Discrete Fourier Transform

Assume we are working with an $N$ length discrete signal and we would like to find its discrete frequencies. This is done through the Discrete Fourier Transform (DFT), which is simply a change of basis to the DFT basis.

First, let us vectorize our signal. If $x[n]$ is our input signal, we model it as a vector by letting the $n^{th}$ coordinate be $x[n]$. In other words,

$$\vec{x} = [x[0], x[1], x[2], \ldots, x[N-1]]^T$$

In order to decompose $\vec{x}$ into its constituent frequencies, we must find the vector representation of these frequencies. A length $N$ signal will have $N$ different discrete frequencies of the following form. The fundamental frequency of the signal is called $\omega_N$ and its value is

$$\omega_N = e^{i \frac{2\pi}{N}}$$

The DFT Basis is built by taking powers of this fundamental frequency. We define the $k^{th}$ basis vector $\vec{u}_k[n]$ as

$$\vec{u}_k[n] = \frac{1}{\sqrt{N}} e^{k \frac{2\pi}{N}} n$$ \quad for $k = 0, 1, \ldots N - 1$$

The matrix $U$ has columns which consist of the $N$ DFT basis vectors

$$U = [\vec{u}_0 \ \vec{u}_1 \ \cdots \ \vec{u}_{N-1}]$$

We choose to normalize all of these vectors by a factor of $\frac{1}{\sqrt{N}}$ so that the DFT basis vectors are orthonormal. Try to verify on your own that

$$\langle \vec{u}_p, \vec{u}_q \rangle = \sum_{n=0}^{N-1} \vec{u}_q[n] \vec{u}_p[n] = \begin{cases} 0, & p \neq q \\ 1, & p = q \end{cases}$$

To represent a signal $x[n]$ in the frequency domain, we can change coordinates to the $U$ basis. We define the matrix $F$ as the matrix that takes our time-domain signal and transforms it into the frequency domain.

$$F = U^{-1} = U^* = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^{-1} & \omega_N^{-2} & \cdots & \omega_N^{-1(N-1)} \\ 1 & \omega_N^{-1} & \omega_N^{-2} & \cdots & \omega_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{(N-1)-1} & \omega_N^{(N-1)-2} & \cdots & \omega_N^{(N-1)(N-1)} \end{bmatrix}$$

Similarly, the matrix $U$ takes a signal $X[k]$ in the frequency domain and converts it back to the time-domain.

$$U = F^{-1} = F^* = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^{1} & \omega_N^{1} & \cdots & \omega_N^{1(N-1)} \\ 1 & \omega_N^{2} & \omega_N^{2} & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{(N-1)-1} & \omega_N^{(N-1)-1} & \cdots & \omega_N^{(N-1)(N-1)} \end{bmatrix}$$

The relationship between a time-domain signal $x[n]$ and its frequency components $X[k]$ can be written as

$$x[n] = X[0] \vec{u}_0 + \ldots + X[N-1] \vec{u}_{N-1} = UX[k]$$
1 Roots of Unity

The DFT is a coordinate transformation to a basis made up of roots of unity. In this problem we explore some properties of the roots of unity. An Nth root of unity is a complex number z satisfying the equation \( z^N = 1 \) (or equivalently \( z^N - 1 = 0 \)).

a) Show that \( z^N - 1 \) factors as

\[
z^N - 1 = (z - 1)(\sum_{k=0}^{N-1} z^k).
\]

b) Show that any complex number of the form \( \omega_k = e^{j \frac{2\pi}{N} k} \) for \( k \in \mathbb{Z} \) is an N-th root of unity.

c) Draw the fifth roots of unity in the complex plane. How many of them are there?

d) Let \( \omega_1 = e^{j \frac{2\pi}{5}} \). What is \( \omega_1^2 \)? What is \( \omega_1^3 \)? What is \( \omega_1^4 \)?

e) What is the complex conjugate of \( \omega_1 \)? What is the complex conjugate of \( \omega_4^2 \)?

f) Compute \( \sum_{k=0}^{N-1} \omega^k \) where \( \omega \) is some root of unity. Does the answer make sense in terms of the plot you drew?
2 DFT of pure sinusoids

a) Consider the continuous-time signal $x(t) = \cos\left(\frac{2\pi}{3} t\right)$. Suppose that we sampled it every 1 second to get (for $n = 3$ time steps):

$$x[n] = \begin{bmatrix} \cos\left(\frac{2\pi}{3} (0)\right) & \cos\left(\frac{2\pi}{3} (1)\right) & \cos\left(\frac{2\pi}{3} (2)\right) \end{bmatrix}^T.$$  

Compute $\tilde{X}[k]$ and the basis vectors $\tilde{u}_k$ for this signal.

b) Now for the same signal as before, suppose that we took $n = 6$ samples. In this case we would have:

$$x[n] = \begin{bmatrix} \cos\left(\frac{2\pi}{3} (0)\right) & \cos\left(\frac{2\pi}{3} (1)\right) & \cos\left(\frac{2\pi}{3} (2)\right) & \cos\left(\frac{2\pi}{3} (3)\right) & \cos\left(\frac{2\pi}{3} (4)\right) & \cos\left(\frac{2\pi}{3} (5)\right) \end{bmatrix}^T.$$  

Repeat what you did above. What are $X[k]$ and the basis vectors $\tilde{u}_k$ for this signal.

c) Let’s do this more generally. For the signal $x(t) = \cos\left(\frac{2\pi m}{N} t\right)$, where $m$ is an integer between 0 and $N - 1$, compute the frequency components $X[k]$ where $x[n]$ is a time-domain signal of length $N$.

$$x[n] = \begin{bmatrix} \cos\left(\frac{2\pi m}{N} (0)\right) & \cos\left(\frac{2\pi m}{N} (1)\right) & \cdots & \cos\left(\frac{2\pi m}{N} (N - 1)\right) \end{bmatrix}^T.$$