Midterm

Exam Location: AAPB

PRINT your student ID: ____________________

PRINT AND SIGN your name: ____________________, ____________________ ____________________

(last) (first) (sign)

PRINT your discussion sections and (u)GSIs (the ones you attend): ____________________

Row Number: ____________________ Seat Number: ____________________

Name and SID of the person to your left: ____________________

Name and SID of the person to your right: ____________________

Name and SID of the person in front of you: ____________________

Name and SID of the person behind you: ____________________

1. **Honor Code (0 pts.)**

   Please copy the following statement in the space provided below and sign your name.

   As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I will follow the rules and do this exam on my own.

   Note that if you do not copy the honor code and sign your name, you will get a 0 on the exam.

2. **What's something you're proud of having done this last year? (2 pts.)**

3. **What fall classes or plans are you excited for? (2 pts.)**

Do not turn this page until the proctor tells you to do so.
You can work on the above problems before time starts.

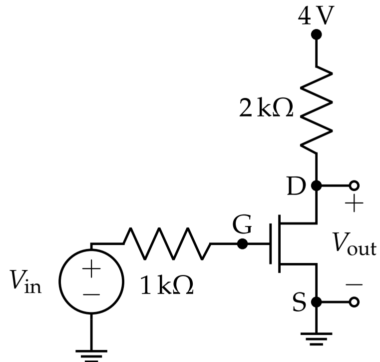
4. Complex Numbers (7 pts.)

(a) (3 pts.) Let $z_1 = 4e^{\mathrm{j}\frac{\pi}{12}}$ and $z_2 = 2e^{\mathrm{j}\frac{\pi}{2}}$. **What is** $\left|\frac{z_1}{z_2}\right|$**? What is** $\angle(z_1 \cdot z_2)$**?**

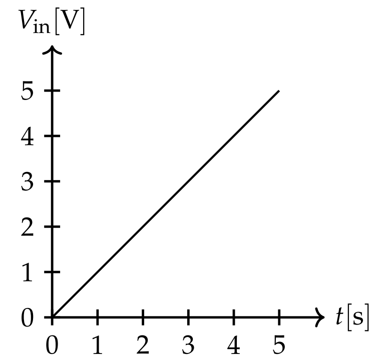
(b) (4 pts.) **Convert the voltage phasor** $\widetilde{V}_{\mathbf{out}} = 3 + 3\mathrm{j}$ **into a sinusoidal signal** $V_{\mathbf{out}}(t) = A\cos(\omega t + \phi)$**. Specifically, solve for the values of** A **and** ϕ**.**

5. NMOS Logic Inverter (14 pts.)

(a) (14 pts.) We have an NMOS logic implementation of an inverter shown below. The circuit has a voltage input $V_{\text{in}}(t) = t$, $t \geq 0$, ($V_{\text{in}}(t) = 0\,\text{V}$ for $t \leq 0$) seen below.



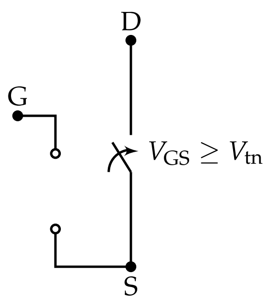
(a) NMOS Inverter



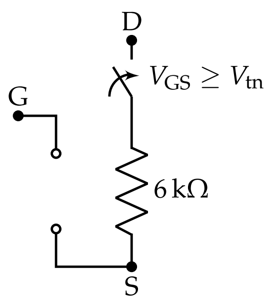
(b) Voltage input, V_{in}

Figure 1: Circuit figure and input signal.

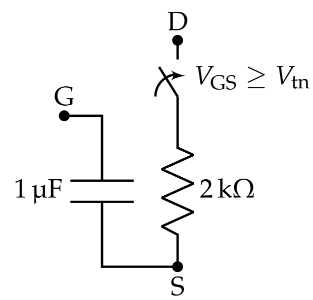
For the transistor models below, define the threshold voltage as $V_{\text{tn}} = 2\,\text{V}$. **Match each NMOS transistor model, plugged into the NMOS inverter circuit, with its corresponding V_{out} plot on the next page.** (Note: All capacitors are fully discharged at $t = 0$.)



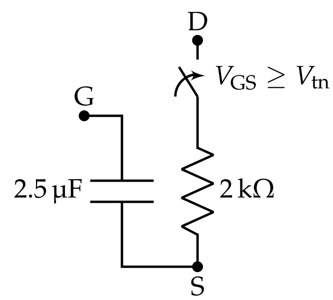
(I) Model I



(II) Model II

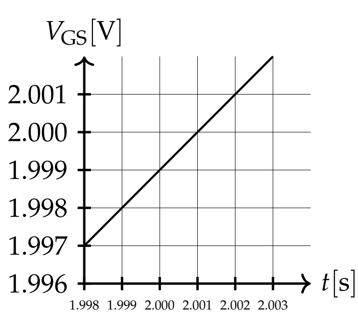


(III) Model III

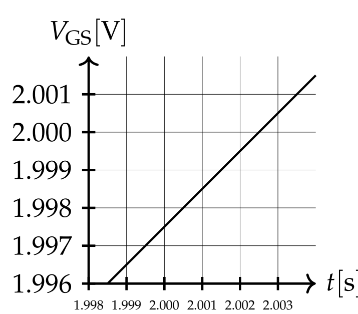


(IV) Model IV

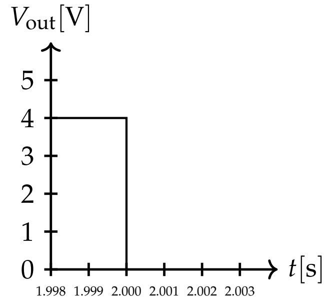
(HINT: You can use the below graphs to evaluate V_{GS} for Models III and IV. We recommend using a scratch page to draw out the NMOS Inverter circuit with the various transistor models plugged in.)



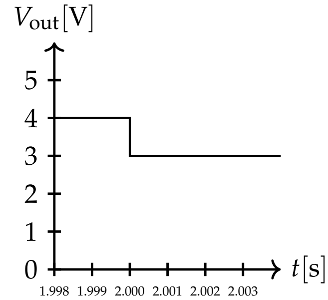
(a) V_{GS} for Model III



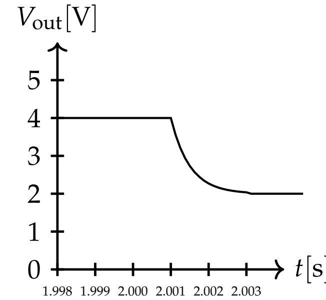
(b) V_{GS} for Model IV



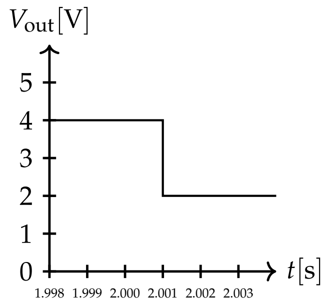
(A) Plot A



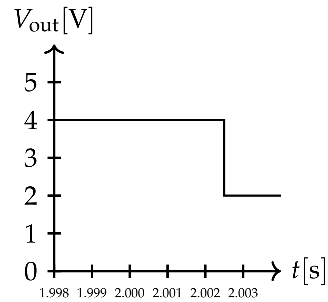
(B) Plot B



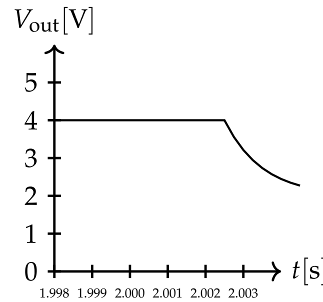
(C) Plot C



(D) Plot D



(E) Plot E



(F) Plot F

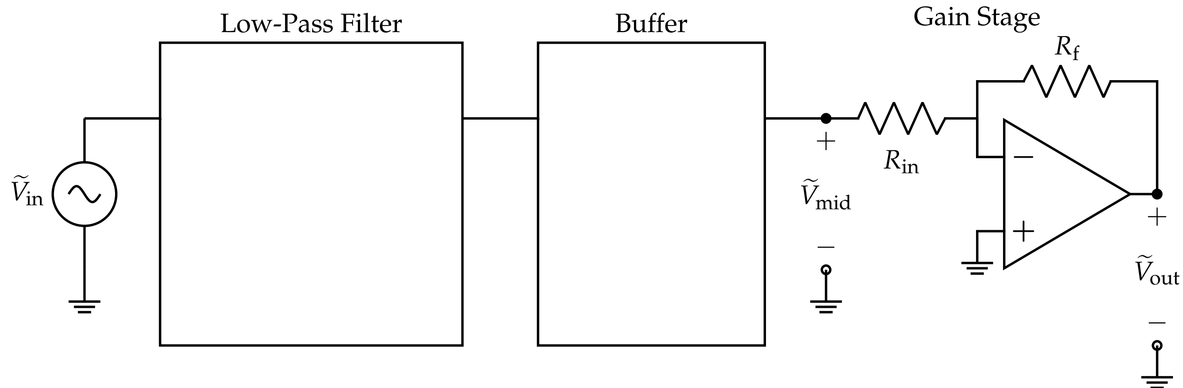
Model #	Plot A	Plot B	Plot C	Plot D	Plot E	Plot F
I	◯	◯	◯	◯	◯	◯
II	◯	◯	◯	◯	◯	◯
III	◯	◯	◯	◯	◯	◯
IV	◯	◯	◯	◯	◯	◯

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

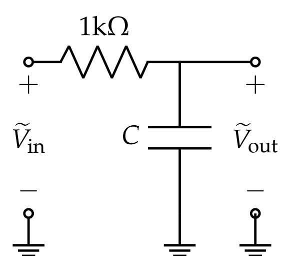
6. Bass Speaker Pre-amplifier (23 pts.)

(a) (4 pts.) Let's design a bass speaker pre-amplifier. We want our pre-amplifier to amplify lower frequencies and attenuate higher frequencies.

In our toolkit, we have **one inductor**, **one op-amp**, and **one resistor** (in addition to the circuit elements already used to implement the Gain Stage in the figure below). Do not worry about the exact values of the inductance and resistance just yet. **Use these components and draw in the circuits that implement the rest of our pre-amplifier in the boxes in the figure below.**



(b) (2 pts.) Instead of using an inductor, consider the low-pass filter constructed using a resistor and capacitor shown below. **If we want a cutoff frequency of $\omega = 10^3 \frac{\text{rad}}{\text{s}}$ what should the capacitance C be, given that the resistance is $1\,\text{k}\Omega$?**



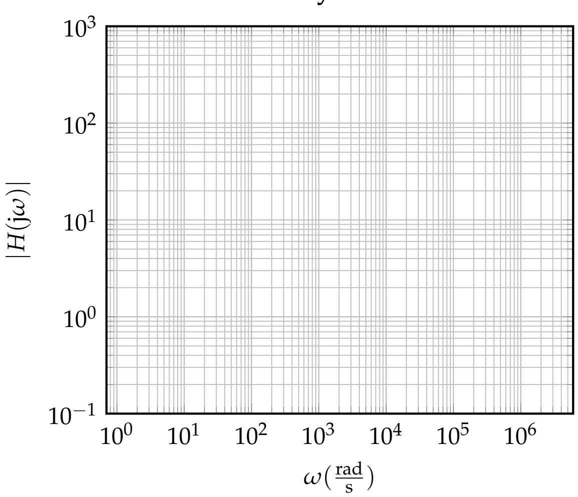
(c) (7 pts.) We want to achieve a transfer function magnitude of 10 in the Gain Stage of our pre-amplifier circuit. We will find what R_f should be, given resistance $R_{in} = 20\Omega$.

i. **Solve for the transfer function** $H(j\omega) = \dfrac{\widetilde{V}_{\mathbf{out}}}{\widetilde{V}_{\mathbf{mid}}}$**.**

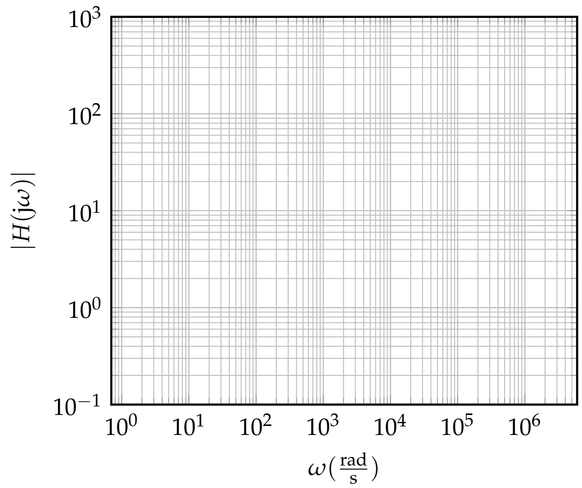
ii. **Based on your transfer function from the previous subpart what should** R_f **be to achieve a transfer function magnitude of** $|H(j\omega)| = 10$**?**

(d) (6 pts.) We have decided that we will select our inductance and resistance values for the elements from part (a) so that our pre-amplifier can pass all frequencies less than $\omega_c = 10^2 \frac{\text{rad}}{\text{s}}$, and subsequently, amplify all the output by $A_V = 100$.

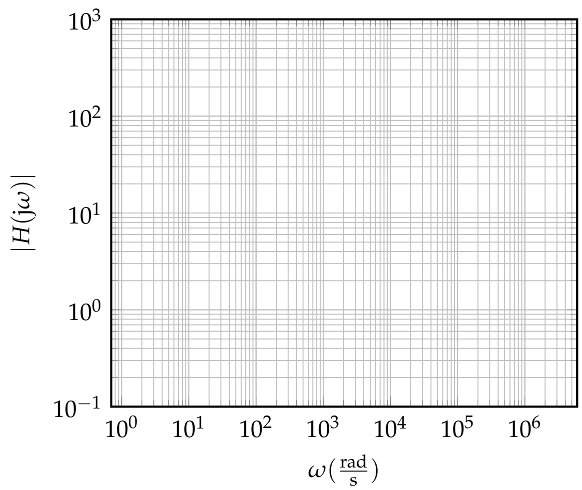
i. **Depict the desired low-pass behavior in a Bode magnitude plot that result from the inductor and the resistor only.**



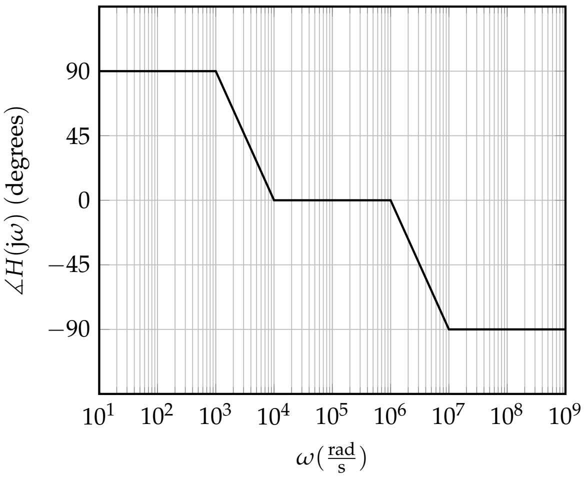
ii. **Depict the desired Gain Stage behavior in a Bode magnitude plot.**

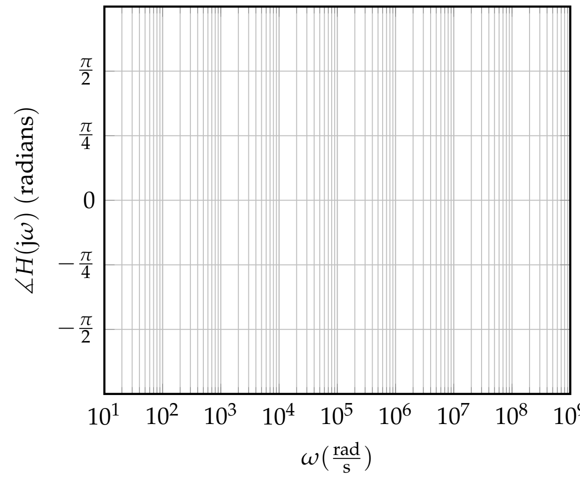


iii. **Depict the desired combined behavior from the previous two plots in a single Bode magnitude plot.**



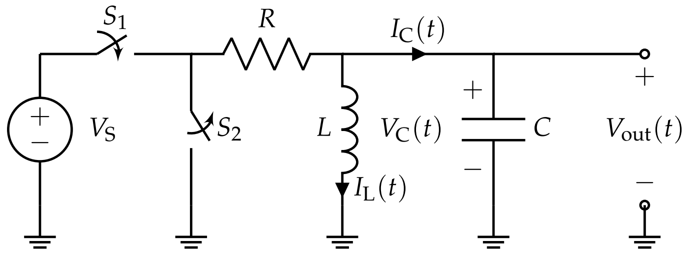
(e) (4 pts.) Let's say we want to eliminate any phase shift caused by imperfections in the hardware; we'll hence follow the bass speaker pre-amplifier with a hypothetical "phase unshifter" circuit component. **What should the phase plot of the phase unshifter (right plot) look like given the following phase plot (left plot) for the hardware imperfections so that the net phase is 0 for** $\omega = 10^1$ **to** $\omega = 10^9$**?** (Note: Both plots share the same y-axis values and scaling.)





7. RLC Circuit from Time to Frequency (34 pts.)

Consider the following circuit fed by a constant voltage source V_S.



The switch S_1, open for $t < 0$, closes at $t = 0$, and the switch S_2, closed for $t < 0$, opens at $t = 0$. Assume $V_C(0) = 0$ and $I_L(0) = 0$.

(a) (8 pts.) **Derive a set of two differential equations, one for $I_L(t)$, the current through the inductor, and one for $V_C(t)$, the voltage across the capacitor.** Write your answer in terms of R, L, C, V_S, and constants.

(b) (3 pts.) **Using your answers from the previous part, create a vector differential equation with the state vector being $\vec{x}(t) = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}$.** Write your answers in terms of R, L, C, V_S, and constants.

(c) (15 pts.) Regardless of your answer to the previous part, suppose the vector differential equation is given by

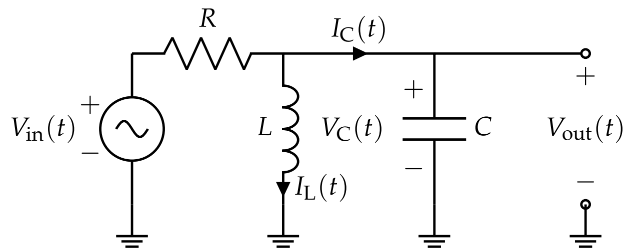
$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \underbrace{\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{\vec{b}} V_S \tag{1}$$

You may use the fact that A is diagonalized as follows:

$$\underbrace{\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} -6 & -2 \\ 1 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix}}_{V^{-1}} \tag{2}$$

With $\vec{x}(0) = \vec{0}$, solve for $\vec{x}(t)$ and find the asymptotic/steady-state behavior as $t \to \infty$.

(d) (5 pts.) Now, consider the same circuit but with an arbitrary sinusoidal voltage source instead of the constant voltage source from part (a). Specifically, let $V_{\text{in}}(t) = A\cos(\omega t + \phi)$, for some arbitrary constants A, ω, ϕ. The circuit below reflects this change.



Solve for the transfer function, $H(\mathrm{j}\omega) = \dfrac{\widetilde{V}_{\textbf{out}}}{\widetilde{V}_{\textbf{in}}}$**, of this circuit.** You may write your answer in terms of R, L, C, ω, and constants.

(e) (3 pts.) **Finally, find** $|H(\mathrm{j}\omega)|$ **for** $V_{\text{in}}(t) = V_S$**, where** V_S **is the voltage supplied by the constant voltage source in part (a).** *(HINT: What would ω be for a constant value?)*

8. Straight Line Stability (24 pts.)

We define a discrete and continuous time system respectively as follows:

$$\underbrace{\vec{x}[i+1] = A_d\vec{x}[i] + \vec{b}_d u[i]}_{\text{discrete time}} \qquad \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)}_{\text{continuous time}}$$

where the state is $\vec{x} \in \mathbb{R}^2$, the input to the system is $u \in \mathbb{R}$, and we have parameters $A_d, A \in \mathbb{R}^{2x2}$, and $\vec{b}_d, \vec{b} \in \mathbb{R}^2$.

(a) (8 pts.) **For the following problems, determine whether the system is stable or unstable.**

$$\vec{x}[i+1] = A_d\vec{x}[i] + \vec{b}_d u[i]$$

A_d	**Stable**	**Unstable**
$\begin{bmatrix} -1 & 0 \\ 0 & -0.5 \end{bmatrix}$	◯	◯
$\begin{bmatrix} 0 & 0.25 \\ 0.5 & 0 \end{bmatrix}$	◯	◯

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t)$$

A	**Stable**	**Unstable**
$\begin{bmatrix} 0 & 7 \\ 0 & -10 \end{bmatrix}$	◯	◯
$\begin{bmatrix} -0.25j & 0 \\ 0 & 0.25j \end{bmatrix}$	◯	◯

(b) (8 pts.) Assume that we are operating on a discrete time model ($\vec{x}[i+1] = A_d\vec{x}[i] + \vec{b}_d u[i]$), and our control matrices A_d and $\vec{b}_d$ are fixed as follows:

$$A_d = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}, \vec{b}_d = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Let $u[i] = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}[i]$. **Solve for the characteristic polynomial of our new feedback-controlled system in the form $C\lambda^2 + D\lambda + E$. You may leave your answer in terms of k_1 and k_2.**

(c) (8 pts.) With a discrete system different from that in part (b), we assume a fixed value of k_2 such that our characteristic polynomial becomes the following:

$$\lambda^2 - 2\lambda - (k_1 + 2)$$

Find a range of values for k_1 for which the system is stable. Please write your answers as either a number or interval(s) of number(s) (i.e. 8 inclusive to ∞ would be $[8, \infty)$); if there is no solution, you may say so. **Justify your answer with your work.**

9. Continuous Time Discretization and Back (14 pts.)

In this problem we will examine how to perform system ID on a continuous time system. Consider a car with a two-dimensional state, $\vec{x}$, whose dynamics are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{3}$$

where $A \in \mathbb{R}^{2\times 2}$ and $\vec{b} \in \mathbb{R}^2$ are unknown. The state's entries are written as $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

(a) (6 pts.) Suppose we discretized our continuous time system and obtained the following dynamics for our system with state $\vec{x}_d[i] = \begin{bmatrix} x_{d,1}[i] \\ x_{d,2}[i] \end{bmatrix}$.

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] \tag{4}$$

The parameters $A_d \in \mathbb{R}^{2\times 2}$ and $\vec{b}_d \in \mathbb{R}^2$ are unknown. We apply discrete input $u_d[i]$ from $i = 0$ to $i = 3$, and obtain the following state observations.

i	$u_d[i]$	$x_{d,1}[i]$	$x_{d,2}[i]$
0	-1	2	3
1	0	7	8
2	1	4	6
3	0	5	9
4	N/A	8	13

Figure 5: Data Collected from Sampling the System

We want to identify A_d and $\vec{b}_d$ by setting up a least squares problem of the form $DP \approx S$ where $P = \begin{bmatrix} A_d^\top \\ \vec{b}_d^\top \end{bmatrix}$.

Express the D and S matrices in terms of numerical values of $x_{d,1}[i]$, $x_{d,2}[i]$, and $u_d[i]$ from the table.

(b) (3 pts.) Suppose now that you want to solve for an estimate of A, the matrix in the continuous system, using your estimate of A_d, the matrix in the discretized system. We will try this by first looking at a scalar system.

Consider the scalar continuous time system in eq. (5) and its corresponding discretization with $x_d[i] = x(i\Delta)$ and $u_d[i] = u(i\Delta)$ in eq. (6).

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \underbrace{\lambda}_{a} x(t) + bu(t) \tag{5}$$

$$x_d[i+1] = \underbrace{e^{\lambda\Delta}}_{a_d} x_d[i] + \underbrace{\frac{e^{\lambda\Delta} - 1}{\lambda} b}_{b_d} u_d[i] \tag{6}$$

You know $a_d \approx e^{\lambda\Delta}$ by system identification. **Express a in terms of a_d and Δ.**

(c) (5 pts.) Let's return to the original matrix-vector system. It is true that if A can be diagonalized as $A = V\Lambda V^{-1}$, then $A_d = V\Lambda_d V^{-1}$. Suppose Λ_d has entries $\begin{bmatrix} a_{d,1} & 0 \\ 0 & a_{d,2} \end{bmatrix}$. **Using the entries of Λ_d, solve for Λ, the matrix of eigenvalues of A. Then, express A in terms of V, V^{-1}, $a_{d,1}$, $a_{d,2}$, and Δ.**

(HINT: Use your result from part (b) to express the entries of Λ in terms of the entries of Λ_d.)

[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.

If needed, you can also use this space to work on problems. But if you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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