Lecture 3

Adjoint and self-adjoint operators and matrices

An inner product structure on a $\mathbb{C}$-vector spaces induces a “mirrored” twin for every linear transformation, called the adjoint. Linear operators equal their own adjoints have many important properties.

3.1 Adjoint of an operator or matrix

Definition 15 (Adjoint of a linear map). Let $f : U \rightarrow V$ be a linear map between two inner product spaces. The adjoint of $f$, denoted by $f^* : V \rightarrow U$, is the unique linear map such that $\langle f(u), v \rangle = \langle u, f^*(v) \rangle$ for all $u \in U$ and $v \in V$.

Theorem 13 (Technical facts about adjoints). Let $f$ and $g$ be two linear operators on $V$.

1. $(\alpha f + g)^* = \overline{\alpha} f^* + g^*$ (conjugate linear)
2. $(fg)^* = g^* f^*$ (reverses composition)
3. $(f^*)^* = f^*$ (involutive)
4. $I^* = I$ (identity operator is its own adjoint)

Proof.

1. Let $x, y \in V$. We need to show that $\langle (\alpha f + g) x, y \rangle = \langle x, (\overline{\alpha} f^* + g^*) y \rangle$.

\[
\begin{align*}
\langle (\alpha f + g) x, y \rangle &= \alpha \langle fx, y \rangle + \langle gx, y \rangle \\
&= \alpha \langle x, f^* y \rangle + \langle x, g^* y \rangle \\
&= \langle x, (\overline{\alpha} f^* + g^*) y \rangle
\end{align*}
\]

2. Same setup as before. $\langle fgy, y \rangle = \langle gx, f^* y \rangle = \langle x, g^* f^* y \rangle$.

3. Same setup as before. $\langle f^* x, y \rangle = \overline{\langle y, f^* x \rangle} = \overline{\langle y, f x \rangle} = \langle x, f y \rangle$.

4. Same setup as before. $\langle I x, y \rangle = \langle x, I y \rangle$.

Definition 16 (Adjoint of a matrix). Let $A \in \mathbb{C}^{m \times n}$. The adjoint or conjugate transpose of $A$ is the matrix $A^*$ such that $A^*_ij = \overline{A}_{ji}$.

Theorem 14. Over an inner product space of finite dimension, adjoints exist. In $\mathbb{C}^n$ with the standard inner product, an adjoint matrix is the matrix of the adjoint of the linear operator it represents.
3.2 Self-adjoint operators

In this section we’ll prove a condition for matrices to be orthogonally (-normally) diagonalizable.

Definition 17 (Self-adjoint). A linear operator or matrix is called self-adjoint or Hermitian if it is equal to its own adjoint.

Lemma 1 (Real eigenvalues). Let \( f: V \to V \) be a Hermitian operator and \( \lambda \) an eigenvalue of \( f \). Then \( \lambda \) is real.

Proof. Let \( v \) be a unit eigenvector of \( f \) satisfying \( f(v) = \lambda v \). Then
\[
\lambda = \langle f v, v \rangle = \langle v, f v \rangle = \langle f v, v \rangle = \bar{\lambda}.
\]
\( \square \)

A large class of complex matrices with complex eigenvectors, but real eigenvalues. Wow! Wow!

Definition 18 (Restriction). Let \( W \) be a subspace of \( V \) and \( f \) a linear operator on \( V \). If \( f(W) \subseteq W \), then \( W \) is called \( f \)-invariant or \( f \)-stable. The linear operator \( f|_W : W \to W \) defined by \( f|_W(w) = f(w) \) is called the restriction of \( f \) to \( W \).

Theorem 15 (Spectral\textsuperscript{1} Theorem). A linear operator \( f: V \to V \) is self-adjoint if and only if it is diagonal and real in an orthonormal basis of \( V \).

Proof. First we show that \((\text{diagonal and real in an orthonormal basis}) \implies \text{(self-adjoint)}\). Suppose that the matrix of \( f \) is diagonal and real in an orthonormal basis. A diagonal real matrix is equal to its conjugate transpose. Therefore, as orthonormal bases faithfully represent inner product spaces and maps between them, \( f \) is self-adjoint.

Next we show that \((\text{self-adjoint}) \implies \text{(diagonal and real in an orthonormal basis)}\). We will use induction on \( n \), the dimension of \( V \). If \( n = 1 \) then \( f \) is already diagonal in any basis.

Next we need to show that if the Spectral Theorem holds on vector spaces of dimension \( n - 1 \), then it holds on vector spaces of dimension \( n \).

By the Fundamental Theorem of Algebra, \( f \) has an eigenvalue \( \lambda \). Because \( f \) is self-adjoint, \( \lambda \) is real. Let \( v \) be an eigenvector such that \( f(v) = \lambda v \). Both \( \text{Span} v \) and its orthogonal complement \((\text{Span } v)^\perp\) are stable under \( f \), the former because it is an eigenspace, the latter in this way: let \( \langle v', v \rangle = 0 \). Then \( \langle f(v'), v \rangle = \langle v', f(v) \rangle = \bar{\lambda} \langle v', v \rangle = 0 \).

By the induction hypothesis, \( f|_{(\text{Span } v)^\perp} \) has \( n - 1 \) orthogonal eigenvectors in \( W \). They are still eigenvectors, and still orthogonal, when treated as members of \( V \). Furthermore, they are orthogonal to \( v \) by construction.

As such \( f \) has a basis of orthogonal eigenvectors, so it has a basis of orthonormal eigenvectors as well. \( \square \)

Lemma 2 (Spectral theorem, factorization version). Let \( A \in \mathbb{C}^{n \times n} \) satisfy \( A = A^* \). Then there exist a unitary matrix \( U \) and a real diagonal matrix \( \Lambda \) such that \( A = U\Lambda U^* \).

Notice that \( U^* = U^{-1} \), so no manual inversion necessary.

\hspace{1em}^1\text{The word spectral imparts an aura of magic and mystery. That is fitting because this theorem is very, very powerful.}
Lemma 3 (Spectral theorem, projection version). Let $f : V \to V$ satisfy $f = f^\ast$. Let \( \{v_1, v_2, \ldots, v_n\} \) be orthonormal eigenvectors of $f$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Then $f = \sum_{i=1}^n \lambda_i \text{proj}_{v_i}$. That is, every self-adjoint operator is a real linear combination of orthogonal projections.

Lemma 4 (Spectral theorem, dyad version). The last and final form of the Spectral Theorem can be seen either as an expansion of the factorization version into outer products or as a translation of the projection version into orthonormal coordinates. Let $A \in \mathbb{C}^n$ satisfy $A = A^\ast$. Let \( \{v_1, v_2, \ldots, v_n\} \) be orthonormal eigenvectors of $f$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Then $f = \sum_{i=1}^n \lambda_i v_i v_i^\ast$.

Application: direction of maximum amplification

Given a nonzero matrix $A \in \mathbb{C}^{n \times n}$, we might wonder what spatial direction gives you the best bang for your buck under left multiplication by $A$. That is, we are interested in the maximum amplification that $A$ can exert on any vector:

$$\max_v \frac{\|Av\|}{\|v\|} \quad (3.10)$$

We can narrow our search to unit vectors.

$$\max_{\|v\|=1} \sqrt{\langle Av, Av \rangle} = \max_{\|v\|=1} \sqrt{\langle A^\ast Av, v \rangle} \quad (3.11)$$

$A^\ast A$ is self-adjoint: $(A^\ast A)^\ast = (A^\ast)^\ast (A^\ast)^\ast = A^\ast A$. Diagonalize it as $A^\ast A = U \Lambda U^\ast$.

$$= \max_{\|v\|=1} \sqrt{\langle U \Lambda U^\ast v, v \rangle} \quad (3.12)$$

$$= \max_{\|v\|=1} \sqrt{\langle \Lambda U^\ast v, U^\ast v \rangle} \quad (3.13)$$

Change variables to $w = U^\ast v$.

$$= \max_{\|w\|=1} \sqrt{\langle \Lambda w, w \rangle} \quad (3.14)$$

This maximum is achieved when $w = e_i$, where $\lambda_i$ is a maximal entry of $\Lambda$.

$$= \sqrt{\lambda_{\text{max}}(A^\ast A)} \quad (3.15)$$

The furthest that $A$ can magnify any vector is the square root of $\lambda_{\text{max}}$, a maximal eigenvalue of $A^\ast A$. It quantifies how “big” $A$ is, and is sometimes called are $\sigma_1$ or $\|A\|_2$, the operator norm of $A$.  

---

15