

Exam Location: Draft

PRINT your student ID: \_\_\_\_\_

PRINT AND SIGN your name: \_\_\_\_\_, \_\_\_\_\_  
(last) (first) (sign)

PRINT your discussion sections and (u)GSIs (the ones you attend): \_\_\_\_\_

Row Number: \_\_\_\_\_ Seat Number: \_\_\_\_\_

Name and SID of the person to your left: \_\_\_\_\_

Name and SID of the person to your right: \_\_\_\_\_

Name and SID of the person in front of you: \_\_\_\_\_

Name and SID of the person behind you: \_\_\_\_\_

**1. Honor Code (0 pts.)**

**Please copy the following statement in the space provided below and sign your name.**

*As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.*

**Note that if you do not copy the honor code and sign your name, you will get a 0 on the exam.**

**Solution:** Any attempt to copy the honor code and sign should get full points.

**2. Tell us something you're excited about. (2 pts.)**

**Solution:** Any answer is sufficient.

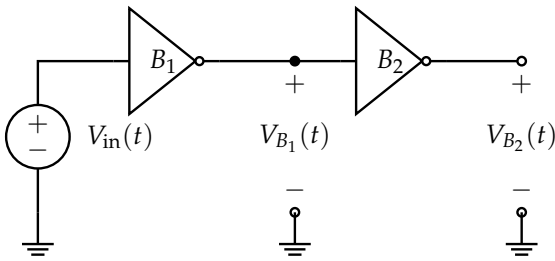
**3. What are you looking forward to this weekend? (2 pts.)**

**Solution:** Any answer is sufficient.

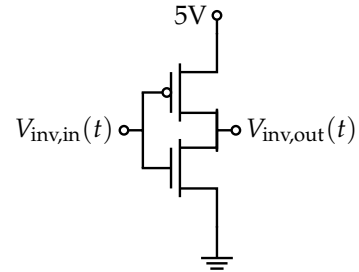
Do not turn this page until the proctor tells you to do so.  
You can work on the above problems before time starts.

**4. CMOS Threshold Engineering (12 pts.)**

In this problem, you will analyze the behavior of the inverter chain in fig. 1 below.



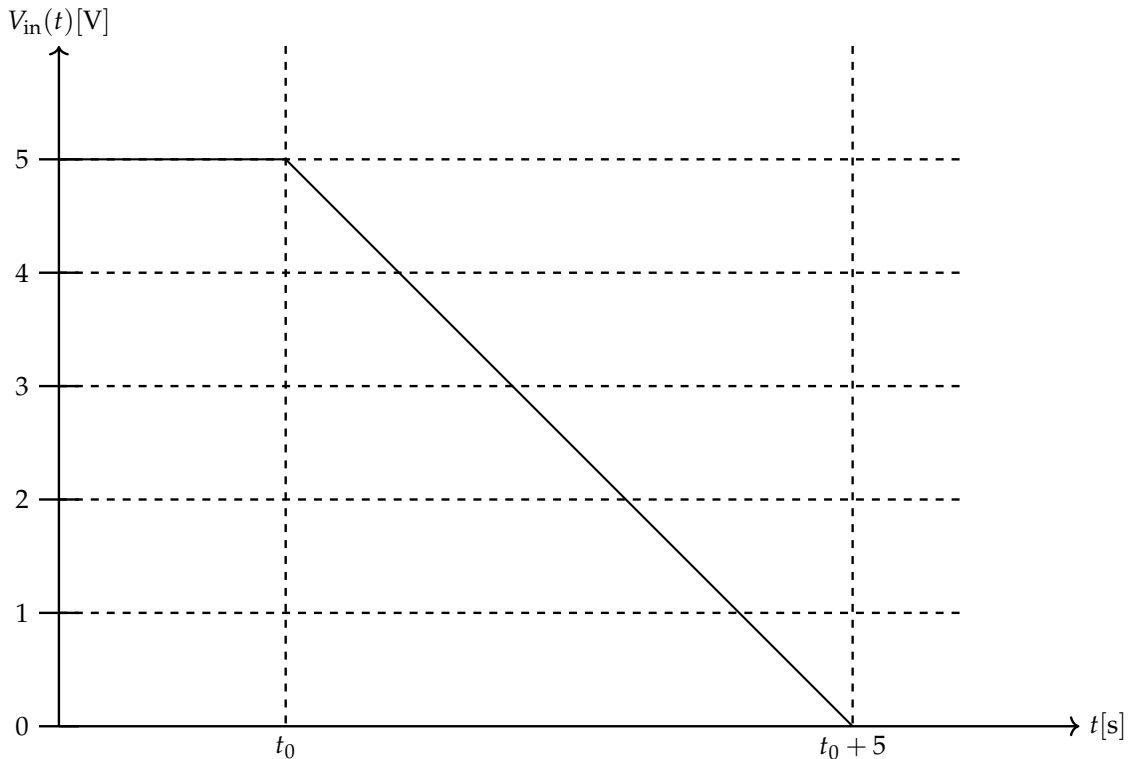
**Figure 1:** Inverter chain circuit.



**Figure 2:** Standard CMOS inverter.

The inverter chain in fig. 1 has two inverters,  $B_1$  and  $B_2$ . These inverters are standard CMOS inverters made of a PMOS and NMOS transistor (fig. 2). The PMOS transistors have threshold voltage  $|V_{tp}| = 2\text{ V}$  and the NMOS transistors have threshold voltage  $V_{tn} = 1\text{ V}$ .

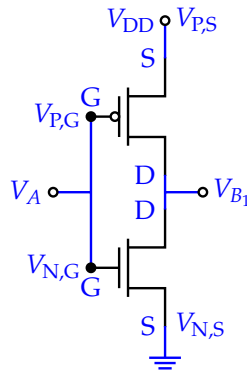
- (a) (4 pts.) The input voltage  $V_{in}(t)$  is fed to  $B_1$  in fig. 1. Starting at time  $t_0$ ,  $V_{in}(t)$  goes from 5V to 0V with a delay of 5 seconds.  $V_{in}(t)$  is plotted below in fig. 3. **Mark the time the PMOS of  $B_1$  turns on,  $t_P$ , and the time the NMOS of  $B_1$  turns off,  $t_N$ , with vertical lines in the graph below.**



**Figure 3:** Input voltage to inverter  $B_1$ .

**Solution:** The PMOS of  $B_1$  turns on when  $V_{P,GS} \leq -|V_{tp}|$  and the NMOS of  $B_1$  turns off when  $V_{N,GS} \leq V_{tn}$ . Looking at the CMOS circuit for  $B_1$  labeled below with gate (G), drain (D), and

source (S) and their corresponding voltages for both NMOS and PMOS, we can examine the conditions further.



We have for the PMOS condition:

$$V_{P,GS} \leq -|V_{tp}| \tag{1}$$

$$V_{P,G} - V_{P,S} \leq -|V_{tp}| \tag{2}$$

$$V_A - V_{DD} \leq -|V_{tp}| \tag{3}$$

$$V_A \leq V_{DD} - |V_{tp}| \tag{4}$$

$$V_A \leq 5V - 2V \tag{5}$$

$$V_A \leq 3V \tag{6}$$

We have for the NMOS condition:

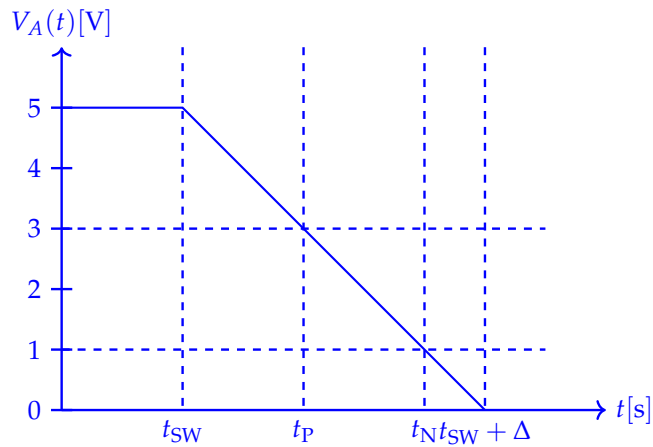
$$V_{N,GS} \leq V_{tn} \tag{7}$$

$$V_{N,G} - V_{N,S} \leq V_{tn} \tag{8}$$

$$V_A - 0V \leq 1V \tag{9}$$

$$V_A \leq 1V \tag{10}$$

The PMOS turns on when  $V_A$  drops below 3V, and the NMOS turns off when  $V_A$  drops below 1V. We can depict this on the graph as follows:



(b) (8 pts.) You should have found that  $t_P < t_N$ . For times,  $t$ , such that  $t_P \leq t \leq t_N$ , we have both transistors of the  $B_1$  inverter on, leading to the following model of the output of the  $B_1$  inverter as shown in fig. 4.

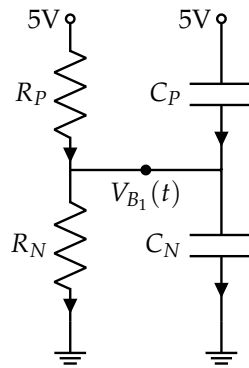
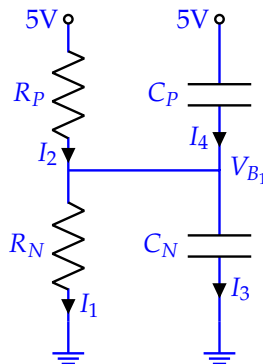


Figure 4:  $B_1$  Inverter's output model.

Setup a differential equation for  $V_{B_1}(t)$  from fig. 4 in the form of eq. (11) below and identify  $\lambda$  and  $c$  in terms of  $R_P, R_N, C_N, C_P$ , and known constants. Do not solve the differential equation.

$$\frac{d}{dt} V_{B_1}(t) = \lambda V_{B_1}(t) + c \tag{11}$$

**Solution:**



Labeling the circuit with current directions and writing the KCL equation at the  $V_{B_1}$  node, we have:

$$I_1 + I_3 = I_2 + I_4 \quad (12)$$

$$\frac{V_{B_1} - 0V}{R_N} + C_N \frac{d}{dt} (V_{B_1} - 0V) = \frac{V_{DD} - V_{B_1}}{R_P} + C_P \frac{d}{dt} (V_{DD} - V_{B_1}) \quad (13)$$

$$(C_N + C_P) \frac{d}{dt} V_{B_1} = - \left( \frac{1}{R_N} + \frac{1}{R_P} \right) V_{B_1} + \frac{V_{DD}}{R_P} \quad (14)$$

$$\frac{d}{dt} V_{B_1} = - \frac{V_{B_1}}{(R_N || R_P)(C_N + C_P)} + \frac{V_{DD}}{R_P(C_N + C_P)} \quad (15)$$

Thus  $x(t) = V_{B_1}(t)$ ,  $\lambda = -\frac{1}{(R_N || R_P)(C_N + C_P)}$ , and  $c = \frac{V_{DD}}{R_P(C_N + C_P)}$ .

**5. Matrix Differential Equation Excited by Eigenvector Input (15 pts.)**

Consider the differential equation, eq. (16), in terms of  $\vec{x}(t), \vec{u}(t) \in \mathbb{R}^N$ .

$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + \vec{u}(t) \tag{16}$$

Let  $A$  be a  $N \times N$  matrix with  $N$  distinct, real, non-zero eigenvalues, and eigenvalue-eigenvector pairs

$(\lambda_i, \vec{v}_i)$  for  $i = 1, \dots, N$ . With  $V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_N \\ | & & | \end{bmatrix}$  as the matrix of the eigenvectors, let  $\tilde{\vec{x}}(t) = V^{-1}\vec{x}(t)$ ,

which is  $\vec{x}(t)$  represented in the  $V$  basis. The differential equation, eq. (16), written in terms of  $\tilde{\vec{x}}(t)$  appears below in eq. (17).

$$\frac{d\tilde{\vec{x}}(t)}{dt} = \tilde{A}\tilde{\vec{x}}(t) + \tilde{B}\vec{u}(t) \tag{17}$$

$\tilde{A}$  and  $\tilde{B}$  are  $N \times N$  matrices.

(a) (6 pts.) Express  $\tilde{A}$  and  $\tilde{B}$  from eq. (17) in terms of  $\lambda_i, \vec{v}_i, V$ , and  $V^{-1}$ . No need to show work.

**Solution:** We can state that  $A$  is diagonalizable as  $A = V \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} V^{-1}$ , and that our coordinate change is defined by  $\vec{x} = V\tilde{\vec{x}}$  and  $\tilde{\vec{x}} = V^{-1}\vec{x}$ . Starting from the first equation:

$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + \vec{u}(t) \tag{18}$$

$$V \frac{d\tilde{\vec{x}}(t)}{dt} = V \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} V^{-1}V\tilde{\vec{x}}(t) + \vec{u}(t) \tag{19}$$

$$V^{-1} \left( V \frac{d\tilde{\vec{x}}(t)}{dt} \right) = V^{-1} \left( V \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} \tilde{\vec{x}}(t) + \vec{u}(t) \right) \tag{20}$$

$$I \frac{d\tilde{\vec{x}}(t)}{dt} = I \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} \tilde{\vec{x}}(t) + V^{-1}\vec{u}(t) \tag{21}$$

$$\frac{d\tilde{\vec{x}}(t)}{dt} = \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}}_{\tilde{A}} \tilde{\vec{x}}(t) + \underbrace{V^{-1}\vec{u}(t)}_{\tilde{B}} \tag{22}$$

So  $\tilde{A} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}$ , the diagonal matrix of the eigenvalues, and  $\tilde{B} = V^{-1}$ .

- (b) (3 pts.) For a square matrix,  $M$ , let  $\vec{m}_i$  denote its  $i$ -th column, and let  $(M^{-1})_i$  denote the  $i$ -th column of  $M$ 's inverse,  $M^{-1}$ . **For our system in eq. (16), let  $N = 5$ . We choose our input to be  $\vec{u}(t) = (\tilde{B}^{-1})_3$ . What is  $\tilde{B}\vec{u}(t)$ ?** Recall that  $\tilde{B}$  is  $N \times N$ .

(HINT: If  $P = Q^{-1}$ ,  $PQ = P \begin{bmatrix} | & & | \\ \vec{q}_1 & \cdots & \vec{q}_N \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ P\vec{q}_1 & \cdots & P\vec{q}_N \\ | & & | \end{bmatrix} = I.$ )

**Solution:** We must calculate  $\tilde{B}\vec{u}(t) = \tilde{B}(\tilde{B}^{-1})_k$ . From the hint, we have that if  $A = \tilde{B}$  and  $B = \tilde{B}^{-1}$ , that the  $k$ -th column of  $\tilde{B}\tilde{B}^{-1}$  is  $\tilde{B}(\tilde{B}^{-1})_k$ , but that this is also the  $k$ -th column of the identity matrix,  $I$ . Thus  $\tilde{B}\vec{u}(t) = \vec{e}_k$  where  $\vec{e}_k$  is the  $k$ -th column of  $I$ . ( $\vec{e}_k$  has a 1 in the  $k$ -th entry, zeroes everywhere else.)

- (c) (6 pts.) For the system represented in the  $V$  basis in eq. (17), let  $\tilde{B}\vec{u}(t) = \vec{w}$ , where  $\vec{w}$  has  $k$ -th entry  $w_k = c \neq 0$  and all other entries  $w_i = 0$ .

**With the initial condition that  $\vec{x}(0) = \vec{0}$ , solve for  $\vec{x}(t)$  for  $t \geq 0$  when  $\tilde{B}\vec{u}(t) = \vec{w}$  in terms of  $\lambda_i, \vec{v}_i, t$ , and other relevant constants.** It may be of use to recall that the  $\lambda_i$  are all non-zero. Additionally, we give you the solution to the differential equation eq. (23) below with constant input  $u$ .

$$\frac{d}{dt}x(t) = \lambda x(t) + u, x(0) = x_0 \tag{23}$$

The solution to eq. (23) is  $x(t) = x_0 e^{\lambda t} - \frac{u}{\lambda}(1 - e^{\lambda t})$ .

**Solution:** We can state the initial condition in the eigenvector basis coordinates,  $\tilde{\vec{x}}(0) = V^{-1}\vec{x}(0) = V^{-1}\vec{0} = \vec{0}$ . Then, from the work done in part (a), we can state the differential equation for each of the rows of the decoupled differential equation system.

$$\frac{d\tilde{\vec{x}}(t)}{dt} = \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}}_{\tilde{A}} \tilde{\vec{x}}(t) + \underbrace{V^{-1}}_{\tilde{B}} \vec{u}(t) \tag{24}$$

$$\frac{d\tilde{\vec{x}}(t)}{dt} = \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}}_{\tilde{A}} \tilde{\vec{x}}(t) + \vec{w} \tag{25}$$

$$\implies \begin{cases} \frac{d\tilde{x}_i(t)}{dt} = \lambda_i \tilde{x}_i(t) + c & i = k \\ \frac{d\tilde{x}_i(t)}{dt} = \lambda_i \tilde{x}_i(t) & i \neq k \end{cases} \tag{26}$$

For  $\tilde{x}_i$ , we can guess a solution of the form  $\tilde{x}_i(t) = \alpha_i + \beta_i e^{\gamma_i t}$ . The  $\alpha_i$  is needed due to possible non-zero input ( $i = k$  case). From the initial conditions we have that  $\tilde{x}_i(0) = \alpha_i + \beta_i = 0 \implies \beta_i = -\alpha_i$ . Thus our solution looks like  $\tilde{x}_i(t) = \alpha_i(1 - e^{\gamma_i t})$ . We get the following when we substitute this solution into the differential equation:

$$\frac{d\tilde{x}_i(t)}{dt} = -\alpha_i \gamma_i e^{\gamma_i t} = \begin{cases} \lambda_i \alpha_i (1 - e^{\gamma_i t}) + c & i = k \\ \lambda_i \alpha_i (1 - e^{\gamma_i t}) & i \neq k \end{cases} \tag{27}$$

$$0 = \begin{cases} \lambda_i \alpha_i + c + \alpha_i (\gamma_i - \lambda_i) e^{\gamma_i t} & i = k \\ \lambda_i \alpha_i + \alpha_i (\gamma_i - \lambda_i) e^{\gamma_i t} & i \neq k \end{cases} \quad (28)$$

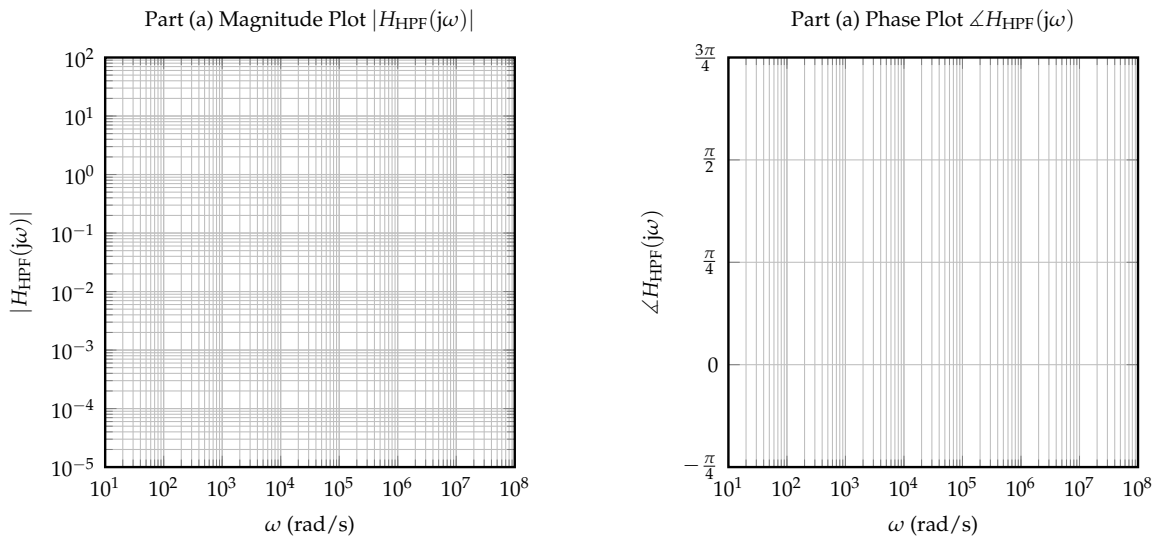
To make both statements true for all time, we should make  $\gamma_i = \lambda_i$  as the exponential term will be non-zero for at least one time, and we know that the  $\lambda_i$  are non-zero. In the  $i = k$  case, what is leftover is that we must satisfy that  $\lambda_i \alpha_i + c = 0 \implies \alpha_i = -\frac{c}{\lambda_i}$  whereas in the  $i \neq k$  case, we have that  $\lambda_i \alpha_i = 0$  so  $\alpha_i = 0$ . Thus  $\tilde{x}_i(t) = 0$  for  $i \neq k$  and  $\tilde{x}_k(t) = -\frac{c}{\lambda_k} (1 - e^{\lambda_k t})$ .

To reconstitute  $\vec{x}(t)$ , we know that it is  $\vec{x}(t) = V \vec{\tilde{x}}(t)$ . Since the only non-zero entry of  $\vec{\tilde{x}}(t)$  is  $\tilde{x}_k(t)$ , we should only pick out  $\vec{v}_k$ , the  $k$ -th column of  $V$ , giving us that our differential equation solution is  $\vec{x}(t) = \tilde{x}_k(t) \vec{v}_k = -\frac{c}{\lambda_k} (1 - e^{\lambda_k t}) \vec{v}_k$ .



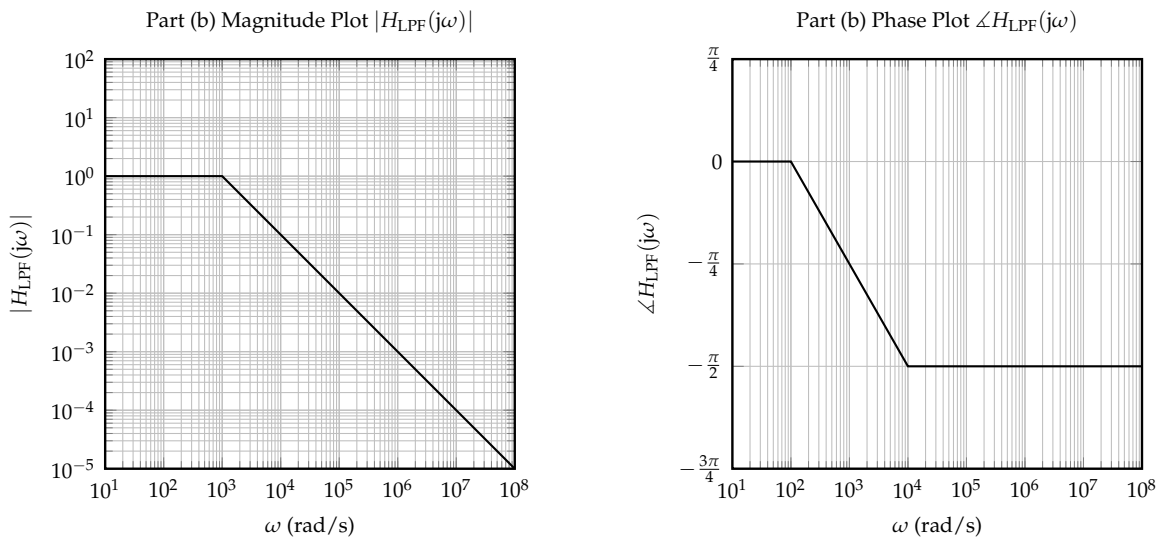
**6. To Bode Or Not To Bode (10 pts.)**

(a) (4 pts.) Draw the magnitude and phase Bode plots of the high-pass filter  $H_{\text{HPF}}(j\omega) = 10 \frac{j\omega}{1 + j\frac{\omega}{10^4}}$ .



**Figure 5:** Part (a) Magnitude and Phase Bode Plots for the transfer function  $H_{\text{HPF}}(j\omega)$ .

(b) (6 pts.) Given the following Bode plots describing a system's behavior, what would the output of the system be if the input were  $20 \cos(10^5 t) + 5 \cos(10^3 t + \frac{\pi}{4})$ ?



**Figure 6:** Part (b) Magnitude and Phase Bode Plots for the transfer function  $H_{\text{LPF}}(j\omega)$ .

**7. Minimum Energy Solutions for Phasors (25 pts.)**

Suppose we have three voltage sources with voltage values  $V_1(t)$ ,  $V_2(t)$ , and  $V_3(t)$  to be utilized in a circuit. For an angular frequency,  $\omega$ ,  $\tilde{V}_{i,\omega}$  denotes the phasor of  $V_i(t)$  corresponding to that  $\omega$ .

(a) (3 pts.) The voltages,  $V_i(t)$ , are defined in the following way.

$$V_1(t) = A_1 \cos(\omega_1 t + \phi_1) \tag{29}$$

$$V_2(t) = A_2 \cos(\omega_2 t + \phi_2) \tag{30}$$

$$V_3(t) = A_3 \cos(\omega_1 t + \phi_3) \tag{31}$$

**Fill in the table below with the phasors of the  $V_i(t)$  for each of the two frequencies  $\omega_1$  and  $\omega_2$  in terms of  $A_i$ ,  $\omega_i$ ,  $\phi_i$ , and any other relevant constants.**

$\tilde{V}_{1,\omega_1}$	$\tilde{V}_{2,\omega_1}$	$\tilde{V}_{3,\omega_1}$		$\tilde{V}_{1,\omega_2}$	$\tilde{V}_{2,\omega_2}$	$\tilde{V}_{3,\omega_2}$

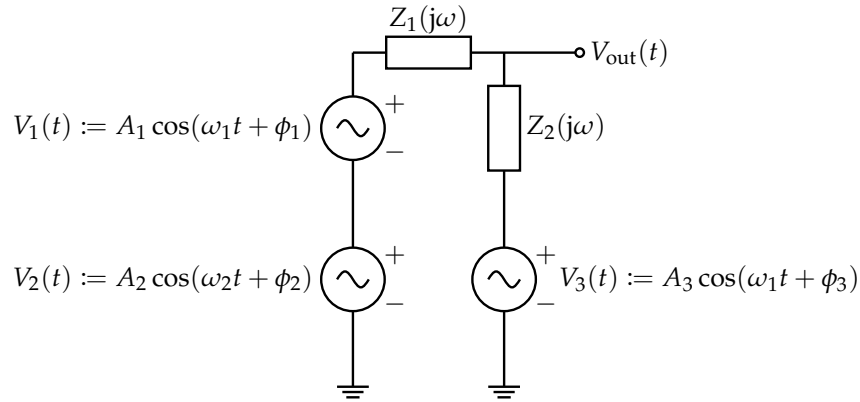
**Solution:** We can convert from time domain to find the phasor forms for these respective elements:

$$\tilde{V}_1 = \frac{A_1}{2} e^{j\phi_1} \tag{32}$$

$$\tilde{V}_2 = \frac{A_2}{2} e^{j\phi_2} \tag{33}$$

$$\tilde{V}_3 = \frac{A_3}{2} e^{j\phi_3} \tag{34}$$

(b) (5 pts.) Now consider a circuit using the three voltage sources shown below.



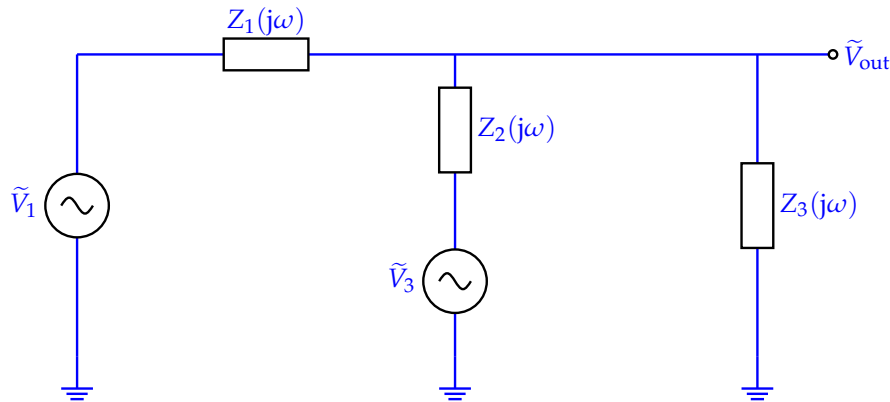
**Solve for the output voltage phasor,  $\tilde{V}_{out, \omega_1}$ , the phasor of  $V_{out}(t)$  associated with the frequency  $\omega_1$ .** You may express your answer using any of  $\omega_i$ ,  $Z_i(j\omega_i)$ , and  $\tilde{V}_{i, \omega_1}$ .

(HINT: Superposition may be useful here.)

**Solution:** As the hint implies, we can solve for the frequency response of  $\omega_1$  separately by shorting the  $\omega_2$  voltage source and solving the resultant phasor system.

Strategy #1:

Apply Superposition to remove  $\omega_2$  and solve both  $\omega_1$  voltage sources together. The resulting circuit looks like this:



We can then apply KCL on the  $V_{out}$  node:

$$\sum I_{in} = \sum I_{out} \tag{35}$$

$$\frac{\tilde{V}_1 - \tilde{V}_{out}}{Z_1(j\omega)} + \frac{\tilde{V}_3 - \tilde{V}_{out}}{Z_2(j\omega)} = \frac{\tilde{V}_{out}}{Z_3(j\omega)} \tag{36}$$

$$\frac{\tilde{V}_1}{Z_1(j\omega)} + \frac{\tilde{V}_3}{Z_2(j\omega)} = \tilde{V}_{out} \left( \frac{1}{Z_1(j\omega)} + \frac{1}{Z_2(j\omega)} + \frac{1}{Z_3(j\omega)} \right) \tag{37}$$

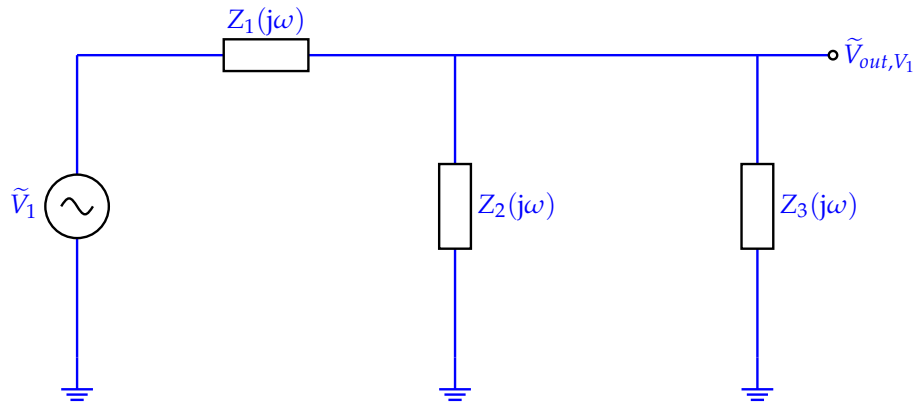
$$\tilde{V}_{out} = \left( \frac{1}{Z_1(j\omega)} + \frac{1}{Z_2(j\omega)} + \frac{1}{Z_3(j\omega)} \right)^{-1} \left( \frac{\tilde{V}_1}{Z_1(j\omega)} + \frac{\tilde{V}_3}{Z_2(j\omega)} \right) \tag{38}$$

Since our circuit zeroed out the  $\omega_2$  sources, we know  $\tilde{V}_{out, \omega_1} = \tilde{V}_{out}$ .

Strategy #2:

Apply Superposition to remove  $\omega_2$ ; apply superposition individually on both  $\omega_1$  voltage sources and add them together.

Let's zero out  $\tilde{V}_3$  first. We will get the following circuit:



We can then apply KCL on the  $V_{out}$  node:

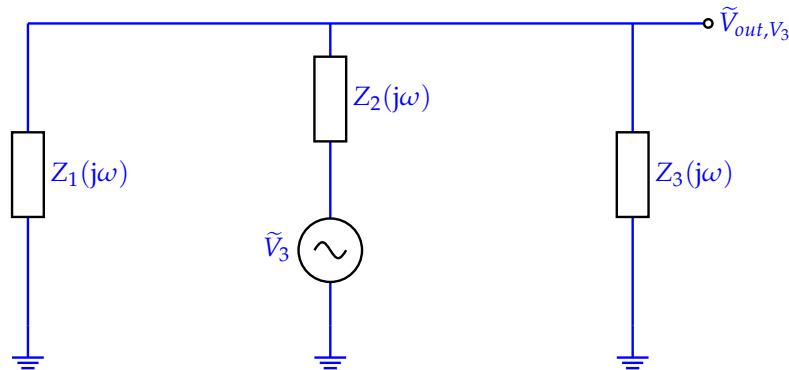
$$\sum I_{in} = \sum I_{out} \tag{39}$$

$$\frac{\tilde{V}_1 - \tilde{V}_{out,V_1}}{Z_1(j\omega)} = \frac{\tilde{V}_{out,V_1}}{Z_3(j\omega)} + \frac{\tilde{V}_{out,V_1}}{Z_2(j\omega)} \tag{40}$$

$$\frac{\tilde{V}_1}{Z_1(j\omega)} = \tilde{V}_{out,V_1} \left( \frac{1}{Z_1(j\omega)} + \frac{1}{Z_2(j\omega)} + \frac{1}{Z_3(j\omega)} \right) \tag{41}$$

$$\tilde{V}_{out,V_1} = \left( \frac{1}{Z_1(j\omega)} + \frac{1}{Z_2(j\omega)} + \frac{1}{Z_3(j\omega)} \right)^{-1} \left( \frac{\tilde{V}_1}{Z_1(j\omega)} \right) \tag{42}$$

Now, let's zero out  $\tilde{V}_1$ . We will get the following circuit:



We can then apply KCL on the  $V_{out}$  node:

$$\sum I_{in} = \sum I_{out} \tag{43}$$

$$\frac{\tilde{V}_3 - \tilde{V}_{out,V_3}}{Z_2(j\omega)} = \frac{\tilde{V}_{out,V_3}}{Z_1(j\omega)} + \frac{\tilde{V}_{out,V_3}}{Z_3(j\omega)} \tag{44}$$

$$\frac{\tilde{V}_3}{Z_2(j\omega)} = \tilde{V}_{out,V_3} \left( \frac{1}{Z_1(j\omega)} + \frac{1}{Z_2(j\omega)} + \frac{1}{Z_3(j\omega)} \right) \quad (45)$$

$$\tilde{V}_{out,V_3} = \left( \frac{1}{Z_1(j\omega)} + \frac{1}{Z_2(j\omega)} + \frac{1}{Z_3(j\omega)} \right)^{-1} \left( \frac{\tilde{V}_3}{Z_2(j\omega)} \right) \quad (46)$$

By the linearity property of superposition, we can add the individual voltages of  $\tilde{V}_{out}$  from sources  $\tilde{V}_{out,V_1}$  and  $\tilde{V}_{out,V_3}$  together to get  $\tilde{V}_{out}$ .

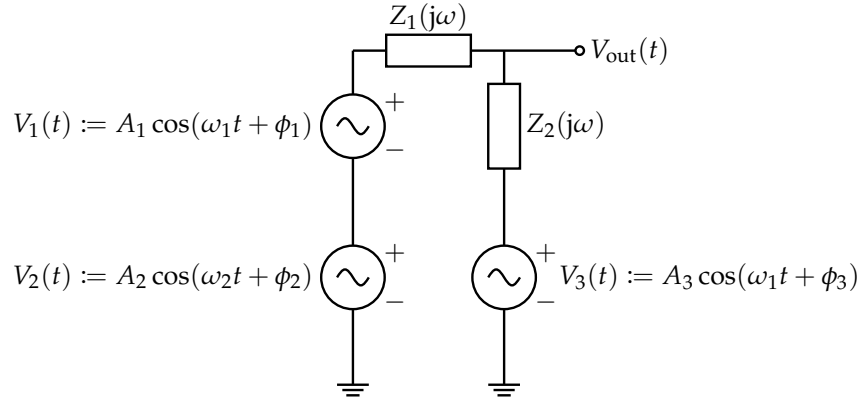
$$\tilde{V}_{out} = \tilde{V}_{out,V_1} + \tilde{V}_{out,V_3} \quad (47)$$

$$= \left( \frac{1}{Z_1(j\omega)} + \frac{1}{Z_2(j\omega)} + \frac{1}{Z_3(j\omega)} \right)^{-1} \left( \frac{\tilde{V}_3}{Z_2(j\omega)} \right) + \left( \frac{1}{Z_1(j\omega)} + \frac{1}{Z_2(j\omega)} + \frac{1}{Z_3(j\omega)} \right)^{-1} \left( \frac{\tilde{V}_1}{Z_1(j\omega)} \right) \quad (48)$$

$$= \left( \frac{1}{Z_1(j\omega)} + \frac{1}{Z_2(j\omega)} + \frac{1}{Z_3(j\omega)} \right)^{-1} \left( \frac{\tilde{V}_1}{Z_1(j\omega)} + \frac{\tilde{V}_3}{Z_2(j\omega)} \right) \quad (49)$$

Note: Even though in both parts we use superposition to eliminate the second voltage source, since we are left with voltage sources with the same frequency, we do not need to apply superposition on each of the individual voltage sources that rely on  $\omega_1$ , making the problem that much simpler.

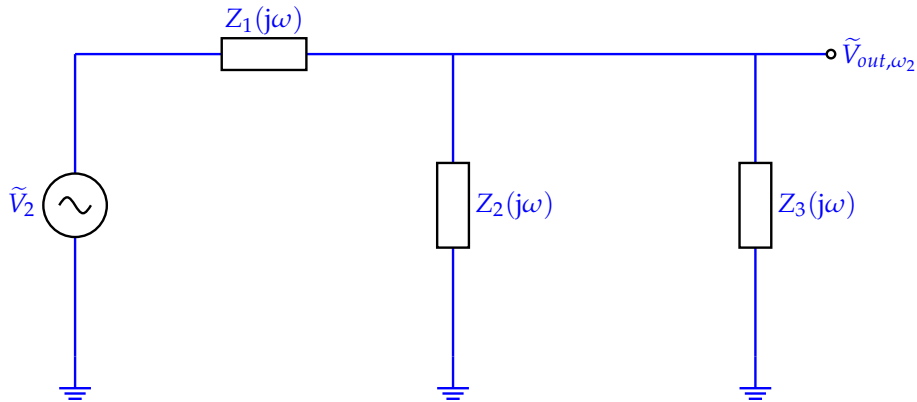
(c) (3 pts.) The circuit from part (b) is repeated below.



**Solve for the output voltage phasor,  $\tilde{V}_{out,\omega_2}$ , the phasor of  $V_{out}(t)$  associated with the frequency  $\omega_2$ . You may express your answer using any of  $\omega_i$ ,  $Z_i(j\omega_2)$ , and  $\tilde{V}_{i,\omega_2}$ . **Solution:****

As the hint implies, we can solve for the frequency response of  $\omega_2$  separately by shorting the blue  $\omega_1$  voltage sources and solving the resultant phasor system.

Since we are left with a single voltage source with frequency  $\omega_2$ ,  $\tilde{V}_{out,\omega_2} = \tilde{V}_{out}$ . The resultant circuit is shown here:



We can again run KCL on the  $\tilde{V}_{out,\omega_2}$  node to solve for  $\tilde{V}_{out,\omega_2}$ .

$$\sum I_{in} = \sum I_{out} \tag{50}$$

$$\frac{\tilde{V}_2 - \tilde{V}_{out,\omega_2}}{Z_1(j\omega)} = \frac{\tilde{V}_{out,\omega_2}}{Z_3(j\omega)} + \frac{\tilde{V}_{out,\omega_2}}{Z_2(j\omega)} \tag{51}$$

$$\frac{\tilde{V}_2}{Z_1(j\omega)} = \tilde{V}_{out,\omega_2} \left( \frac{1}{Z_1(j\omega)} + \frac{1}{Z_2(j\omega)} + \frac{1}{Z_3(j\omega)} \right) \tag{52}$$

$$\tilde{V}_{out,\omega_2} = \left( \frac{1}{Z_1(j\omega)} + \frac{1}{Z_2(j\omega)} + \frac{1}{Z_3(j\omega)} \right)^{-1} \left( \frac{\tilde{V}_2}{Z_1(j\omega)} \right) \tag{53}$$

(d) (10 pts.) Our goal is to find the values of  $V_i(t)$  that give us a desired value of  $V_{out}(t)$  such that  $\left\| \begin{bmatrix} \tilde{V}_{1,\omega_1} & \tilde{V}_{2,\omega_2} & \tilde{V}_{3,\omega_1} \end{bmatrix}^T \right\|$  is minimized. There are two output phasor values,  $\tilde{V}_{out,\omega_1}$  and  $\tilde{V}_{out,\omega_2}$ , that corresponded to a desired value of  $V_{out}(t)$ . We stack our answers from parts (b) and (c) to get the following equation:

$$\underbrace{\begin{bmatrix} \tilde{V}_{out,\omega_1} \\ \tilde{V}_{out,\omega_2} \end{bmatrix}}_{\vec{\tilde{V}}_{out}} = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \tilde{V}_{1,\omega_1} \\ \tilde{V}_{2,\omega_2} \\ \tilde{V}_{3,\omega_1} \end{bmatrix}}_{\vec{\tilde{V}}_{in}} \quad (54)$$

For your convenience, we also provide the following information:

$$\begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_U \underbrace{\begin{bmatrix} \frac{\sqrt{5}}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}}_\Sigma \underbrace{\begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}}_{V^T} \quad (55)$$

Express the optimal values of  $\tilde{V}_{1,\omega_1}$ ,  $\tilde{V}_{2,\omega_2}$ , and  $\tilde{V}_{3,\omega_1}$  achieving  $\tilde{V}_{out,\omega_1}$  and  $\tilde{V}_{out,\omega_2}$  while also minimizing  $\left\| \begin{bmatrix} \tilde{V}_{1,\omega_1} & \tilde{V}_{2,\omega_2} & \tilde{V}_{3,\omega_1} \end{bmatrix}^T \right\|$ .

**Solution:** By definition, we know that for wide matrices, we have an underdetermined system with infinitely many solutions. In order to solve for the unique solution that minimizes the norm of our input vector, we can multiply by the pseudoinverse of  $A$  to get the  $V$  vector that has the minimum norm.

First, we can solve for the pseudoinverse from the Singular Value Decomposition of  $A$  provided to us.

$$A^\dagger = V\Sigma^\dagger U^T \quad (56)$$

$$= \begin{bmatrix} 0 & \frac{3}{5} & -\frac{4}{5} \\ 1 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (57)$$

$$= \begin{bmatrix} 0 & \frac{3}{25} \\ \frac{1}{5} & 0 \\ 0 & \frac{4}{25} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (58)$$

$$= \begin{bmatrix} \frac{3}{25} & 0 \\ 0 & \frac{1}{5} \\ \frac{4}{25} & 0 \end{bmatrix} \quad (59)$$

Using the pseudoinverse, we can now solve for the optimum  $\vec{\tilde{V}}^*$ :

$$\vec{\tilde{V}}^* = A^\dagger \vec{\tilde{V}}_{out} \quad (60)$$

$$= \begin{bmatrix} \frac{3}{25} & 0 \\ 0 & \frac{1}{5} \\ \frac{4}{25} & 0 \end{bmatrix} \begin{bmatrix} \tilde{V}_{out,\omega_1} \\ \tilde{V}_{out,\omega_2} \end{bmatrix} \quad (61)$$

$$= \begin{bmatrix} \frac{3}{25} \tilde{V}_{\text{out},\omega_1} \\ \frac{1}{5} \tilde{V}_{\text{out},\omega_2} \\ \frac{4}{25} \tilde{V}_{\text{out},\omega_1} \end{bmatrix} \quad (62)$$

$$= \begin{bmatrix} \tilde{V}_{1,\omega_1}^* \\ \tilde{V}_{2,\omega_2}^* \\ \tilde{V}_{3,\omega_1}^* \end{bmatrix} \quad (63)$$

(e) (4 pts.) Suppose we find that the optimum  $\vec{\tilde{V}}_{\text{in}} = \begin{bmatrix} \tilde{V}_1^* \\ \tilde{V}_2^* \\ \tilde{V}_3^* \end{bmatrix} = \begin{bmatrix} e^{j\pi/2} \\ 2e^{j3\pi/4} \\ 3e^{j\pi/2} \end{bmatrix}$  and that  $\omega_1 = 10 \frac{\text{rad}}{\text{s}}$  and  $\omega_2 = 20 \frac{\text{rad}}{\text{s}}$ . The relationship between  $\vec{\tilde{V}}_{\text{out}}$  and  $\vec{\tilde{V}}_{\text{in}}$  is still as it appeared in the last part:

$$\vec{\tilde{V}}_{\text{out}} = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \vec{\tilde{V}}_{\text{in}} \quad (64)$$

**Write  $V_{\text{out}}(t)$ , the output voltage of our circuit in the time domain.**

**Solution:** From the previous part, we know the matrix-vector relationship between  $\vec{\tilde{V}}_{\text{in}}$  and  $\vec{\tilde{V}}_{\text{out}}$ , so we can solve for  $\vec{\tilde{V}}_{\text{out}}$ .

$$\vec{\tilde{V}}_{\text{out}} = A \vec{\tilde{V}}_{\text{in}} \quad (65)$$

$$= \begin{bmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} e^{j\pi/2} \\ 2e^{j3\pi/4} \\ 3e^{j\pi/2} \end{bmatrix} \quad (66)$$

$$= \begin{bmatrix} 3e^{j\pi/2} + 12e^{j\pi/2} \\ 10e^{j3\pi/4} \end{bmatrix} \quad (67)$$

$$= \begin{bmatrix} 15e^{j\pi/2} \\ 10e^{j3\pi/4} \end{bmatrix} \quad (68)$$

By definition of superposition, we can convert these phasors back to time domain (remembering to multiply by 2) and add the entries together:

$$V_{\text{out}} = 30 \cos(10t + \pi/2) + 20 \cos(20t + 3\pi/4)$$



**8. SVD Puzzle (10 pts.)**

Given the matrix  $A$  in eq. (69), write out a singular value decomposition of matrix  $A$  in the form  $U\Sigma V^T$ .

$$A = \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \end{bmatrix} \quad (69)$$

Note that you should order the singular values in  $\Sigma$  from largest to smallest.

(HINT:  $\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| = \sqrt{2}$       $\left\| \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\| = \sqrt{18}$       $\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$       $\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = 0.$ )

**9. Compressed Learning (13 pts.)**

(a) (6 pts.) Suppose we have a matrix  $X \in \mathbb{R}^{d \times n}$  of noisy data points

$$X := \begin{bmatrix} | & | & \cdots & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & \cdots & | \end{bmatrix} \quad (70)$$

where each  $\vec{x}_i \in \mathbb{R}^d$ , and  $n > d$ . Suppose further that we observed some outputs  $y_i$ , where each  $y_i = \vec{w}^\top \vec{x}_i$ , for some unknown vector  $\vec{w} \in \mathbb{R}^d$ .

Now, your friend Adam wants to estimate new outputs  $y_{n+1}, y_{n+2}, \dots$  on his TI-Launchpad device as he sees new data points,  $\vec{x}_{n+1}, \vec{x}_{n+2}, \dots \in \mathbb{R}^d$ , but it takes too much memory to store these  $\vec{x}_i$ . He compresses these  $\vec{x}_i$  by projecting them onto the subspace spanned by the singular vectors corresponding to the largest  $\ell$  singular values. Here, we can assume  $\ell \ll d$ . Suppose that  $X$  has SVD  $X = U\Sigma V^\top$ , and let  $U_k$  denote the sub-matrix of the first  $k$  columns of  $U$  (and likewise, let  $V_k$  denote the sub-matrix of the first  $k$  columns of  $V$ ). Answer the following questions in order:

i. To save memory, instead of storing vectors  $\vec{x}_i \in \mathbb{R}^d$ , we can store smaller vectors of the  $\ell$  coordinates of  $\vec{x}_i$  projected onto the  $\ell$ -dimensional subspace. Call these smaller vectors  $\vec{\tilde{x}}_i \in \mathbb{R}^\ell$ . **For a new data point  $\vec{x}_{\text{new}}$ , what is the corresponding  $\vec{\tilde{x}}_{\text{new}}$ ?** Express  $\vec{\tilde{x}}_{\text{new}}$  in terms of the quantities provided.

(HINT: Based on how the data is arranged in  $X$ , should you project onto columns of  $U$  or columns of  $V$ ?)

ii. Using the initial  $n$  data points, Adam gives you an estimate of the unknown  $\vec{w} \in \mathbb{R}^d$ :  $\hat{\vec{w}}$ .  $\hat{\vec{w}}$  should allow you to estimate  $y_i$  for new  $\vec{x}_i$ . However, we cannot store the incoming  $\vec{x}_i$  to compute  $y_i$  and must use  $\vec{\tilde{x}}_i$  instead. **What is the output,  $y_{\text{new}}$ , corresponding to a new data point  $\vec{x}_{\text{new}}$ , in terms of  $\hat{\vec{w}}$ ,  $\vec{\tilde{x}}_{\text{new}}$ , and other provided quantities?** You may not use  $\vec{x}_{\text{new}}$  in your answer.

(HINT: You may want to consider how to compute a projection, since  $\vec{\tilde{x}}_{\text{new}}$  are the  $\ell$  coordinates of the projection into the subspace spanned by relevant singular vectors.)

**Solution: [Part 9.\(a\)i](#)**

The principal components that we would project onto are  $U_\ell$ , the first  $\ell$  columns of  $U$ . Using the least squares formula for obtaining the projection coefficients, we have

$$\vec{\tilde{x}}_i = \left( \underbrace{U_\ell^\top U_\ell}_I \right)^{-1} U_\ell^\top \vec{x}_i \quad (71)$$

$$= U_\ell^\top \vec{x}_i \quad (72)$$

**[Part 9.\(a\)ii](#)**

We can use the projection coefficients  $\vec{\tilde{x}}_i$  to find an estimate of  $\vec{x}_i$ , namely

$$\vec{x}_i \approx U_\ell \vec{\tilde{x}}_i \quad (73)$$

With this, we can estimate

$$y_i \approx \hat{\vec{w}}^\top \vec{x}_i \approx \hat{\vec{w}}^\top U_\ell \vec{\tilde{x}}_i \quad (74)$$

- (b) (3 pts.) Now, we will consider the problem of classification. Your friend Rohit has built a classifier where he finds the coordinates of his data on a 1-dimensional subspace generated from a single singular vector. Let  $s_i$  be the coordinate of  $\vec{x}_i$ . His classifier outputs '+1' if  $s_i > 0$ , '-1' if  $s_i < 0$ , and '0' if  $s_i = 0$ . Consider the following data matrix:

$$X := \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \vec{x}_4 \end{bmatrix} \tag{75}$$

$$= \begin{bmatrix} 1 & 0.5 & -0.5 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{76}$$

**For each data point  $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4$ , what would Rohit's classifier output?** If writing the SVD of  $X$ , use the following (eigenvalue, eigenvector) pairs for  $XX^\top$ :  $\left(2.5, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$ .

**Solution:** We can find our  $U$  matrix in the SVD  $X = U\Sigma V^\top$  by finding the spectral decomposition of  $XX^\top$ , using the provided (eigenvalue, eigenvector) pairs:

$$XX^\top = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_U \begin{bmatrix} 2.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{77}$$

Note that we took care to sort the entries in our  $\Lambda$  matrix in decreasing order. We need to find the 1-dimensional projection coefficients of the data, so we can multiply

$$\vec{y}^\top = U_1^\top X = \begin{bmatrix} 1 & 0.5 & -0.5 & -1 \end{bmatrix} \tag{78}$$

where  $U_1$  is the first column of  $U$ , namely  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ . Based on these projection coefficients, we conclude that  $\vec{x}_1$  and  $\vec{x}_2$  are classified as '+1', and  $\vec{x}_3$  and  $\vec{x}_4$  are classified as '-1'.

(c) (4 pts.) Rohit accidentally drops his measurement device and now his data looks like

$$X_{\text{buggy}} := \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \vec{x}_4 \end{bmatrix} \quad (79)$$

$$= \begin{bmatrix} 1 & 0.5 & -0.5 & -1 \\ 100 & 100 & 100 & 100 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (80)$$

i. **What is the classification of the data points now, using the same classification strategy with  $X_{\text{buggy}}$  as in part (b)?** If writing the SVD of  $X_{\text{buggy}}$ , use the following (eigenvalue, eigenvector) pairs for  $X_{\text{buggy}}X_{\text{buggy}}^\top$ :

$$\left( 2.5, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right), \left( 40000, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right), \left( 0, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

ii. Suppose you were to “center” your data matrix  $X_{\text{buggy}}$ . The centered version of  $X_{\text{buggy}} \in \mathbb{R}^{d \times n}$  can be written as

$$X_{\text{center}} = X_{\text{buggy}} - \begin{bmatrix} \mu_1 & \mu_1 & \cdots & \mu_1 \\ \mu_2 & \mu_2 & \cdots & \mu_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_d & \mu_d & \cdots & \mu_d \end{bmatrix} \quad (81)$$

where  $\mu_i$  is the average value of all elements in row  $i$  of  $X_{\text{buggy}}$ . **Find  $X_{\text{center}}$ . How does it compare to  $X$  in part (b)?**

**Solution:** Part 9.(c)i We can find our new  $U$  matrix by finding the spectral decomposition of  $XX^\top$ , taking care to sort the elements of  $\Lambda$  appropriately:

$$XX^\top = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_U \begin{bmatrix} 40000 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (82)$$

Now, to find the 1-dimensional projection coefficients,

$$\vec{y}^\top = U_1^\top X = \begin{bmatrix} 100 & 100 & 100 & 100 \end{bmatrix} \quad (83)$$

where  $U_1$  is the first column of  $U$ , namely  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ . Based on these projection coefficients, we conclude that all vectors are classified as ‘+1’.

Part 9.(c)ii We know here that  $d = 3$ , so we can take the average value across columns as follows:

$$\mu_1 = \frac{1 + 0.5 + (-0.5) + (-1)}{4} = 0 \quad (84)$$

$$\mu_2 = \frac{100 + 100 + 100 + 100}{4} = 100 \quad (85)$$

$$\mu_3 = \frac{0 + 0 + 0 + 0}{4} = 0 \quad (86)$$

so we have that

$$X_{\text{center}} = X - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 100 & 100 & 100 & 100 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & -0.5 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (87)$$

This is the same matrix as part (a). Based on the classifications that we found in that part, we see that  $\vec{x}_3$  and  $\vec{x}_4$  are misclassified.

**10. Reachability for Nonlinear Systems (19 pts.)**

Consider a pendulum with some applied torque,  $u(t)$ , as a control input. The coordinate system for  $\theta$  has the downward vertical as  $\theta = 0$  with counterclockwise rotation as  $\theta > 0$ . This pendulum has the following dynamics equation:

$$ml^2 \frac{d^2\theta}{dt^2} + mgl \sin(\theta) = u(t) \tag{88}$$

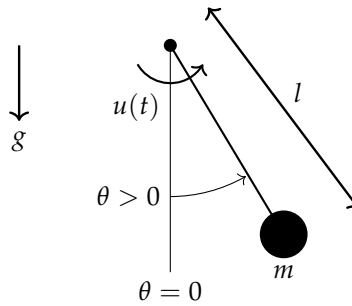


Figure 7: Pendulum

- (a) (4 pts.) Let us define the state as  $\vec{x}(t) = \begin{bmatrix} \theta \\ \frac{d\theta}{dt} \end{bmatrix}$ , and the values  $m = l = 1$ . Use the definition of  $\vec{x}(t)$  to rewrite the dynamics in eq. (88) in the form  $\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), u(t))$ . What is  $\vec{f}(\vec{x}(t), u(t))$ ?

**Solution:** We can rearrange eq. (88) as follows:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin(\theta) + \frac{1}{ml^2} u(t) \tag{89}$$

which, rewritten in terms of our state variables, is

$$\frac{dx_2(t)}{dt} = -\frac{g}{l} \sin(x_1(t)) + \frac{1}{ml^2} u(t) \tag{90}$$

The second equation we have is (in terms of our state variables)

$$\frac{dx_1(t)}{dt} = x_2(t) \tag{91}$$

Hence,

$$f(\vec{x}(t), u(t)) = \begin{bmatrix} x_2(t) \\ -\frac{g}{l} \sin(x_1(t)) + \frac{1}{ml^2} u(t) \end{bmatrix} \tag{92}$$

- (b) (3 pts.) Given the diagram of the pendulum above, pick an appropriate operating point  $(\vec{x}^*, u^*)$  such that the pendulum points straight upwards and remains stationary.  $\theta$  has units of radians. (HINT: Take care to note how  $\theta$  is defined above!)

**Solution:** Based on this requirement, we want  $\theta = \pi$ , so  $x_1 = \pi$ . Since we need the first entry of  $f(\vec{x}(t), u(t))$  to be 0, we need to choose  $x_2 = 0$ . Lastly, the second entry of  $f(\vec{x}(t), u(t))$  will be  $\frac{1}{ml^2} u(t)$  if  $x_1 = \pi$ , so we need to set  $u(t) = 0$ . This yields  $\vec{x}^* = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$  and  $u^* = 0$ .

(c) (8 pts.) A linearized model of the system for variables  $\delta\vec{x} := \vec{x} - \vec{x}^*$  and  $\delta u := u - u^*$  is given by:

$$\frac{d}{dt}\delta\vec{x}(t) = \delta A\delta\vec{x}(t) + \delta B\delta u(t) \quad (93)$$

$\delta A$  and  $\delta B$  are matrices that depend on the operating point,  $(\vec{x}^*, u^*)$ . We choose the operating point for this linearization to be the one you found in the previous part. **Finish the linearization by finding the matrices  $\delta A$  and  $\delta B$ .**

**Solution:** We have the following Jacobians:

$$J_{\vec{x}}f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \quad (94)$$

$$= \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} \cos(x_1) & 0 \end{bmatrix} \quad (95)$$

and

$$J_u f = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} \quad (96)$$

$$= \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \quad (97)$$

Now, we can plug in for our operating point and obtain

$$\delta A = J_{\vec{x}}f|_{(\vec{x}^*, u^*)} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix} \quad (98)$$

$$\delta B = J_u f|_{(\vec{x}^*, u^*)} = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \quad (99)$$

which means our linearized dynamics are

$$\frac{d\vec{x}}{dt} = \delta A\vec{x} + \delta B u \quad (100)$$

(d) (2 pts.) Regardless of your answers to the previous parts, suppose that the linearized dynamics are

$$\frac{d}{dt}\delta\vec{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \delta\vec{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u \quad (101)$$

**Is this operating point locally stable?**

**Solution:** We can calculate the eigenvalues of our  $A$  matrix as follows:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) \quad (102)$$

$$= \lambda^2 - 1 = 0 \quad (103)$$

so  $\lambda = \pm 1$ . Since there is a  $\lambda$  such that  $\text{Re}(\lambda) > 0$ , the system is unstable at that operating point.

(e) (2 pts.) Suppose we discretized this system and obtained the following discrete, linear system:

$$\delta\vec{x}[i+1] = \underbrace{\begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{5}{2} & -1 \end{bmatrix}}_{A_d} \delta\vec{x}[i] + \underbrace{\begin{bmatrix} -1 \\ \frac{5}{2} \end{bmatrix}}_{\vec{b}_d} \delta u[i] \quad (104)$$

**Is this system controllable?** You may use the fact that

$$A_d \vec{b}_d = \begin{bmatrix} -\frac{9}{4} \\ -5 \end{bmatrix} \quad (105)$$

**Solution:** We can compute the controllability matrix, i.e.,

$$C = [A_d \vec{b}_d \quad \vec{b}_d] = \begin{bmatrix} -\frac{9}{4} & -1 \\ -5 & \frac{5}{2} \end{bmatrix} \quad (106)$$

To check if it is full rank, we can compute the determinant, i.e.

$$\det(C) = -\frac{45}{8} - 5 = -\frac{85}{8} \neq 0 \quad (107)$$

so it is full rank, and hence controllable.



**11. A Complex Triangularization of the Schur Variety (18 pts.)**

Upper triangularization is the process of expressing a square matrix in terms of an upper triangular matrix via an orthonormal basis. Let's explore what happens when we try the procedure on a matrix with complex entries.

Our matrix we want to upper triangularize is  $A = \begin{bmatrix} \frac{1}{2} + j & \frac{j}{2} \\ \frac{j}{2} & -\frac{1}{2} + j \end{bmatrix}$ .

(a) (3 pts.) Show that  $\vec{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix}$  is an eigenvector of  $A$ . What is its corresponding eigenvalue?

**Solution:** Since we are given that  $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix}$  is an eigenvector of  $A$ , in order to prove that this is true, we must show that it satisfies the eigenvalue-eigenvector pair definition:  $A\vec{v} = \lambda\vec{v}$ .

$$\begin{bmatrix} \frac{1}{2} + j & \frac{j}{2} \\ \frac{j}{2} & -\frac{1}{2} + j \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + j\right) \cdot \frac{1}{\sqrt{2}} + \frac{j}{2} \cdot \frac{j}{\sqrt{2}} \\ \frac{j}{2} \cdot \frac{1}{\sqrt{2}} + \left(-\frac{1}{2} + j\right) \cdot \frac{j}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}} + \frac{j}{\sqrt{2}} + \frac{j^2}{2\sqrt{2}} \\ \frac{j}{2\sqrt{2}} - \frac{j}{2\sqrt{2}} + \frac{j^2}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{j}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

We can observe that this outputted vector is scaled by a factor of  $j$  from our original eigenvector, deeming  $j$  our eigenvalue.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix} \cdot j = \begin{bmatrix} \frac{j}{\sqrt{2}} \\ \frac{j^2}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{j}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

(b) (5 pts.) Use Gram-Schmidt to generate a vector  $\vec{u}_2$  that is orthogonal to  $\vec{u}_1$  from part (a) and has norm  $\|\vec{u}_2\|$  equal to 1.

**Solution:** Recall that Gram-Schmidt is an algorithm which takes in a set of vectors and outputs a set of orthonormal vectors that span the same space as the input. The question asks us to form a complex, orthonormal basis in  $\mathbb{C}^2$ . As input to the algorithm, then, we should provide a set of vectors that we know completely span  $\mathbb{C}^2$ . So, we append the identity matrix to the given vector,

and then run Gram-Schmidt on the set  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ .

$$\vec{s}_1 = \vec{v}_1$$

$$\vec{q}_1 = \frac{\vec{s}_1}{\|\vec{s}_1\|} = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix}$$

$$\vec{s}_2 = \vec{v}_2 - \text{proj}_{\vec{q}_1}(\vec{v}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{\langle \vec{v}_2, \vec{q}_1 \rangle}{\|\vec{q}_1\|^2} \vec{q}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{j}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{j}{2} \end{bmatrix}$$

$$\vec{q}_2 = \frac{\vec{s}_2}{\|\vec{s}_2\|} = \frac{\begin{bmatrix} \frac{1}{2} \\ -\frac{j}{2} \end{bmatrix}}{\left\| \begin{bmatrix} \frac{1}{2} \\ -\frac{j}{2} \end{bmatrix} \right\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{j}{\sqrt{2}} \end{bmatrix}$$

At this point, we already have two orthonormal vectors that, due to their linear independence, undoubtedly span  $\mathbb{C}^2$ . So, we can terminate the algorithm here. (If we had run continued to execute Gram-Schmidt with  $\vec{v}_3$  in order to find  $\vec{s}_3$  and  $\vec{q}_3$ , we'd end up getting  $\vec{0}$ .) Now, to prove that our basis is in fact orthonormal, we must show that the vectors are orthogonal to each other and that they each have norm 1. In other words, we must show that  $\langle \vec{q}_1, \vec{q}_2 \rangle = 0$ ,  $\|\vec{q}_1\| = 1$ , and  $\|\vec{q}_2\| = 1$ .

$$\langle \vec{q}_1, \vec{q}_2 \rangle = \vec{q}_2^* \vec{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{j}{\sqrt{2}}\right)^2 = \frac{1}{2} - \frac{1}{2} = 0$$

$$\|\vec{q}_1\| = \sqrt{\langle \vec{q}_1, \vec{q}_1 \rangle} = \sqrt{\vec{q}_1^* \vec{q}_1} = \sqrt{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix}} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{j}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$$

$$\|\vec{q}_2\| = \sqrt{\langle \vec{q}_2, \vec{q}_2 \rangle} = \sqrt{\vec{q}_2^* \vec{q}_2} = \sqrt{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{j}{\sqrt{2}} \end{bmatrix}} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{j}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$$

**Note:** Depending on the order of the input vectors, Gram-Schmidt may output different vectors. As long as you prove their orthonormality, they are a valid basis.

(c) (10 pts.) We will proceed with one iteration of the Schur Decomposition Algorithm. **Calculate**

$U^*AU$ , where  $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{j}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ . **Is the result of your calculation an upper triangular matrix?**

**12. Tracing Trace Proofs (14 pts.)**

In parts (a) through (c), we will prove that the square of the Frobenius norm of  $A \in \mathbb{R}^{n \times m}$  is equal to the sum of the eigenvalues of  $A^\top A$ . In other words, we will prove that  $\|A\|_F^2 = \sum_{i=1}^m \lambda_i$ , where each of the  $\lambda_i$  are one of the  $m$  eigenvalues of  $A^\top A$ . Then, we will synthesize our learnings in parts (d) and

(e). Recall that matrix  $A \in \mathbb{R}^{n \times m}$  has Frobenius norm  $\|A\|_F$  that equals  $\sqrt{\sum_{i=1}^m \sum_{k=1}^n a_{ki}^2}$ .

(a) (4 pts.) **Prove that  $\|A\|_F^2 = \text{tr}\{A^\top A\}$  for  $A \in \mathbb{R}^{2 \times 3}$ .**

(HINT: The trace of matrix  $M$ ,  $\text{tr}\{M\}$ , is the sum over its diagonal. It may help to identify what the entries of  $A^\top A$  are in terms of the columns of  $A$ .)

**Solution:** We can write

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \tag{108}$$

where  $\vec{a}_i \in \mathbb{R}^m$ . This would allow us to write

$$A^\top A = \begin{bmatrix} \vec{a}_1^\top \\ \vec{a}_2^\top \\ \vdots \\ \vec{a}_n^\top \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \tag{109}$$

Using the matrix multiplication algorithm, we have that  $(A^\top A)_{ii} = \langle \vec{a}_i, \vec{a}_i \rangle = \|\vec{a}_i\|^2$ . Hence,

$$\text{tr}\{A^\top A\} = \sum_{i=1}^n \|\vec{a}_i\|^2 = \sum_{i=1}^n \sum_{j=1}^m a_{ij}^2 \tag{110}$$

which is exactly  $\|A\|_F^2$ .

(b) (2 pts.) **Show that  $A^\top A$  simplifies down to  $V\Sigma^\top \Sigma V^\top$  if the SVD of  $A$  is  $U\Sigma V^\top$ .**

**Solution:** Using the given SVD,

$$A^\top A = (U\Sigma V^\top)^\top (U\Sigma V^\top) \tag{111}$$

$$= (V\Sigma^\top U^\top) (U\Sigma V^\top) \tag{112}$$

$$= V\Sigma^\top \Sigma V^\top \tag{113}$$

**Solution:** From the previous part, we have

$$A^T A = V \Sigma^T \Sigma V^T \tag{114}$$

where

$$\Sigma^T \Sigma = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots \\ 0 & \sigma_2^2 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} \tag{115}$$

where all the elements along the diagonal are still in increasing order. Hence,  $\tilde{\Sigma} = \Sigma^T \Sigma$  is a valid choice. Furthermore,  $V$  is orthonormal, so by pattern matching,  $\tilde{U} = V$  and  $\tilde{V} = V$  is a valid choice for an SVD of  $A^T A$ .

- (c) (2 pts.) Given that  $\text{tr}\{QP\} = \text{tr}\{PQ\}$  for  $Q \in \mathbb{R}^{m \times n}$  and  $P \in \mathbb{R}^{n \times m}$ , **show that  $\text{tr}\{A^T A\} = \text{tr}\{\Sigma^T \Sigma\}$  using your result from part (b).**

(HINT: What is  $V^T V$ ?)

**Solution:** By the cyclic property of the trace, we have

$$\text{tr}\{A^T A\} = \text{tr}\{V \Sigma^T \Sigma V^T\} \tag{116}$$

$$= \text{tr}\{V^T V \Sigma^T \Sigma\} \tag{117}$$

$$= \text{tr}\{\Sigma^T \Sigma\} \tag{118}$$

- (d) (3 pts.) **Given that  $B \in \mathbb{R}^{2 \times 2}$ , that  $\|B\|_F^2 = 5$ , and that  $B^T B$  has an eigenvalue equal to 5, what can we say about the invertibility of  $B$ ?** Conclude and justify whether  $B$  is invertible, not invertible, or if we do not have enough information.

(HINT: What should the  $\Sigma$  matrix of  $B$  look like if it is invertible? In parts (a) through (c), we proved that  $\text{tr}\{A^T A\} = \sum_k \sigma_k^2 = \sum_i \lambda_i$ , where  $\sigma_k$  describes the singular values of  $A$  and  $\lambda_i$  describes the eigenvalues of  $A^T A$ .)

**Solution:** Since we know that  $\lambda_{\max, B^T B} = n$ , it must be the case that  $\sigma_{\max, B} = \sqrt{n}$ , where  $\sigma_{\max, B}$  is the largest singular value of  $B$ . since  $B \in \mathbb{R}^{2 \times 2}$ , it must be the case that we have at most 2 nonzero singular values, denoted  $\sigma_1$  and  $\sigma_2$ . We know that  $\sigma_1 = \sigma_{\max, B} = \sqrt{n}$ , and we know that  $\sigma_1^2 + \sigma_2^2 = \|B\|_F^2 = n$ . Hence,  $\sigma_2 = 0$  and  $B$  cannot be full rank due to there being fewer than 2 singular values.

- (e) (3 pts.) We want to understand the stability of the following continuous system, where  $M \in \mathbb{R}^{n \times n}$ ,  $\vec{x} \in \mathbb{R}^n$ , and  $\vec{b} \neq \vec{0}_n$ .

$$\frac{d}{dt}\vec{x}(t) = M\vec{x}(t) + \vec{b}u(t) \quad (119)$$

**Given that  $M$  has  $\text{tr}\{M^\top M\} = 0$ , what can we say about the stability of our system?** Conclude and justify whether the system is stable, not stable, or if we do not have enough information.

*(HINT: Try using the statement in the previous part to answer: what are the singular values of  $M$ , if  $\text{tr}\{M^\top M\} = 0$ ? Given this, what will  $M = U\Sigma V^\top$  be?)*

**Solution:** We have that the sum of squares of singular values of  $M$  are 0, so  $\Sigma = 0_{n \times n}$  in  $M = U\Sigma V^\top$ . As a result, it must be the case that  $M = 0_{n \times n}$ , and all the eigenvalues are 0. This system is not stable since  $\vec{b} \neq \vec{0}_n$ .