Circulant Matrices

A square matrix $C_h$ is circulant if each row vector is rotated one element to the right relative to the preceding row vector.

$$C_h = \begin{bmatrix} h_0 & h_{N-1} & \cdots & h_2 & h_1 \\ h_1 & h_0 & h_{N-1} & \ddots & h_2 \\ \vdots & h_1 & h_0 & \ddots & \vdots \\ h_{N-2} & \ddots & \ddots & \ddots & h_{N-1} \\ h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{bmatrix}$$  \hspace{1cm} (1)

Recall from lecture that we can describe the input-output relationship of a periodic discrete-time LTI system via a circulant matrix.

$$\tilde{y} = C_h \tilde{x}$$  \hspace{1cm} (2)

In this case, the first column of $C_h$ is the impulse response $h[n]$ of the system.

$$\tilde{h} = \begin{bmatrix} h_0 & h_1 & \cdots & h_{N-2} & h_{N-1} \end{bmatrix}$$  \hspace{1cm} (3)

Rather beautifully, the DFT basis vectors are eigenvectors of $C_h$. We will have $N$ DFT vectors, since that is the dimensionality of our model.

$$\tilde{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ e^{j2\pi k/1} \\ \vdots \\ e^{j2\pi k/(N-1)} \end{bmatrix}$$  \hspace{1cm} (4)

Letting $H[k]$ be the $k^{th}$ DFT coefficient of $h[n]$, we can write the following eigenvalue equation for $k = 0, 1, \cdots, N-1$.

$$C_h \tilde{u}_k = \left( \sqrt{N} \times H[k] \right) \tilde{u}_k$$  \hspace{1cm} (5)

In this discussion you’ll see why this is useful by representing convolution as a circulant matrix $C_h$, and then diagonalizing it. This will draw the connection between the DFT and LTI systems.
**Sampling theorem**

Let $x$ be continuous signal bandlimited by frequency $\omega_{\text{max}}$. We sample $x$ with a period of $T_s$.

Given the discrete samples, we can try reconstructing the original signal $f$ through sinc-interpolation where $\Phi(t) = \text{sinc} \left( \frac{t}{T_s} \right)$

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x[n] \Phi(t - nT_s)$$

We define the **sampling frequency** as $\omega_s = \frac{2\pi}{T_s}$. The Sampling Theorem says if $\omega_{\text{max}} < \frac{\pi}{T_s}$, or $\omega_s > 2\omega_{\text{max}}$, then we are able to recover the original signal, i.e. $x = \hat{x}$. 
1 Circulant Matrices & Convolution

Consider the signal \( x[n] \) of length 3 and an impulse response \( h[n] \) of length 2. You may assume that they are zero everywhere else.

\[
\tilde{x} = \begin{bmatrix} -1 & 3 & -2 \end{bmatrix}^T \quad \tilde{h} = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}^T
\]  

a) What is the convolution \( y[n] = x[n] * h[n] \)? Also what is the length of this output signal?

b) Now write each term of the output signal \( y[n] \) as a sum using the convolution formula and set up a matrix equation \( \tilde{y} = \tilde{A} \tilde{x} \). What is the size of this matrix?

c) Add elements to the matrix \( \tilde{A} \) and zeros to the vector \( \tilde{x} \) to create a square matrix \( \tilde{C}_h \) that is circulant.

d) Since the DFT diagonalizes circulant matrices, lets try to solve for the output signal \( y[n] \) using the DFT instead of convolution.
   - Step 1: Compute the DFT of \( x[n] \) and \( h[n] \): \( \tilde{X} = F \tilde{x} \), \( \tilde{H} = F \tilde{h} \).
   - Step 2: Take the elementwise product of the DFTs and scale: \( \tilde{Y} = \sqrt{N} \tilde{X} \odot \tilde{H} \).
   - Step 3: Perform the inverse DFT to get the result \( \tilde{y} = F^* \tilde{Y} \).

e) What is the importance behind this result? Compare the runtimes between convolution and the Fast Fourier Transform (FFT) which takes \( O(N \log N) \) operations.
2 Sampling Theorem basics

Consider the following signal, \( x(t) \) defined as,

\[
x(t) = \cos(2\pi t)
\]

a) Find the maximum frequency, \( \omega_{\text{max}} \), in radians per second? In Hertz? (From now on, frequencies will refer to radians per second.)

b) If I sample every \( T \) seconds, what is the sampling frequency?

c) What is the smallest sampling period \( T \) that would result in an imperfect reconstruction?

3 More Sampling

Let’s sample the signal from the previous question \( x \) with sampling period \( T_m = \frac{1}{4} \) s and \( T_n = 1 \) s and perform sinc interpolation on the resulting samples. Let the reconstructed functions be \( f_m \) and \( f_n \).

a) Have we satisfied the Nyquist limit (i.e. the sampling theorem) in any case?

b) What is the highest frequency we can reconstruct with the sampling rate \( T_n \)?

c) Based on this answer, can you think of any periodic function that has a frequencies less than or equal to \( \pi \) that samples the same as \( f_n \)?

4 Aliasing

Consider the signal \( x(t) = \sin(0.2\pi t) \).

a) At what period \( T \) should we sample so that sinc interpolation recovers a function that is identically zero?

b) At what period \( T \) can we sample at so that sinc interpolation recovers the function \( f(t) = -\sin \left( \frac{\pi}{15} t \right) \)?