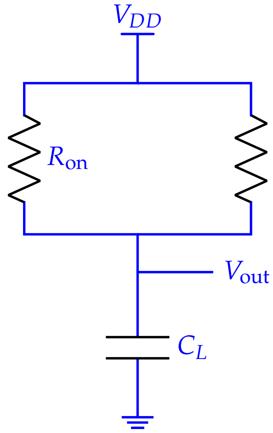
In which scenario will V_{out} transition between voltages the fastest? Explain your reasoning using your knowledge of time constants.

Solution: For Scenario 1, we know based on the previous part that transistor M1 is on while transistor M2 is off. Thus, the equivalent circuit will look like this (notice that we do not include the gate capacitance for the transistors because they are independent of the output node of the circuit; both are connected between an input terminal and V_{DD}):



There is just one resistor R_{on} and one capacitor C_L connected to the output node so we can say that the time constant is $\tau_1 = R_{\text{on}}C_L$ (you could also write the differential equation for this).

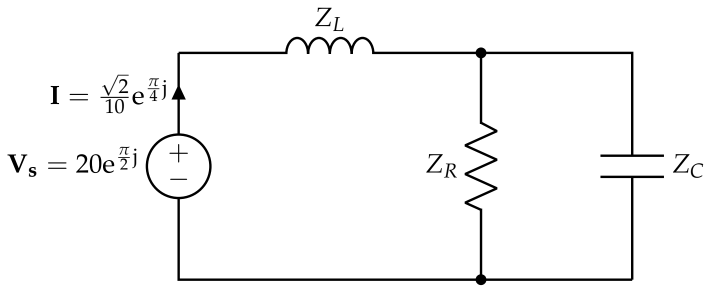
For Scenario 2, we know that transistor M1 and M2 are both on. Thus, the equivalent circuit will look like this:



This time, there are two resistors connected to the output node in parallel, so the time constant is $\tau_2 = \frac{1}{2}R_{\text{on}}C_L$.

Thus, since $\tau_2 < \tau_1$, V_{out} will transition between voltages the fastest in Scenario 2.

6. AC Power (5 pts.)



Find the *average power* and *reactive power* taken from the source. Show your work. Remember that complex power follows the equation

$$P_{\text{complex}} = \frac{1}{2}VI^* \tag{15}$$

and (average) power is $P_{\text{avg}} = \text{Re}\{P_{\text{complex}}\}$, while reactive power is $P_{\text{reactive}} = \text{Im}\{P_{\text{complex}}\}$.

Solution:

$$\begin{aligned}
\mathbf{P} &= \frac{1}{2}\widetilde{\mathbf{V}_{\mathbf{s}}}\widetilde{\mathbf{I}}^* && (16)\\
&= \frac{1}{2}\left(20e^{\frac{\pi}{2}j}\right)\left(\frac{\sqrt{2}}{10}e^{\frac{\pi}{4}j}\right)^* && (17)\\
&= \frac{1}{2}\left(20e^{\frac{\pi}{2}j}\right)\left(\frac{\sqrt{2}}{10}e^{-\frac{\pi}{4}j}\right) && (18)\\
&= \sqrt{2}e^{\frac{\pi}{4}j} && (19)\\
&= 1 + j && (20)
\end{aligned}$$

Then, we have:

$$\begin{aligned}
P_{\text{avg}} &= \text{Re}\{P\} = 1\,\text{W} && (21)\\
P_{\text{react}} &= \text{Im}\{P\} = 1\,\text{VAR} && (22)
\end{aligned}$$

7. System ID (10 pts.)

(a) (5 pts.) Consider the following *discrete-time* system:

$$x[i+1] = a_1 x[i] + a_2 x[i-1] + bu[i] \tag{23}$$

where $x[i], i \geq 0$ is the "state" of the system and $u[i], i \geq 0$ is the "input" into the system. The constants $a_1, a_2, b \in \mathbb{R}$ are unknown. Consider the following table of collected data points:

i	$x[i]$	$u[i]$
0	0	0
1	1	0
2	1	3
3	1.5	8
4	1.85	8
5	1.7	4
6	1.5	3
7	1.4	4

Set up a least squares problem, of the form $D\vec{p} = \vec{s}$ for some matrix D and vector $\vec{s}$ derived from the data above, where $\vec{p} := \begin{bmatrix} a_1 \\ a_2 \\ b \end{bmatrix}$.

Solution: Based on the given system, we can deduce that the ith row of D would be $\begin{bmatrix} x[i+1] & x[i] & u[i+1] \end{bmatrix}$, since $i \geq 0$. The corresponding element of $\vec{s}$ will be $\vec{x}[i+1]$. Thus, we have

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 1.5 & 1 & 8 \\ 1.85 & 1.5 & 8 \\ 1.7 & 1.85 & 4 \\ 1.5 & 1.7 & 3 \end{bmatrix} \tag{24}$$

$$\vec{s} = \begin{bmatrix} 1 \\ 1.5 \\ 1.85 \\ 1.7 \\ 1.5 \\ 1.4 \end{bmatrix} \tag{25}$$

Thus, the least squares problem can be written as

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 1.5 & 1 & 8 \\ 1.85 & 1.5 & 8 \\ 1.7 & 1.85 & 4 \\ 1.5 & 1.7 & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ b \end{bmatrix}}_{\vec{p}} = \underbrace{\begin{bmatrix} 1 \\ 1.5 \\ 1.85 \\ 1.7 \\ 1.5 \\ 1.4 \end{bmatrix}}_{\vec{s}} \tag{26}$$

(b) (5 pts.) Suppose the actual system was

$$x[i+1] = \underbrace{0.5}_{a_1} x[i] + \underbrace{0.2}_{a_2} x[i-1] + \underbrace{0.1}_{b} u[i] \tag{27}$$

Say you were still interested in "estimating" the values of a_1, a_2, b (even though you know what the values are). **Construct a sequence of control inputs** $u[0], u[1], u[2], u[3], u[4]$ **and initial states** $x[0]$ **and** $x[1]$ **such that:**

i. You can use least squares to estimate a_1, a_2, b**.**

ii. You cannot use least squares to estimate a_1, a_2, b**.**

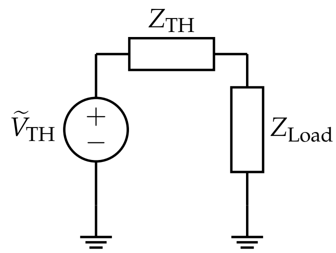
Assume there is no noise/disturbance term.

(HINT: What is the condition needed on the matrix D to apply least squares? Construct a sequence of control inputs so that you can/cannot satisfy this condition.)

Solution: To solve the least squares problem, we need columns of D to be linearly independent. It is the case that, for the D matrix that we derived in the previous part, the columns are linearly independent. If the columns are linearly dependent, we cannot use least squares to solve the system. This would occur when $x[i] = u[i]$, for all $i \geq 0$, for example.

8. Maximum Power Transfer Theorem (15 pts.)

Consider the following phasor domain circuit:



where Z_{TH} is written as its real and complex components as follows:

$$Z_{\text{TH}} = z_r + \mathrm{j}z_i \tag{28}$$

Suppose Z_{Load} is constrained to be purely real (i.e., $\operatorname{Im}\{Z_{\text{Load}}\} = 0$).

(a) (5 pts.) **Find the average power (or, equivalently, active power) delivered to the load in terms of $z_r, z_i, \widetilde{V}_{\text{TH}}, Z_{\text{Load}}$.** Remember that complex power follows the equation

$$P_{\text{complex}} = \frac{1}{2} V I^* \tag{29}$$

and average power is $P_{\text{avg}} = \operatorname{Re}\{P_{\text{complex}}\}$.

Solution: Note that $I_{\text{Load}} = \frac{\widetilde{V}_{\text{TH}}}{z_r + Z_{\text{Load}} + \mathrm{j}z_i}$ and $V_{\text{Load}} = \frac{\widetilde{V}_{\text{TH}} \cdot Z_{\text{Load}}}{z_r + Z_{\text{Load}} + \mathrm{j}z_i}$. Thus, the complex power is

$$P_{\text{Load}} = \frac{1}{2} Z_{\text{Load}} \cdot \frac{\widetilde{V}_{\text{TH}}}{z_r + Z_{\text{Load}} + \mathrm{j}z_i} \cdot \frac{\widetilde{V}_{\text{TH}}^*}{(z_r + Z_{\text{Load}} + \mathrm{j}z_i)^*} \tag{30}$$

$$= \frac{1}{2} Z_{\text{Load}} \cdot \frac{\left|\widetilde{V}_{\text{TH}}\right|^2}{(z_r + Z_{\text{Load}})^2 + z_i^2} \tag{31}$$

which is real (since all the terms in that expression are real), so $P_{\text{avg}} = P_{\text{Load}}$.

(b) (5 pts.) **Show that the value of Z_{Load} that maximizes P_{Load}, the average power delivered to Z_{Load}, is $Z_{\text{Load}} = |Z_{\text{TH}}| = \sqrt{z_r^2 + z_i^2}$.**

(HINT: Take a derivative of your average power expression from the previous part and set it to 0. Use this to solve for the maximizing value of Z_{Load}. Remember, impedance is nonnegative.)

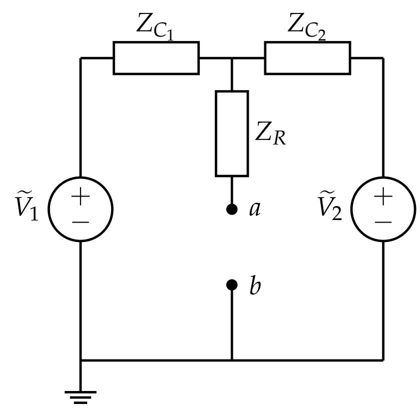
Solution: Following the hint, we take a derivative to obtain

$$\frac{\mathrm{d}}{\mathrm{d}Z_{\text{Load}}} P_{\text{avg}} = \frac{\left|\widetilde{V}_{\text{TH}}\right|^2}{2} \frac{(z_r + Z_{\text{Load}})^2 + z_i^2 - 2Z_{\text{Load}}(z_r + Z_{\text{Load}})}{\left((z_r + Z_{\text{Load}})^2 + z_i^2\right)^2} \tag{32}$$

$$= \frac{z_r^2 - Z_{\text{Load}}^2 + z_i^2}{\left((z_r + Z_{\text{Load}})^2 + z_i^2\right)^2} = 0 \tag{33}$$

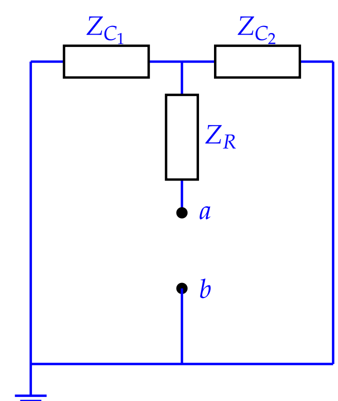
so $z_r^2 - Z_{\text{Load}}^2 + z_i^2 = 0 \implies Z_{\text{Load}} = \pm\sqrt{z_r^2 + z_i^2}$. Since impedance is nonnegative, $Z_{\text{Load}} = \sqrt{z_r^2 + z_i^2} = |Z_{\text{TH}}|$.

(c) (4 pts.) Consider the following circuit:



Here, $Z_R = R$, $Z_{C_1} = \frac{1}{j\omega C_1}$, and $Z_{C_2} = \frac{1}{j\omega C_2}$. You may assume that the value of ω is fixed (i.e., the time domain representations of $\widetilde{V}_1$ and $\widetilde{V}_2$ both have the same angular frequency). **Find the Thevenin equivalent impedance between ports a and b.**

Solution: Zeroing out the voltage sources, we get



Combining $Z_{C_1} \parallel Z_{C_2}$ into a single element, we see that $Z_{\text{TH}} = \left(Z_{C_1} \parallel Z_{C_2}\right) + Z_R = R - \frac{j}{\omega(C_1+C_2)}$.

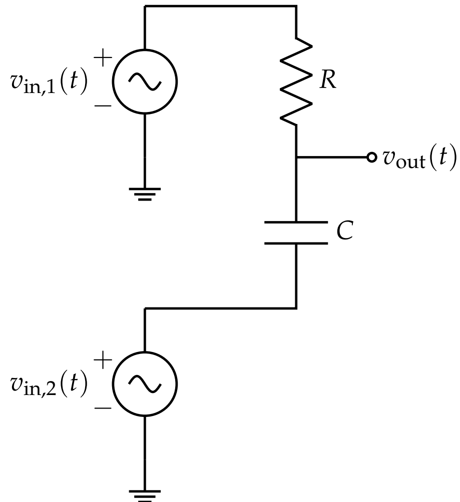
(d) (1 pts.) Again, assume that your angular frequency ω is fixed as was the case in the previous part. Let's say you wanted to connect a lightbulb between ports a and b (a purely resistive/real Z_{Load}). **What should the resistance be, if you wanted to maximize brightness (i.e., maximize power delivered)?**

Solution: From the previous parts, we have $Z_{\text{Load}} = |Z_{\text{TH}}| = \sqrt{R^2 + \frac{1}{\omega^2(C_1+C_2)^2}}$.

9. Transfer Function Superposition (20 pts.)

In this problem, we will explore how superposition relates to transfer functions.

Suppose you have the following circuit with two sinusoidal inputs.

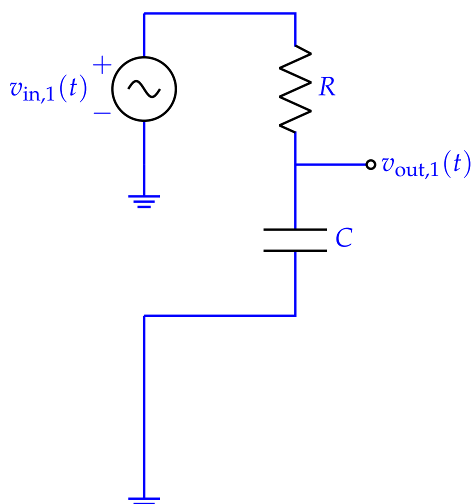


(a) (7 pts.) Find the **transfer function** $H_1(j\omega_1) = \frac{\widetilde{V}_{\text{out},1}}{\widetilde{V}_{\text{in},1}}$ that describes the contribution of $v_{\text{in},1}(t)$ to the output voltage.

(HINT: What happens to the other voltage sources when superposition is used?)

Solution: To use superposition with voltage source $v_{\text{in},1}(t)$, we set $v_{\text{in},2}(t) = 0$, which corresponds to a short circuit.

The circuit we need to focus on this part is shown below.



The transfer function in this case is that of RC low pass filter:

$$H_1(j\omega_1) = \frac{Z_C}{Z_R + Z_C} \tag{34}$$

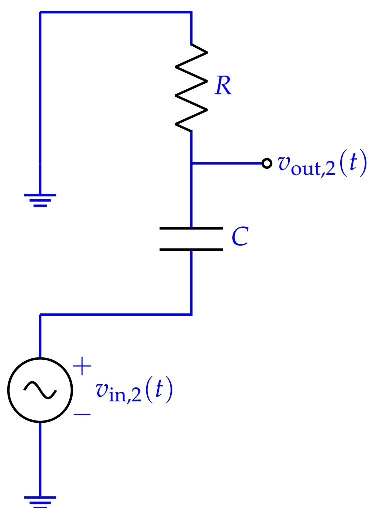
$$= \frac{\frac{1}{j\omega_1 C}}{R + \frac{1}{j\omega_1 C}} \tag{35}$$

$$= \frac{1}{1 + \mathrm{j}\omega_1 RC} \tag{36}$$

(b) (7 pts.) Find the **transfer function** $H_2(\mathrm{j}\omega_2) = \frac{\widetilde{V}_{\text{out},2}}{\widetilde{V}_{\text{in},2}}$ that describes the contribution of $v_{\text{in},2}(t)$ to the output voltage.

Solution: To use superposition with voltage source $v_{\text{in},2}(t)$, we set $v_{\text{in},1}(t) = 0$.

The circuit we need to focus on this part is shown below.



The transfer function in this case is that of CR high pass filter:

$$H_2(\mathrm{j}\omega_2) = \frac{Z_R}{Z_R + Z_C} \tag{37}$$

$$= \frac{R}{R + \frac{1}{\mathrm{j}\omega_2 C}} \tag{38}$$

$$= \frac{\mathrm{j}\omega_2 RC}{1 + \mathrm{j}\omega_2 RC} \tag{39}$$

(c) (3 pts.) Using your answers to the previous parts, **find an expression for** $\widetilde{V}_{\text{out}}$ in terms of $H_1(\mathrm{j}\omega_1)$, $H_2(\mathrm{j}\omega_2)$, $\widetilde{V}_{\text{in},1}$, and $\widetilde{V}_{\text{in},2}$.

Solution: From superposition, we know that

$$\widetilde{V}_{\text{out}} = \widetilde{V}_{\text{out},1} + \widetilde{V}_{\text{out},2} \tag{40}$$

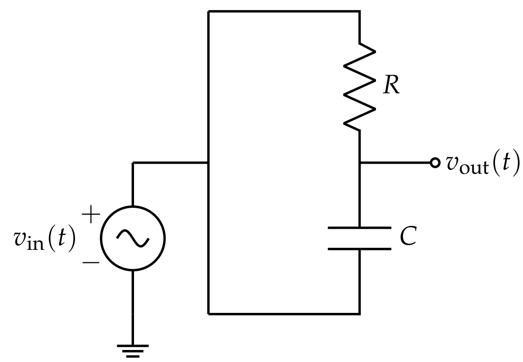
Using the transfer function $H_1(\mathrm{j}\omega_1)$, $\widetilde{V}_{\text{out},1} = H_1(\mathrm{j}\omega_1)\widetilde{V}_{\text{in},1}$.

Similarly, using the transfer function $H_2(\mathrm{j}\omega_2)$, $\widetilde{V}_{\text{out},2} = H_2(\mathrm{j}\omega_2)\widetilde{V}_{\text{in},2}$.

Thus,

$$\widetilde{V}_{\text{out}} = H_1(\mathrm{j}\omega_1)\widetilde{V}_{\text{in},1} + H_2(\mathrm{j}\omega_2)\widetilde{V}_{\text{in},2} \tag{41}$$

(d) (3 pts.) Now, suppose we have the following circuit with one sinusoidal input.



Using your answers to the previous parts, **find** $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$ **for this circuit**. Simplify your answer as much as possible.

Solution: In this case, we can think of this problem as a version of the previous parts where $v_{\text{in},1}(t) = v_{\text{in},2}(t) = v_{\text{in}}(t)$ which implies $\widetilde{V}_{\text{in}} = \widetilde{V}_{\text{in},1} = \widetilde{V}_{\text{in},2}$ and $\omega_1 = \omega_2 = \omega$.

Using the previous part,

$$H_1(\mathrm{j}\omega)\widetilde{V}_{\text{in}} + H_2(\mathrm{j}\omega)\widetilde{V}_{\text{in}} = \widetilde{V}_{\text{out}} \tag{42}$$

$$H_1(\mathrm{j}\omega) + H_2(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = H(\mathrm{j}\omega) \tag{43}$$

The main idea that we learn from this analysis is that

$$H(\mathrm{j}\omega) = H_1(\mathrm{j}\omega) + H_2(\mathrm{j}\omega) \tag{44}$$

when the inputs to both terminals is the same.

Thus, using the answers from the first two parts:

$$\begin{aligned} H(\mathrm{j}\omega) &= H_1(\mathrm{j}\omega) + H_2(\mathrm{j}\omega) && (45) \\ &= \frac{1}{1+\mathrm{j}\omega RC} + \frac{\mathrm{j}\omega RC}{1+\mathrm{j}\omega RC} && (46) \\ &= \frac{1+\mathrm{j}\omega RC}{1+\mathrm{j}\omega RC} && (47) \\ &= 1 && (48) \end{aligned}$$

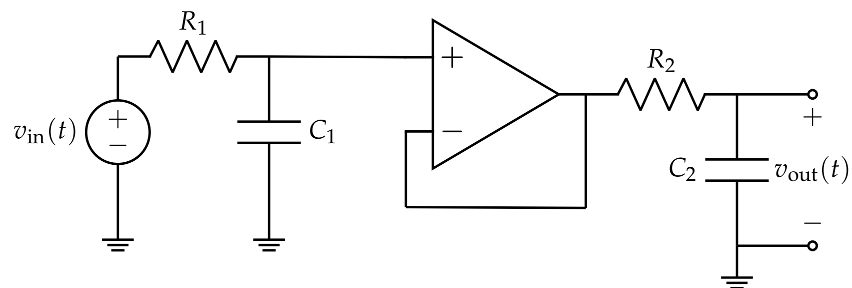
10. Loading Effect Demonstrated (30 pts.)

Consider the following two RC filters:

We know from lecture that these circuits represent low-pass filters. In this problem, we will demonstrate the loading effect and hence motivate the need for a unity gain buffer when cascading filters.

NOTE: For all Bode plotting questions, straight line approximations are fine.

(a) (5 pts.) Consider the following circuit:



Suppose $v_{\text{in}}(t)$ is some sinusoidal input, and the input and output voltage phasors are denoted $\widetilde{V}_{\text{in}}$ and $\widetilde{V}_{\text{out}}$ respectively. **Write the transfer function for this circuit.**

Solution: We know that the first filter has a transfer function of the form

$$H_1(j\omega) = \frac{1}{1 + j\omega R_1 C_1} \tag{49}$$

and similarly, the second filter has a transfer function of the form

$$H_2(j\omega) = \frac{1}{1 + j\omega R_2 C_2} \tag{50}$$

Since these two filters are cascaded and combined by a unity gain buffer, the combined transfer function is

$$H(j\omega) = H_1(j\omega) \cdot H_2(j\omega) \tag{51}$$

$$= \left(\frac{1}{1 + j\omega R_1 C_1}\right)\left(\frac{1}{1 + j\omega R_2 C_2}\right) \tag{52}$$

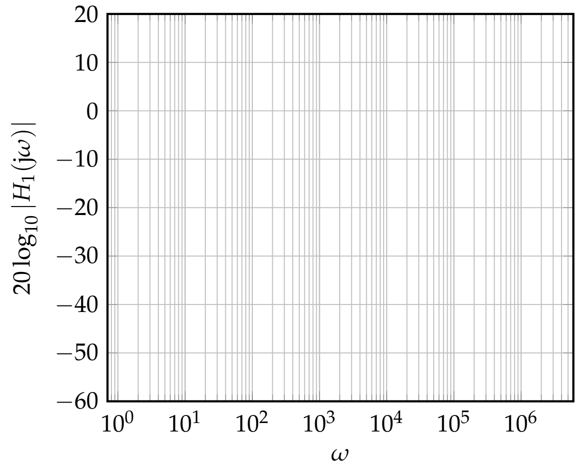
(b) (6 pts.) Regardless of your answer to the previous part, suppose that the transfer function is

$$H(j\omega) = \frac{1}{\left(1 + \frac{j\omega}{10^3}\right)^2} = \frac{1}{\left(1 + \frac{j\omega}{10^3}\right)\left(1 + \frac{j\omega}{10^3}\right)} \tag{53}$$

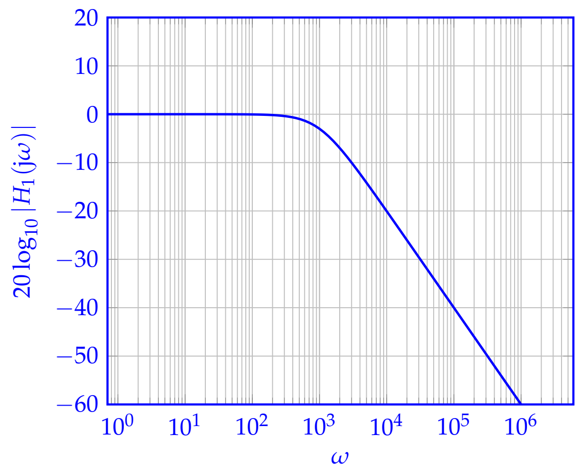
We notice that this transfer function is characterized by two poles at $\omega_c = 10^3$.

i. **First, draw the Bode magnitude plot for**

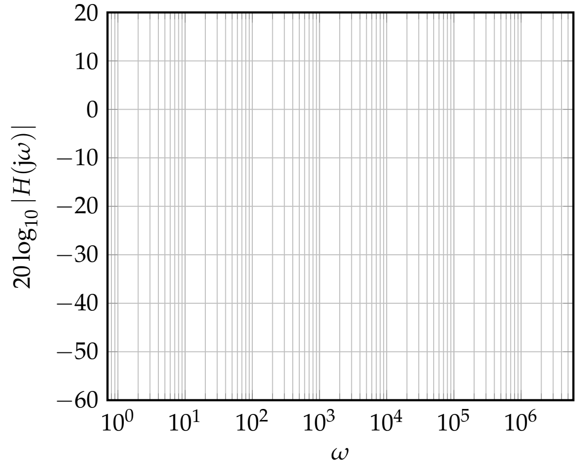
$$H_1(\mathrm{j}\omega) = \frac{1}{1 + \frac{\mathrm{j}\omega}{10^3}} \tag{54}$$



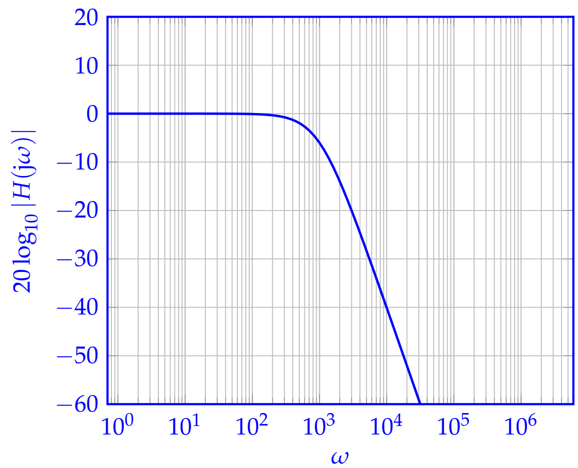
Solution: It is a low-pass filter, so the Bode plot looks like



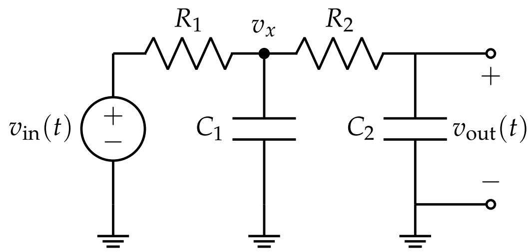
ii. **Next, draw the Bode magnitude plot for** $H(\mathrm{j}\omega)$**.**



Solution: We can pointwise add the answer to the previous part with itself, to obtain the following Bode plot:



(c) (6 pts.) Now, suppose we omitted the unity gain buffer above. This yields the following circuit:



Again, assume $v_{\text{in}}(t)$ is sinusoidal, and the phasor representations of the input and output voltages are $\widetilde{V}_{\text{in}}$ and $\widetilde{V}_{\text{out}}$ respectively. **Find $\widetilde{V}_x$, the phasor representation for v_x in the circuit above, in terms of $\widetilde{V}_{\text{in}}$.**

Solution: From KCL and Ohm's law, we have

$$\frac{\widetilde{V}_{\text{in}} - \widetilde{V}_x}{R_1} = \mathrm{j}\omega C_1 \widetilde{V}_x + \frac{\mathrm{j}\omega C_2}{1 + \mathrm{j}\omega C_2 R_2}\widetilde{V}_x \tag{55}$$

$$\widetilde{V}_{\text{in}} = \left(1 + \mathrm{j}\omega C_1 R_1 + \frac{\mathrm{j}\omega C_2 R_1}{1 + \mathrm{j}\omega C_2 R_2}\right)\widetilde{V}_x \tag{56}$$

$$\widetilde{V}_{\text{in}} = \frac{1 + \mathrm{j}\omega(C_2R_2 + C_1R_1 + C_2R_1) - \omega^2 C_1C_2R_1R_2}{1 + \mathrm{j}\omega C_2 R_2}\widetilde{V}_x \tag{57}$$

$$\widetilde{V}_x = \frac{1 + \mathrm{j}\omega C_2 R_2}{1 + \mathrm{j}\omega(C_2R_2 + C_1R_1 + C_2R_1) - \omega^2 C_1C_2R_1R_2}\widetilde{V}_{\text{in}} \tag{58}$$

(d) (2 pts.) **Using your answer to the previous part, find the transfer function $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}}$ that represents this circuit.**

(HINT: Find $\widetilde{V}_{\text{out}}$ in terms of $\widetilde{V}_x$ first.)

Solution: Following the hint, $\widetilde{V}_{\text{out}} = \frac{1}{1+\mathrm{j}\omega R_2 C_2}\widetilde{V}_x$ by voltage dividers. Thus,

$$\widetilde{V}_{\text{out}} = \frac{1}{1+\mathrm{j}\omega(C_2R_2 + C_1R_1 + C_2R_1) - \omega^2 C_1C_2R_1R_2} V_{\text{in}} \tag{59}$$

and

$$H(\mathrm{j}\omega) = \frac{1}{1+\mathrm{j}\omega(C_2R_2 + C_1R_1 + C_2R_1) - \omega^2 C_1C_2R_1R_2} \tag{60}$$

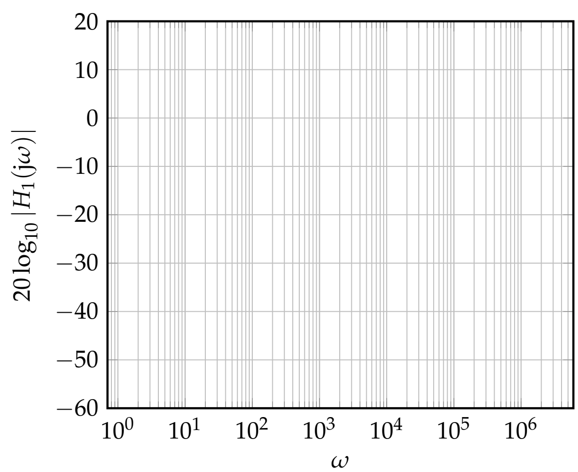
(e) (10 pts.) Regardless of your answer to the previous part, suppose that the transfer function can be written in factored form as

$$H(\mathrm{j}\omega) = \frac{1}{\left(1+\frac{\mathrm{j}\omega}{10^2}\right)\left(1+\frac{\mathrm{j}\omega}{10^4}\right)} \tag{61}$$

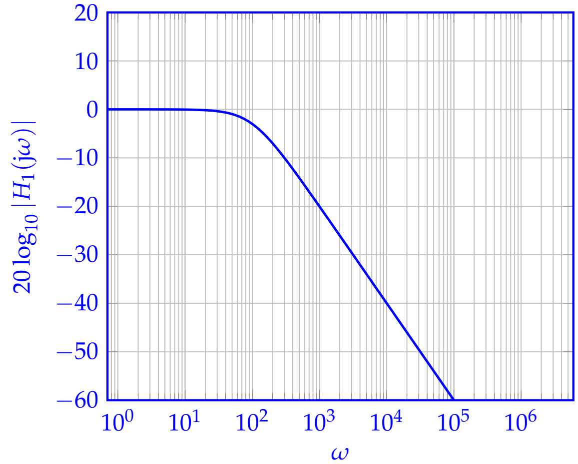
which has poles at $\omega_c = 10^2$ and $\omega_c = 10^4$.

i. **Draw the Bode magnitude plot for the first part of the transfer function (the first pole), i.e.,**

$$H_1(\mathrm{j}\omega) = \frac{1}{1+\frac{\mathrm{j}\omega}{10^2}} \tag{62}$$

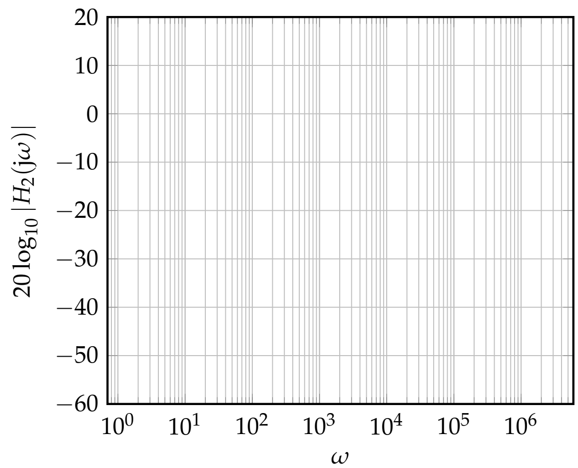


Solution: We have a low-pass filter with cutoff frequency $\omega_c = 10^2$:

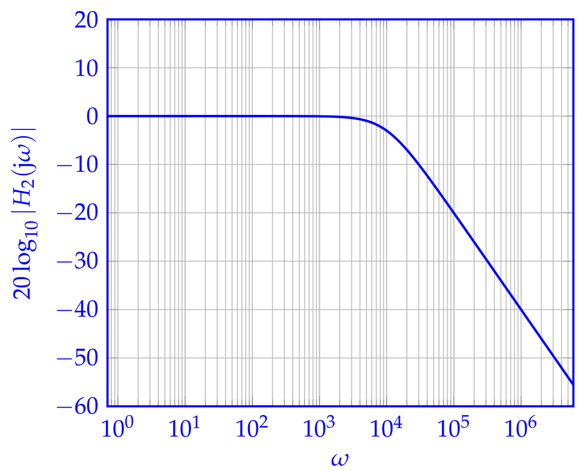


ii. **Draw the Bode magnitude plot for the second part of the transfer function (the second pole), i.e.,**

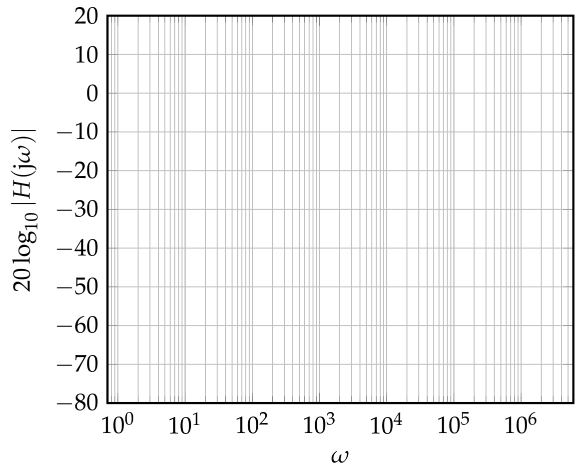
$$H_2(j\omega) = \frac{1}{1 + \frac{j\omega}{10^4}} \tag{63}$$



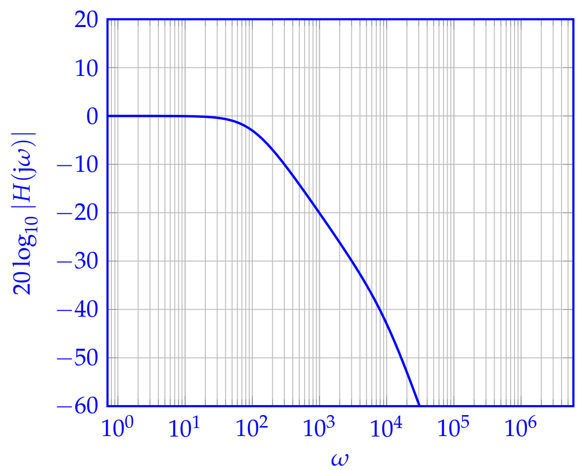
Solution: Again, we have a low-pass filter but with cutoff frequency $\omega_c = 10^4$:



iii. **Draw the combined Bode magnitude plot for $H(\mathrm{j}\omega) = H_1(\mathrm{j}\omega) \cdot H_2(\mathrm{j}\omega)$. Is the Bode plot the same as the one for the circuit with a unity gain buffer?**



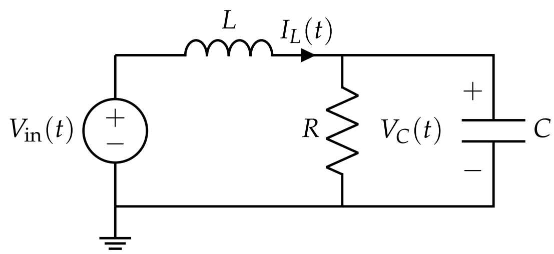
Solution: We can pointwise add the graphs from above, which yields:



This is clearly not the same as the Bode plot which we obtained when we cascaded the two filters with a unity gain buffer.

11. RLC Vector Differential Equation (20 pts.)

In this problem, you will approach solving the following RLC circuit using *the vector differential equation method*.



(a) (6 pts.) **Derive a set of differential equations, one for $I_L(t)$ and another for $V_C(t)$.** **Solution:** We begin by using our circuit analysis methods. Using KVL and replacing the inductor voltage, we obtain:

$$V_{\text{in}}(t) - V_L(t) - V_C(t) = 0 \tag{64}$$

$$V_L(t) = V_{\text{in}}(t) - V_C(t) \tag{65}$$

$$L\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) = V_{\text{in}}(t) - V_C(t) \tag{66}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) = \frac{V_{\text{in}}(t)}{L} - \frac{V_C(t)}{L} \tag{67}$$

Following this, we need one more differential equation. This time we utilize KCL on the node connected to the inductor, resistor, and capacitor:

$$I_L(t) = I_R(t) + I_C(t) \tag{68}$$

$$I_C(t) = I_L(t) - I_R(t) \tag{69}$$

$$C\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) = I_L(t) - \frac{V_C(t)}{R} \tag{70}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) = \frac{I_L(t)}{C} - \frac{V_C(t)}{RC} \tag{71}$$

Thus, we obtain our set of differential equations for this problem:

$$\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) = \frac{V_{\text{in}}(t)}{L} - \frac{V_C(t)}{L} \tag{72}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) = \frac{I_L(t)}{C} - \frac{V_C(t)}{RC} \tag{73}$$

(b) (6 pts.) Suppose our vector differential equation was written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\vec{x}(t) + \begin{bmatrix} e \\ f \end{bmatrix} V_{\text{in}}(t) \tag{74}$$

The state vector is defined as $\vec{x}(t) = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}$. **Find the values of a, b, c, d, e, and f in terms of R, L, C.** **Solution:** Our first step is to obtain a vectorized version of the two differential equations we found in part (a). We do this by stacking complementary terms.

As suggested, we will take $\vec{x}(t) = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}$. With this, we find that the system can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} V_{\text{in}} \tag{75}$$

Thus, $a = -\frac{1}{RC}$, $b = \frac{1}{C}$, $c = -\frac{1}{L}$, $d = 0$, $e = 0$, $f = \frac{1}{L}$.

(c) (6 pts.) Suppose you are now told that the system is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} -6 & 8 \\ -1 & 0 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_{\text{in}}(t) \tag{76}$$

Given that the eigenvalues of the above state matrix are -4 and -2 with eigenvectors $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ respectively, **rewrite the given system of differential equations in the diagonalized basis ($\vec{\tilde{x}}(t) = \begin{bmatrix} \widetilde{V}_C(t) \\ \widetilde{I}_L(t) \end{bmatrix}$).** **Solution:** This problem part asks us to diagnolize the system (i.e. convert it to the V-basis). The columns of V are simply the eigenvalues of the state matrix.

$$V = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \tag{77}$$

We know that the diagonlized system will look like the following (where $\tilde{\vec{x}}$ is our state vector in the diagnalized basis):

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\vec{x}}(t) = V^{-1}AV\tilde{\vec{x}}(t) + V^{-1}\vec{b}V_{\text{in}}(t) \tag{78}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\vec{x}}(t) = \Lambda\tilde{\vec{x}}(t) + V^{-1}\vec{b}V_{\text{in}}(t) \tag{79}$$

where $A = \begin{bmatrix} -6 & 8 \\ -1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ according to our problem.

There are a few ways to find V^{-1}. One is to use the determinant as follows:

$$V^{-1} = \frac{1}{\det(V)}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} \tag{80}$$

$$V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} \tag{81}$$

$$V^{-1} = \begin{bmatrix} 0.5 & -1 \\ -0.5 & 2 \end{bmatrix} \tag{82}$$

Since we are told the eigenvalues of the state matrix, we now have everything for our answer:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\vec{x}}(t) = \Lambda\tilde{\vec{x}}(t) + V^{-1}\vec{b}V_{\text{in}}(t) \tag{83}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\vec{x}}(t) = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix}\tilde{\vec{x}}(t) + \begin{bmatrix} -1 \\ 2 \end{bmatrix}V_{\text{in}}(t) \tag{84}$$

(d) (2 pts.) Your initial conditions in the natural basis are given symbolically as $V_C(0)$ and $I_L(0)$. **Find the initial conditions in the diagonalized basis ($\widetilde{V}_C(0)$ and $\widetilde{I}_L(0)$).** **Solution:**
To perform our change of basis on our original state vector $\vec{x}$, we had to apply the following transformation:

$$\tilde{\vec{x}}(t) = V^{-1}\vec{x}(t) \tag{85}$$

Therfore, to get our initial conditions in the diagonalized basis, we would perform the same transformation:

$$\tilde{\vec{x}}(t) = V^{-1}\vec{x}(t) \tag{86}$$

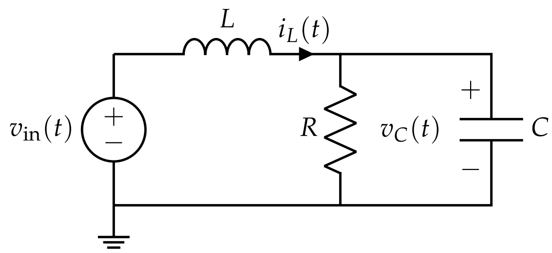
$$\tilde{\vec{x}}(t) = \begin{bmatrix} 0.5 & -1 \\ -0.5 & 2 \end{bmatrix}\begin{bmatrix} V_C(0) \\ I_L(0) \end{bmatrix} \tag{87}$$

$$\tilde{\vec{x}}(t) = \begin{bmatrix} 0.5V_C(0) - I_L(0) \\ -0.5V_C(0) + 2I_L(0) \end{bmatrix} \tag{88}$$

Thus, $\tilde{V}_C(0) = 0.5V_C(0) - I_L(0)$ and $\tilde{I}_L(0) = -0.5V_C(0) + 2I_L(0)$.

12. RLC 2nd Order Differential Equation (30 pts.)

In this problem, you will approach solving the same RLC circuit as the previous problem using your knowledge of 2nd Order Differential Equations.



(a) (8 pts.) **Find the differential equation for** $v_C(t)$ in the following form (where a, b, and c are constants that depend on R, L, and C):

$$\frac{\mathrm{d}^2 v_C(t)}{\mathrm{d}t^2} + a\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + bv_C(t) = cv_{\text{in}}(t) \tag{89}$$

Solution: To start, let's find the KCL equation at the output node.

$$i_L(t) = i_R(t) + i_C(t) \tag{90}$$

$$i_L(t) = \frac{1}{R}v_C(t) + C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} \tag{91}$$

We still have $i_L(t)$ in our equation, but we want a differential equation with just $v_C(t)$. To do this, we can use the inductor equation, which allows us to find that

$$L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = v_{\text{in}}(t) - v_C(t) \tag{92}$$

$$\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = \frac{1}{L}v_{\text{in}}(t) - \frac{1}{L}v_C(t) \tag{93}$$

To use this information, we can take the derivative of both sides of our original equation.

$$\frac{1}{R}\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + C\frac{\mathrm{d}^2 v_C(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}i_L(t)}{\mathrm{d}t} \tag{94}$$

$$\frac{1}{R}\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + C\frac{\mathrm{d}^2 v_C(t)}{\mathrm{d}t^2} = \frac{1}{L}v_{\text{in}}(t) - \frac{1}{L}v_C(t) \tag{95}$$

With some algebra, if we write the differential equation in the desired form, we would have

$$\frac{\mathrm{d}^2 v_C(t)}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + \frac{1}{LC}v_C(t) = \frac{1}{LC}v_{\text{in}}(t) \tag{96}$$

where $a = \frac{1}{RC}$ and $b = c = \frac{1}{LC}$.

(b) (6 pts.) Suppose that you found the differential equation

$$\frac{\mathrm{d}^2 v_C(t)}{\mathrm{d}t^2} + 6\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + 8v_C(t) = 8v_{\text{in}}(t) \tag{97}$$

The general homogeneous solution $v_{C,h}(t)$ to this differential equation can be written in the following form:

$$v_{C,h}(t) = C_1 \mathrm{e}^{s_1 t} + C_2 \mathrm{e}^{s_2 t} \tag{98}$$

Find numerical values for s_1 **and** s_2 for this differential equation (the order of s_1 and s_2 does not matter).

Solution: To start off, we will write the homogeneous differential equation, which is the differential equation without the input term:

$$\frac{\mathrm{d}^2 v_{C,h}(t)}{\mathrm{d}t^2} + 6\frac{\mathrm{d}v_{C,h}(t)}{\mathrm{d}t} + 8v_{C,h}(t) = 0 \tag{99}$$

We will plug in $v_{C,h}(t) = \mathrm{e}^{st}$ to the homogeneous differential equation. This will create a quadratic equation that will provide two solutions for s, which will be the s_1 and s_2 exponent coefficients for the general homogeneous solution.

$$s^2\mathrm{e}^{st} + 6s\mathrm{e}^{st} + 8\mathrm{e}^{st} = 0 \tag{100}$$

$$s^2 + 6s + 8 = 0 \tag{101}$$

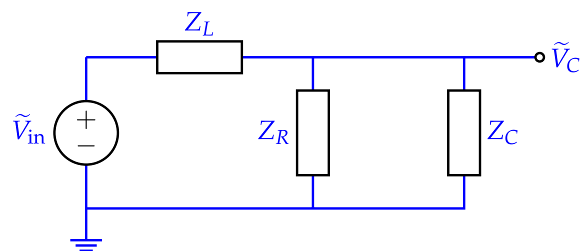
$$(s+4)(s+2) = 0 \tag{102}$$

Thus, $s = -4, -2$ are the two solutions.

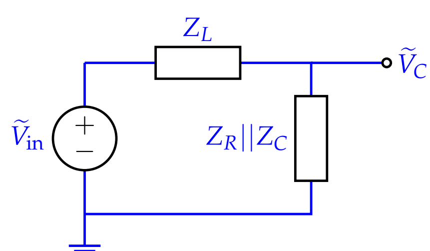
(c) (6 pts.) Suppose $v_{\text{in}}(t) = V_0 \cos\left(\frac{1}{\sqrt{LC}}t\right)$. Find the **particular solution** $v_{C,p}(t)$ for this differential equation (in terms of R, L, C, and V_0).

(HINT: Remember the relationship between the particular solution and steady state solution. What method can you use to find the steady state solution when the input is sinusoidal?)

Solution: When we have a sinusoidal input, we can find the steady state solution using phasor analysis. Here is the equivalent circuit in phasor domain:



If we combine the resistor and capacitor impedances in parallel, we have the following circuit:



where $Z_R||Z_C = \frac{Z_R Z_C}{Z_R+Z_C} = \frac{(R)(\frac{1}{\mathrm{j}\omega C})}{R+\frac{1}{\mathrm{j}\omega C}} = \frac{R}{1+\mathrm{j}\omega RC}$ and $Z_L = \mathrm{j}\omega L$.

We can find $\widetilde{V}_C$ using our knowledge of voltage dividers.

$$\widetilde{V}_C = \frac{Z_R||Z_C}{Z_L + Z_R||Z_C}\widetilde{V}_{\text{in}} \tag{103}$$

$$= \frac{\frac{R}{1+j\omega RC}}{j\omega L + \frac{R}{1+j\omega RC}} \widetilde{V}_{\text{in}} \tag{104}$$

$$= \frac{R}{R - \omega^2 RLC + j\omega L} \widetilde{V}_{\text{in}} \tag{105}$$

For the given input $v_{\text{in}}(t) = V_0 \cos\left(\frac{1}{\sqrt{LC}} t\right)$, $\widetilde{V}_{\text{in}} = V_0$ and $\omega = \frac{1}{\sqrt{LC}}$.

$$\widetilde{V}_C = \left(\frac{R}{R - \frac{1}{LC} RLC + j\frac{1}{\sqrt{LC}} L} \right) (V_0) \tag{106}$$

$$= \frac{V_0 R \sqrt{\frac{C}{L}}}{j} = V_0 R \sqrt{\frac{C}{L}} e^{-j\frac{\pi}{2}} \tag{107}$$

Thus, converting back to time domain, $v_{C,p}(t) = V_0 R \sqrt{\frac{C}{L}} \cos\left(\frac{1}{\sqrt{LC}} t - \frac{\pi}{2}\right) = V_0 R \sqrt{\frac{C}{L}} \sin\left(\frac{1}{\sqrt{LC}} t\right)$.

(d) (4 pts.) To solve the 2nd Order Differential Equation, we need initial conditions for both $v_C(t)$ and $v_C'(t) = \frac{dv_C(t)}{dt}$. Suppose we know that $v_C(0) = 0$ and $i_L(0) = 0$. **Find $v_C'(0)$.**

Solution: From part (a), we found the KCL equation

$$i_L(t) = \frac{1}{R} v_C(t) + C \frac{dv_C(t)}{dt} \tag{108}$$

$$i_L(t) = \frac{1}{R} v_C(t) + C v_C'(t) \tag{109}$$

$$v_C'(t) = \frac{1}{C} i_L(t) - \frac{1}{RC} v_C(t) \tag{110}$$

If we evaluate this equation at $t = 0$, we get

$$v_C'(0) = \frac{1}{C} i_L(0) - \frac{1}{RC} v_C(0) \tag{111}$$

$$= \frac{1}{C}(0) - \frac{1}{RC}(0) \tag{112}$$

$$= 0 \tag{113}$$

(e) (6 pts.) Suppose that you found that the particular solution is $v_{C,p}(t) = \sin(\omega_p t)$. Then, you could write your total solution as

$$v_C(t) = v_{C,p}(t) + v_{C,h}(t) = \sin(\omega_p t) + C_1 e^{s_1 t} + C_2 e^{s_2 t} \tag{114}$$

Using the initial conditions $v_C(0)$ and $v_C'(0)$, **set up a system of equations to solve for C_1 and C_2** (your equations can contain the constants $v_C(0)$, $v_C'(0)$, ω_p, s_1, and s_2).

Solution: We can simply use the equation for $v_C(t)$ with constants C_1 and C_2 to set up the equations. For $v_C(0)$:

$$v_C(0) = v_C(0) \tag{115}$$

$$\sin(\omega_p(0)) + C_1 e^{s_1(0)} + C_2 e^{s_2(0)} = v_C(0) \tag{116}$$

$$C_1 + C_2 = v_C(0) \tag{117}$$

By differentiating $v_C(t)$, we get $v_C'(t) = \omega_p \cos(\omega_p t) + s_1 C_1 e^{s_1 t} + s_2 C_2 e^{s_2 t}$. For $v_C'(0)$:

$$v_C'(0) = v_C'(0) \tag{118}$$

$$\omega_p \cos(\omega_p(0)) + s_1 C_1 e^{s_1(0)} + s_2 C_2 e^{s_2(0)} = v_C'(0) \tag{119}$$

$$\omega_p + s_1 C_1 + s_2 C_2 = v_C'(0) \tag{120}$$

Thus, the system of equations for C_1 and C_2 is:

$$C_1 + C_2 = v_C(0) \tag{121}$$

$$\omega_p + s_1 C_1 + s_2 C_2 = v_C'(0) \tag{122}$$