EECS 16B Designing Information Systems and Devices II UC Berkeley Summer 2022

Final

Exam Location: Draft

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ____________________, ____________________ ____________________
(last) (first) (sign)

PRINT your discussion sections and (u)GSIs (the ones you attend): ____________________

Row Number: ____________________ Seat Number: ____________________

Name and SID of the person to your left: ____________________

Name and SID of the person to your right: ____________________

Name and SID of the person in front of you: ____________________

Name and SID of the person behind you: ____________________

1. **Honor Code (0 pts.)**

   Please copy the following statement in the space provided below and sign your name.

   As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

   Note that if you do not copy the honor code and sign your name, you will get a 0 on the exam.

2. **Tell us something you're excited about. (2 pts.)**

3. **What are you looking forward to this weekend? (2 pts.)**

Do not turn this page until the proctor tells you to do so.
You can work on the above problems before time starts.

4. CMOS Threshold Engineering (12 pts.)

In this problem, you will analyze the behavior of the inverter chain in fig. 1 below.

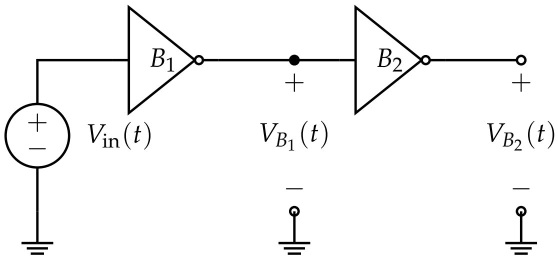


Figure 1: Inverter chain circuit.

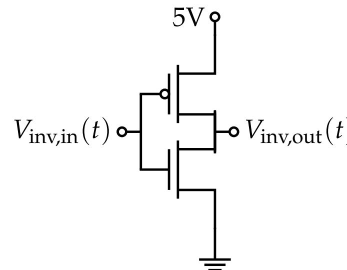


Figure 2: Standard CMOS inverter.

The inverter chain in fig. 1 has two inverters, B_1 and B_2. These inverters are standard CMOS inverters made of a PMOS and NMOS transistor (fig. 2). The PMOS transistors have threshold voltage $|V_{tp}| = 2\,\text{V}$ and the NMOS transistors have threshold voltage $V_{tn} = 1\,\text{V}$.

(a) (4 pts.) The input voltage $V_{in}(t)$ is fed to B_1 in fig. 1. Starting at time t_0, $V_{in}(t)$ goes from 5V to 0V with a delay of 5 seconds. $V_{in}(t)$ is plotted below in fig. 3. **Mark the time the PMOS of B_1 turns on, t_P, and the time the NMOS of B_1 turns off, t_N, with vertical lines in the graph below.**

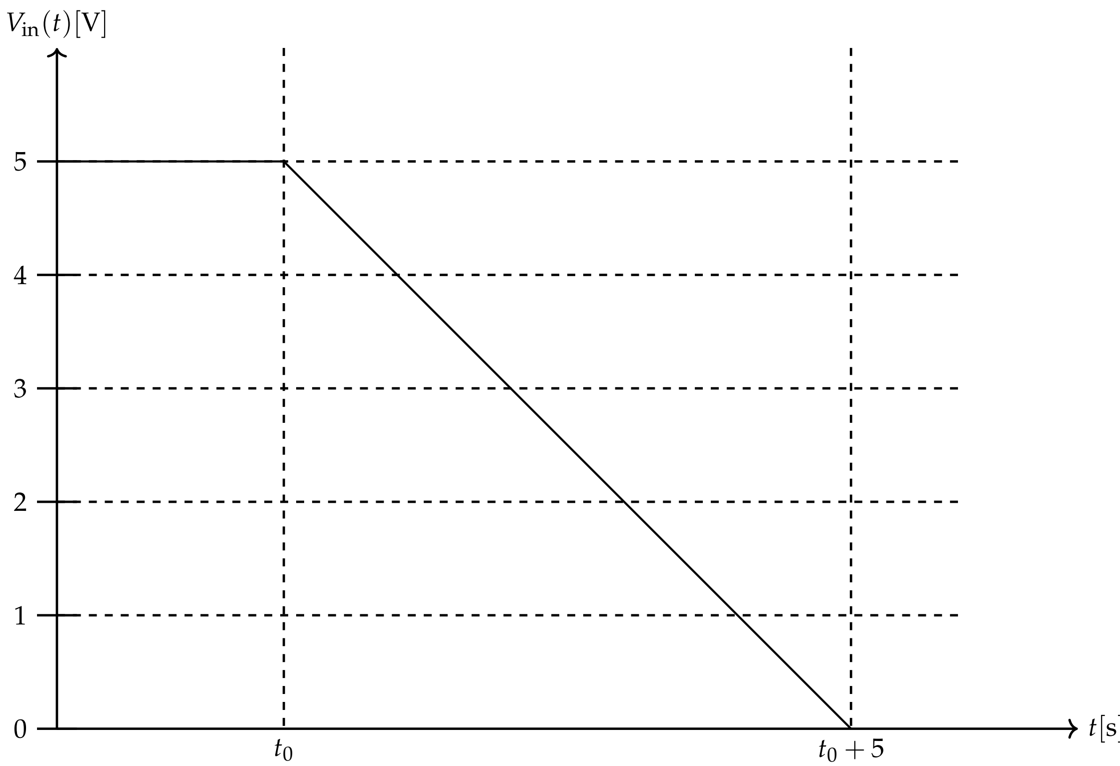


Figure 3: Input voltage to inverter B_1.

(b) (8 pts.) You should have found that $t_\text{P} < t_\text{N}$. For times, t, such that $t_\text{P} \leq t \leq t_\text{N}$, we have both transistors of the B_1 inverter on, leading to the following model of the output of the B_1 inverter as shown in fig. 4.

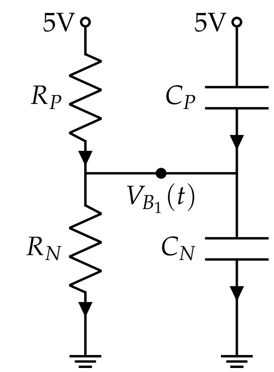


Figure 4: B_1 Inverter's output model.

Setup a differential equation for $V_{B_1}(t)$ from fig. 4 in the form of eq. (1) below and identify λ and c in terms of R_P, R_N, C_N, C_P, and known constants. Do not solve the differential equation.

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{B_1}(t) = \lambda V_{B_1}(t) + c \tag{1}$$

5. Matrix Differential Equation Excited by Eigenvector Input (15 pts.)

Consider the differential equation, eq. (2), in terms of $\vec{x}(t), \vec{u}(t) \in \mathbb{R}^N$.

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = A\vec{x}(t) + \vec{u}(t) \tag{2}$$

Let A be a $N \times N$ matrix with N distinct, real, non-zero eigenvalues, and eigenvalue-eigenvector pairs $(\lambda_i, \vec{v}_i)$ for $i = 1, ..., N$. With $V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_N \\ | & & | \end{bmatrix}$ as the matrix of the eigenvectors, let $\vec{\tilde{x}}(t) = V^{-1}\vec{x}(t)$, which is $\vec{x}(t)$ represented in the V basis. The differential equation, eq. (2), written in terms of $\vec{\tilde{x}}(t)$ appears below in eq. (3).

$$\frac{\mathrm{d}\vec{\tilde{x}}(t)}{\mathrm{d}t} = \widetilde{A}\vec{\tilde{x}}(t) + \widetilde{B}\vec{u}(t) \tag{3}$$

$\widetilde{A}$ and $\widetilde{B}$ are $N \times N$ matrices.

(a) (6 pts.) **Express $\widetilde{A}$ and $\widetilde{B}$ from eq. (3) in terms of λ_i, $\vec{v}_i$, V, and V^{-1}. No need to show work.**

(b) (3 pts.) For a square matrix, M, let $\vec{m}_i$ denote its i-th column, and let $(M^{-1})_i$ denote the i-th column of M's inverse, M^{-1}. **For our system in eq. (2), let $N = 5$. We choose our input to be $\vec{u}(t) = (\widetilde{B}^{-1})_3$. What is $\widetilde{B}\vec{u}(t)$?** Recall that $\widetilde{B}$ is $N \times N$.

(HINT: If $P = Q^{-1}$, $PQ = P \begin{bmatrix} | & & | \\ \vec{q}_1 & \cdots & \vec{q}_N \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ P\vec{q}_1 & \cdots & P\vec{q}_N \\ | & & | \end{bmatrix} = I$.)

(c) (6 pts.) For the system represented in the V basis in eq. (3), let $\widetilde{B}\vec{u}(t) = \vec{w}$, where $\vec{w}$ has k-th entry $w_k = c \neq 0$ and all other entries $w_i = 0$.

With the initial condition that $\vec{x}(0) = \vec{0}$, solve for $\vec{x}(t)$ for $t \geq 0$ when $\widetilde{B}\vec{u}(t) = \vec{w}$ in terms of λ_i, $\vec{v}_i$, t, and other relevant constants. It may be of use to recall that the λ_i are all non-zero. Additionally, we give you the solution to the differential equation eq. (4) below with constant input u.

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u, x(0) = x_0 \tag{4}$$

The solution to eq. (4) is $x(t) = x_0 e^{\lambda t} - \frac{u}{\lambda}(1 - e^{\lambda t})$.

6. To Bode Or Not To Bode (10 pts.)

(a) (4 pts.) **Draw the magnitude and phase Bode plots of the high-pass filter** $H_{\mathrm{HPF}}(\mathrm{j}\omega) = 10\frac{\mathrm{j}\frac{\omega}{10^4}}{1+\mathrm{j}\frac{\omega}{10^4}}$.

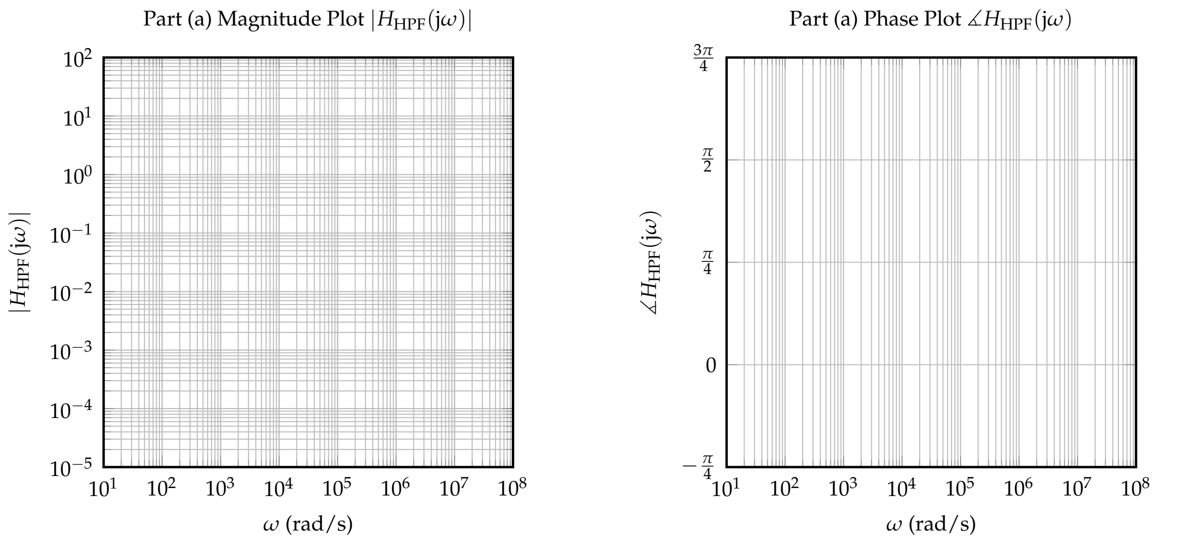


Figure 5: Part (a) Magnitude and Phase Bode Plots for the transfer function $H_{\mathrm{HPF}}(\mathrm{j}\omega)$.

(b) (6 pts.) **Given the following Bode plots describing a system's behavior, what would the output of the system be if the input were** $20\cos(10^5 t) + 5\cos\left(10^3 t + \frac{\pi}{4}\right)$**?**

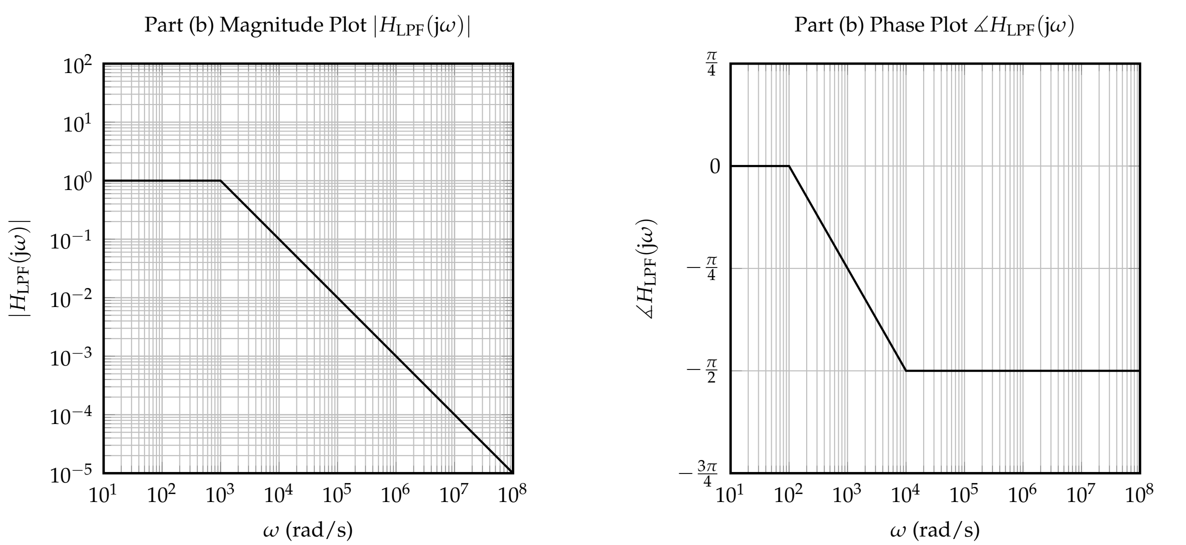


Figure 6: Part (b) Magnitude and Phase Bode Plots for the transfer function $H_{\mathrm{LPF}}(\mathrm{j}\omega)$.

7. Minimum Energy Solutions for Phasors (25 pts.)

Suppose we have three voltage sources with voltage values $V_1(t)$, $V_2(t)$, and $V_3(t)$ to be utilized in a circuit. For an angular frequency, ω, $\widetilde{V}_{i,\omega}$ denotes the phasor of $V_i(t)$ corresponding to that ω.

(a) (3 pts.) The voltages, $V_i(t)$, are defined in the following way.

$$V_1(t) = A_1 \cos(\omega_1 t + \phi_1) \tag{5}$$

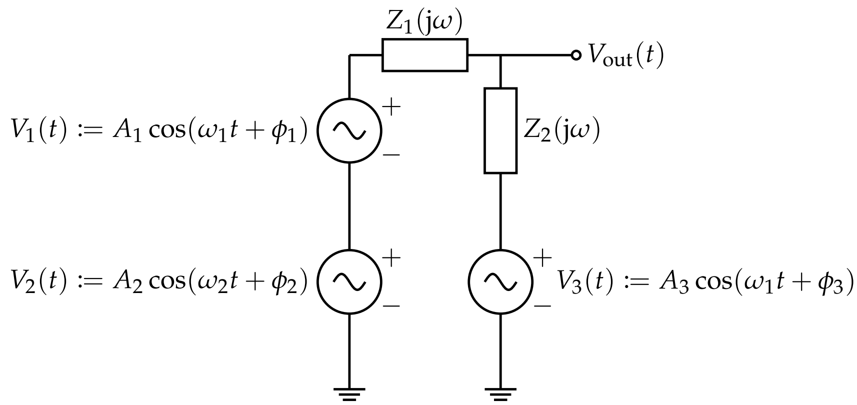
$$V_2(t) = A_2 \cos(\omega_2 t + \phi_2) \tag{6}$$

$$V_3(t) = A_3 \cos(\omega_1 t + \phi_3) \tag{7}$$

Fill in the table below with the phasors of the $V_i(t)$ for each of the two frequencies ω_1 and ω_2 in terms of A_i, ω_i, ϕ_i, and any other relevant constants.

$\widetilde{V}_{1,\omega_1}$	$\widetilde{V}_{2,\omega_1}$	$\widetilde{V}_{3,\omega_1}$	$\widetilde{V}_{1,\omega_2}$	$\widetilde{V}_{2,\omega_2}$	$\widetilde{V}_{3,\omega_2}$

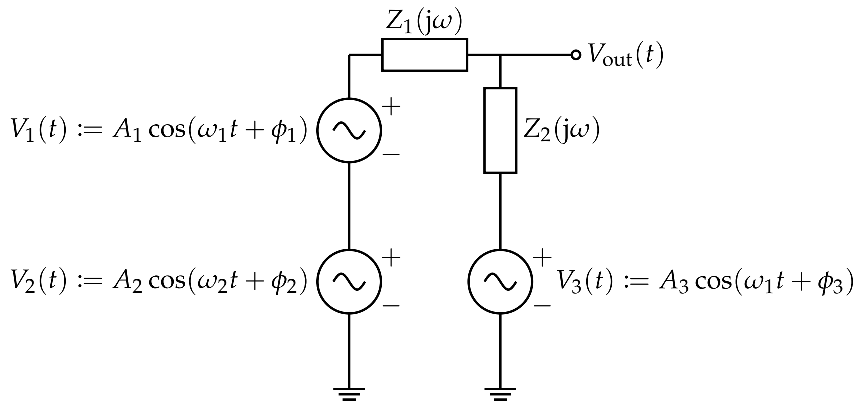
(b) (5 pts.) Now consider a circuit using the three voltage sources shown below.



Solve for the output voltage phasor, $\widetilde{V}_{\mathbf{out},\omega_1}$, the phasor of $V_{\mathbf{out}}(t)$ associated with the frequency ω_1. You may express your answer using any of ω_i, $Z_i(j\omega_1)$, and $\widetilde{V}_{i,\omega_1}$.

(HINT: Superposition may be useful here.)

(c) (3 pts.) The circuit from part (b) is repeated below.



Solve for the output voltage phasor, $\widetilde{V}_{\mathbf{out},\omega_2}$, the phasor of $V_{\mathbf{out}}(t)$ associated with the frequency ω_2. You may express your answer using any of ω_i, $Z_i(j\omega_2)$, and $\widetilde{V}_{i,\omega_2}$.

(d) (10 pts.) Our goal is to find the values of $V_i(t)$ that give us a desired value of $V_{\text{out}}(t)$ such that $\left\| \begin{bmatrix} \widetilde{V}_{1,\omega_1} & \widetilde{V}_{2,\omega_2} & \widetilde{V}_{3,\omega_1} \end{bmatrix}^\top \right\|$ is minimized. There are two output phasor values, $\widetilde{V}_{\text{out},\omega_1}$ and $\widetilde{V}_{\text{out},\omega_2}$, that corresponed to a desired value of $V_{\text{out}}(t)$. We stack our answers from parts (b) and (c) to get the following equation:

$$\underbrace{\begin{bmatrix} \widetilde{V}_{\text{out},\omega_1} \\ \widetilde{V}_{\text{out},\omega_2} \end{bmatrix}}_{\vec{\widetilde{V}}_{\text{out}}} = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \widetilde{V}_{1,\omega_1} \\ \widetilde{V}_{2,\omega_2} \\ \widetilde{V}_{3,\omega_1} \end{bmatrix}}_{\vec{\widetilde{V}}_{\text{in}}} \tag{8}$$

For your convenience, we also provide the following information:

$$\begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \frac{\sqrt{5}}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}}_{V^T} \tag{9}$$

Express the optimal values of $\widetilde{V}_{1,\omega_1}$, $\widetilde{V}_{2,\omega_2}$, and $\widetilde{V}_{3,\omega_1}$ achieving $\widetilde{V}_{\text{out},\omega_1}$ and $\widetilde{V}_{\text{out},\omega_2}$ while also minimizing $\left\| \begin{bmatrix} \widetilde{V}_{1,\omega_1} & \widetilde{V}_{2,\omega_2} & \widetilde{V}_{3,\omega_1} \end{bmatrix}^\top \right\|$.

(e) (4 pts.) Suppose we find that the optimum $\vec{\widetilde{V}}_{\text{in}} = \begin{bmatrix} \widetilde{V}_1^* \\ \widetilde{V}_2^* \\ \widetilde{V}_3^* \end{bmatrix} = \begin{bmatrix} e^{j\pi/2} \\ 2e^{j3\pi/4} \\ 3e^{j\pi/2} \end{bmatrix}$ and that $\omega_1 = 10\frac{\text{rad}}{\text{s}}$ and $\omega_2 = 20\frac{\text{rad}}{\text{s}}$. The relationship between $\vec{\widetilde{V}}_{\text{out}}$ and $\vec{\widetilde{V}}_{\text{in}}$ is still as it appeared in the last part:

$$\vec{\widetilde{V}}_{\text{out}} = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \vec{\widetilde{V}}_{\text{in}} \tag{10}$$

Write $V_{\text{out}}(t)$, the output voltage of our circuit in the time domain.

8. SVD Puzzle (10 pts.)

Given the matrix A in eq. (11), write out a singular value decomposition of matrix A in the form $U\Sigma V^\top$.

$$A = \frac{5}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1 & 4\end{bmatrix} \tag{11}$$

Note that you should order the singular values in Σ from largest to smallest.

(HINT: $\left\|\begin{bmatrix}1\\1\end{bmatrix}\right\| = \sqrt{2}$ $\quad \left\|\begin{bmatrix}1\\-1\\4\end{bmatrix}\right\| = \sqrt{18}$ $\quad \begin{bmatrix}2\\-2\\-1\end{bmatrix}^\top \begin{bmatrix}1\\1\\0\end{bmatrix} = 0$ $\quad \begin{bmatrix}2\\-2\\-1\end{bmatrix}^\top \begin{bmatrix}1\\-1\\4\end{bmatrix} = 0.$*)*

9. Compressed Learning (13 pts.)

(a) (6 pts.) Suppose we have a matrix $X \in \mathbb{R}^{d \times n}$ of noisy data points

$$X := \begin{bmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{bmatrix} \tag{12}$$

where each $\vec{x}_i \in \mathbb{R}^d$, and $n > d$. Suppose further that we observed some outputs y_i, where each $y_i = \vec{w}^\top \vec{x}_i$, for some unknown vector $\vec{w} \in \mathbb{R}^d$.

Now, your friend Adam wants to estimate new outputs $y_{n+1}, y_{n+2}, \ldots$ on his TI-Launchpad device as he sees new data points, $\vec{x}_{n+1}, \vec{x}_{n+2}, \cdots \in \mathbb{R}^d$, but it takes too much memory to store these $\vec{x}_i$. He compresses these $\vec{x}_i$ by projecting them onto the subspace spanned by the singular vectors corresponding to the largest ℓ singular values. Here, we can assume $\ell \ll d$. Suppose that X has SVD $X = U\Sigma V^\top$, and let U_k denote the sub-matrix of the first k columns of U (and likewise, let V_k denote the sub-matrix of the first k columns of V). Answer the following questions in order:

i. To save memory, instead of storing vectors $\vec{x}_i \in \mathbb{R}^d$, we can store smaller vectors of the ℓ coordinates of $\vec{x}_i$ projected onto the ℓ-dimensional subspace. Call these smaller vectors $\vec{\vec{x}}_i \in \mathbb{R}^\ell$. **For a new data point $\vec{x}_{\text{new}}$, what is the corresponding $\vec{\vec{x}}_{\text{new}}$?** Express $\vec{\vec{x}}_{\text{new}}$ in terms of the quantities provided.

(HINT: Based on how the data is arranged in X, should you project onto columns of U or columns of V?)

ii. Using the initial n data points, Adam gives you an estimate of the unknown $\vec{w} \in \mathbb{R}^d$: $\widehat{\vec{w}}$. $\widehat{\vec{w}}$ should allow you to estimate y_i for new $\vec{x}_i$. However, we cannot store the incoming $\vec{x}_i$ to compute y_i and must use $\vec{\vec{x}}_i$ instead. **What is the output, y_{new}, corresponding to a new data point $\vec{x}_{\text{new}}$, in terms of $\widehat{\vec{w}}$, $\vec{\vec{x}}_{\text{new}}$, and other provided quantities?** You may not use $\vec{x}_{\text{new}}$ in your answer.

(HINT: You may want to consider how to compute a projection, since $\vec{\vec{x}}_{new}$ are the ℓ coordinates of the projection into the subspace spanned by relevant singular vectors.)

(b) (3 pts.) Now, we will consider the problem of classification. Your friend Rohit has built a classifier where he finds the coordinates of his data on a 1-dimensional subspace generated from a single singular vector. Let s_i be the coordinate of $\vec{x}_i$. His classifier outputs '+1' if $s_i > 0$, '-1' if $s_i < 0$, and '0' if $s_i = 0$. Consider the following data matrix:

$$X := \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \vec{x}_4 \end{bmatrix} \tag{13}$$

$$= \begin{bmatrix} 1 & 0.5 & -0.5 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

For each data point $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4$, what would Rohit's classifier output? If writing the SVD of X, use the following (eigenvalue, eigenvector) pairs for $XX^\top$: $\left(2.5, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$.

(c) (4 pts.) Rohit accidentally drops his measurement device and now his data looks like

$$X_{\text{buggy}} := \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \vec{x}_4 \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} 1 & 0.5 & -0.5 & -1 \\ 100 & 100 & 100 & 100 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

i. **What is the classification of the data points now, using the same classification strategy with X_{buggy} as in part (b)?** If writing the SVD of X_{buggy}, use the following (eigenvalue, eigenvector) pairs for $X_{\text{buggy}} X_{\text{buggy}}^{\top}$: $\left(2.5, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \left(40000, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$.

ii. Suppose you were to "center" your data matrix X_{buggy}. The centered version of $X_{\text{buggy}} \in \mathbb{R}^{d \times n}$ can be written as

$$X_{\text{center}} = X_{\text{buggy}} - \begin{bmatrix} \mu_1 & \mu_1 & \cdots & \mu_1 \\ \mu_2 & \mu_2 & \cdots & \mu_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_d & \mu_d & \cdots & \mu_d \end{bmatrix} \tag{17}$$

where μ_i is the average value of all elements in row i of X_{buggy}. **Find X_{center}. How does it compare to X in part (b)?**

10. Reachability for Nonlinear Systems (19 pts.)

Consider a pendulum with some applied torque, $u(t)$, as a control input. The coordinate system for θ has the downward vertical as $\theta = 0$ with counterclockwise rotation as $\theta > 0$. This pendulum has the following dynamics equation:

$$ml^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mgl\sin(\theta) = u(t) \tag{18}$$

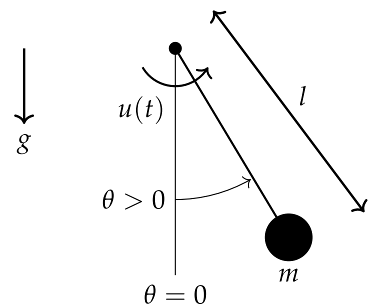


Figure 7: Pendulum

(a) (4 pts.) Let us define the state as $\vec{x}(t) = \begin{bmatrix} \theta \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} \end{bmatrix}$, and the values $m = l = 1$. **Use the definition of $\vec{x}(t)$ to rewrite the dynamics in eq. (18) in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{f}(\vec{x}(t), u(t))$. What is $\vec{f}(\vec{x}(t), u(t))$?**

(b) (3 pts.) **Given the diagram of the pendulum above, pick an appropriate operating point $(\vec{x}^\star, u^\star)$ such that the pendulum points straight upwards and remains stationary.** θ has units of radians. *(HINT: Take care to note how θ is defined above!)*

(c) (8 pts.) A linearized model of the system for variables $\delta\vec{x} := \vec{x} - \vec{x}^\star$ and $\delta u := u - u^\star$ is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{x}(t) = \delta A\delta\vec{x}(t) + \delta B\delta u(t) \tag{19}$$

δA and δB are matrices that depend on the operating point, $(\vec{x}^\star, u^\star)$. We choose the operating point for this linearization to be the one you found in the previous part. **Finish the linearization by finding the matrices δA and δB.**

(d) (2 pts.) Regardless of your answers to the previous parts, suppose that the linearized dynamics are

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\delta\vec{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\delta u \tag{20}$$

Is this operating point locally stable?

(e) (2 pts.) Suppose we discretized this system and obtained the following discrete, linear system:

$$\delta\vec{x}[i+1] = \underbrace{\begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{5}{2} & -1 \end{bmatrix}}_{A_d} \delta\vec{x}[i] + \underbrace{\begin{bmatrix} -1 \\ \frac{5}{2} \end{bmatrix}}_{\vec{b}_d} \delta u[i] \tag{21}$$

Is this system controllable? You may use the fact that

$$A_d\vec{b}_d = \begin{bmatrix} -\frac{9}{4} \\ -5 \end{bmatrix} \tag{22}$$

11. A Complex Triangularization of the Schur Variety (18 pts.)

Upper triangularization is the process of expressing a square matrix in terms of an upper triangular matrix via an orthonormal basis. Let's explore what happens when we try the procedure on a matrix with complex entries.

Our matrix we want to upper triangularize is $A = \begin{bmatrix} \frac{1}{2} + \mathrm{j} & \frac{\mathrm{j}}{2} \\ \frac{\mathrm{j}}{2} & -\frac{1}{2} + \mathrm{j} \end{bmatrix}$.

(a) (3 pts.) **Show that $\vec{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{\mathrm{j}}{\sqrt{2}} \end{bmatrix}$ is an eigenvector of A. What is its corresponding eigenvalue?**

(b) (5 pts.) **Use Gram-Schmidt to generate a vector $\vec{u}_2$ that is orthogonal to $\vec{u}_1$ from part (a) and has norm $\|\vec{u}_2\|$ equal to** 1.

(c) (10 pts.) We will proceed with one iteration of the Schur Decomposition Algorithm. **Calculate U^*AU, where $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{j}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. Is the result of your calculation an upper triangular matrix?**

12. Tracing Trace Proofs (14 pts.)

In parts (a) through (c), we will prove that the square of the Frobenius norm of $A \in \mathbb{R}^{n\times m}$ is equal to the sum of the eigenvalues of $A^\top A$. In other words, we will prove that $\|A\|_F^2 = \sum_{i=1}^{m} \lambda_i$, where each of the λ_i are one of the m eigenvalues of $A^\top A$. Then, we will synthesize our learnings in parts (d) and (e). Recall that matrix $A \in \mathbb{R}^{n\times m}$ has Frobenius norm $\|A\|_F$ that equals $\sqrt{\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{n} a_{ki}^2}$.

(a) (4 pts.) **Prove that $\|A\|_{\mathbf{F}}^2 = \operatorname{tr}\{A^\top A\}$ for $A \in \mathbb{R}^{2\times 3}$.**

(HINT: The trace of matrix M, $\operatorname{tr}\{M\}$, is the sum over its diagonal. It may help to identify what the entries of $A^\top A$ are in terms of the columns of A.)

(b) (2 pts.) **Show that $A^\top A$ simplifies down to $V\Sigma^\top \Sigma V^\top$ if the SVD of A is $U\Sigma V^\top$.**

(c) (2 pts.) Given that $\text{tr}\{QP\} = \text{tr}\{PQ\}$ for $Q \in \mathbb{R}^{m\times n}$ and $P \in \mathbb{R}^{n\times m}$, **show that** $\text{tr}\{A^\top A\} = \text{tr}\{\Sigma^\top \Sigma\}$ **using your result from part (b).**

(HINT: What is $V^\top V$?)

(d) (3 pts.) **Given that** $B \in \mathbb{R}^{2\times 2}$**, that** $\|B\|_F^2 = 5$**, and that** $B^\top B$ **has an eigenvalue equal to 5, what can we say about the invertibility of** B**?** Conclude and justify whether B is invertible, not invertible, or if we do not have enough information.

(HINT: What should the Σ matrix of B look like if it is invertible? In parts (a) through (c), we proved that $\text{tr}\{A^\top A\} = \sum_k \sigma_k^2 = \sum_i \lambda_i$, where σ_k describes the singular values of A and λ_i describes the eigenvalues of $A^\top A$.)

(e) (3 pts.) We want to understand the stability of the following continuous system, where $M \in \mathbb{R}^{n\times n}$, $\vec{x} \in \mathbb{R}^n$, and $\vec{b} \neq \vec{0}_n$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = M\vec{x}(t) + \vec{b}u(t) \tag{23}$$

Given that M has $\mathrm{tr}\{M^\top M\} = 0$, what can we say about the stability of our system? Conclude and justify whether the system is stable, not stable, or if we do not have enough information.

(HINT: Try using the statement in the previous part to answer: what are the singular values of M, if $\mathrm{tr}\{M^\top M\} = 0$? Given this, what will $M = U\Sigma V^\top$ be?)

[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.

If needed, you can also use this space to work on problems. But if you want the work on this page to be graded, make sure you tell us on the problem's main page.]

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]