1 Stability

Discrete time systems

A discrete time system is of the form:

\[ x[t + 1] = Ax[t] + Bu[t] \]

This system is stable if \( |\lambda_i| < 1 \) for all \( \lambda_i \)'s are the eigenvalues of \( A \). If we plot all \( \lambda_i \) for \( A \) on the complex plane, if all \( \lambda_i \) lie within (not on) the unit circle, then the system is stable.

If \( |\lambda| = 1 \), we say the system is marginally stable and with respect to bounded-input bounded-output stability, this system would be unstable.
**Continuous time systems**

A continuous time system is of the form:

\[
\frac{d\tilde{x}}{dt}(t) = A\tilde{x}(t) + B\tilde{u}(t)
\]

This system is stable if \(\text{Re}\{\lambda_i\} < 0\) for all \(\lambda_i\), where \(\lambda_i\)'s are the eigenvalues of \(A\). If we plot all \(\lambda_i\) for \(A\) on the complex plane, if all \(\lambda_i\) lie to the left of \(\text{Re}\{\lambda_i\} = 0\), then the system is stable.

If \(\text{Re}\{\lambda_i\} = 0\), the system is marginally stable and it is again unstable with respect to bounded-input bounded-output stability.
2 Stability in continuous time system

Remember the spring-mass system introduced in Discussion 8A:

We assumed that each spring is linear with spring constant $k$ and resting length $X_0$. The differential equation modeling this system was $\frac{d^2y}{dt^2} = -\frac{2k}{m} (y - X_0 \frac{y}{\sqrt{y^2 + a^2}})$. We built a state space model that describes how the displacement $y$ of the mass from the spring base evolves. The state variables were $x_1 = y$ and $x_2 = \dot{y}$. Then we linearized the model around the equilibrium point $(x_1, x_2) = (0, 0)$, assuming $X_0 < a$. The linearized model is presented below.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m} \left( 1 - \frac{X_0}{a} \right) & 0 \end{bmatrix} x.$$

Compute the eigenvalues of your linearized model. Is this equilibrium stable?
3 Stability in discrete time system

Determine which values of $\alpha$ and $\beta$ will make the following discrete-time state space models stable. Assume, $\alpha$ and $\beta$ are real numbers and $b \neq 0$.

a)  
\[ x[t + 1] = \alpha x[t] + bu(t) \]

b)  
\[ \bar{x}[t + 1] = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \bar{x}[t] + b\bar{u}(t) \]

c)  
\[ \tilde{x}[t + 1] = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \tilde{x}[t] + b\tilde{u}(t) \]