Discussion 4A

by Rebecca Won

EECS 16B Su22

Q1.

Recall from Discussion 2A:

→ Given $\frac{dx(t)}{dt} = \lambda x(t) + bu(t)$,

$$x(t) = e^{\lambda(t-t_0)} x(t_0) + b\int_{t_0}^{t} u(\theta)\, e^{\lambda(t-\theta)}\, d\theta$$

a) $u(t)$ is a piecewise constant

i.e. $u(t) = u(i\Delta) = u_d[i]$ over $t \in [\, i\Delta, (i+1)\Delta]$

$$x(t) = e^{\lambda(t-t_0)} x(t_0) + b\int_{t_0}^{t} \cancel{u(\theta)}^{\,u_d[i]}\, e^{\lambda(t-\theta)}\, d\theta$$, $t_0 = i\Delta$ (starting time)

$$x(t) = e^{\lambda(t-i\Delta)} x_d[i] + b u_d[i] \int_{i\Delta}^{t} e^{\lambda(t-\theta)}\, d\theta$$

Goal: $x_d[i+1] = ??? \, x_d[i]$

→ Plug in $t = (i+1)\Delta$

$$x((i+1)\Delta) = e^{\lambda\Delta} x_d[i] + b u_d[i] \int_{i\Delta}^{(i+1)\Delta} e^{\lambda((i+1)\Delta-\theta)}\, d\theta$$

$$x_d[i+1] = e^{\lambda\Delta} x_d[i] + b u_d[i] \left(\frac{e^{\lambda\Delta} - e^{0}}{\lambda} \right)$$

$$\therefore\ x_d[i+1] = e^{\lambda\Delta} x_d[i] + b u_d[i] \left(\frac{e^{\lambda\Delta} - 1}{\lambda} \right)$$

⇒ Conclusion :

$$\boxed{\begin{aligned} &\frac{d}{dt}x(t) = \lambda x(t) + b u_d[i] \qquad t \in [i\Delta, (i+1)\Delta] \\ &\quad\Big\downarrow \ \text{discretization} \\ &x_d[i+1] = e^{\lambda\Delta} x_d[i] + b u_d[i] \left(\frac{e^{\lambda\Delta}-1}{\lambda} \right) \end{aligned}}$$

1) constant piecewise input

2) $t = (i+1)\Delta$

b) $\frac{d}{dt}\vec{x}(t) = \underline{A_c}\vec{x}(t) + \underline{\vec{b_c}}\, u_d[i]$

continuous

$\downarrow$

$\vec{x}_d[i+1] = A_d \vec{x}_d[i] + \vec{b_d}\, u_d[i]$

↳ Goal: Find A_d and $\vec{b_d}$ given A_c and $\vec{b_d}$ in continuous time

i. Diagonalize continuous time system w/ a change of basis

$$\frac{d}{dt}\vec{x}(t) = A_c\vec{x}(t) + \vec{b_c}\, u_d[i]$$

$\vec{x}(t) = V\vec{\tilde{x}}(t)$

$$\frac{d}{dt}V\vec{\tilde{x}}(t) = A_c V\vec{\tilde{x}}(t) + \vec{b_c}\, u_d[i]$$

$$\frac{d}{dt}\vec{\tilde{x}}(t) = V^{-1}A_cV\vec{\tilde{x}}(t) + V^{-1}\vec{b_c}\, u_d[i]$$

$$= \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}\vec{\tilde{x}}(t) + \vec{\tilde{b}_c}\, u_d[i]$$

$$\begin{tikzcd} \vec{x}(t) \arrow[r, bend left, "A_c"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x}(t) \arrow[d, bend right, "V^{-1}"'] \\ \vec{\tilde{x}}(t) \arrow[u, bend right, "V"'] \arrow[r] & \frac{d}{dt}\vec{\tilde{x}}(t) \arrow[u, bend right, "V"'] \end{tikzcd}$$

$D = V^{-1}AV$

look at the kth row $\rightarrow$

Index k of the vector $\vec{\tilde{b}_c}\, u_d[i]$

$$\frac{d}{dt}\tilde{x}_k(t) = \lambda_k\tilde{x}_k(t) + (\vec{\tilde{b}_c}\, u_d[i])_k$$

$\downarrow$ discretization from part (a)

$$\tilde{x}_{dk}[i+1] = e^{\lambda_k\Delta}\tilde{x}_{dk}[i] + (\vec{\tilde{b}_c}\, u_d[i])_k\left(\frac{e^{\lambda_k\Delta}-1}{\lambda}\right)$$

Index k of the vector $\vec{\tilde{b}}_c u_d[i]$

$$\frac{d}{dt}\tilde{x}_k(t) = \lambda_k \tilde{x}_k(t) + \left(\vec{\tilde{b}}_c u_d[i]\right)_k$$

$$\Big\downarrow \text{ discretization from part (a)}$$

$$\tilde{x}_{dk}[i+1] = e^{\lambda_k \Delta}\tilde{x}_{dk}[i] + \left(\vec{\tilde{b}}_c u_d[i]\right)_k \left(\frac{e^{\lambda_k\Delta}-1}{\lambda}\right)$$

→ Stack the discretized row as a vector

$$\vec{\tilde{x}}_d[i+1] = \underbrace{\begin{bmatrix} e^{\lambda_1\Delta} & & & 0 \\ & \ddots & & \\ & & e^{\lambda_k\Delta} & \\ & & & \ddots \\ 0 & & & e^{\lambda_n\Delta} \end{bmatrix}}_{e^{\Lambda\Delta}} \vec{\tilde{x}}_d[i] + \underbrace{\begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{e^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix}}_{\Lambda^{-1}(e^{\Lambda\Delta}-I)} \vec{\tilde{b}}_c u_d[i]$$

→ Convert back to original coordinates using $\vec{\tilde{x}}_d[i] = V^{-1}\vec{x}_d[i]$, $\vec{\tilde{b}}_c = V^{-1}\vec{b}_c$

$$V^{-1}\vec{x}_d[i+1] = e^{\Lambda\Delta}V^{-1}\vec{x}_d[i] + \Lambda^{-1}(e^{\Lambda\Delta}-I)V^{-1}\vec{b}_c u_d[i]$$

$$\Rightarrow \vec{x}_d[i+1] = \underbrace{Ve^{\Lambda\Delta}V^{-1}}_{A_d}\vec{x}_d[i] + \underbrace{V\Lambda^{-1}(e^{\Lambda\Delta}-I)V^{-1}\vec{b}_c}_{\vec{b}_d} u_d[i]$$

c) $\vec{x_d}[i+1] = A_d\vec{x_d}[i] + \vec{b_d}\, u_d[i]$ ← unroll the equations

$\vec{x_d}[1] = A_d\vec{x_d}[0] + \vec{b_d}\, u_d[0]$

$\vec{x_d}[2] = A_d\vec{x_d}[1] + \vec{b_d}\, u_d[1]$

$= A_d(A_d\vec{x_d}[0] + \vec{b_d}\, u_d[0]) + \vec{b_d}\, u_d[1]$

$= A_d^2\vec{x_d}[0] + (A_d u_d[0]\vec{b_d} + u_d[1]\vec{b_d})$

$= A_d^2\vec{x_d}[0] + (A_d u_d[0] + u_d[1])\vec{b_d}$

$\vec{x_d}[3] = A_d(A_d^2\vec{x_d}[0] + (A_d u_d[0] + u_d[1])\vec{b_d}) + u_d[2]\vec{b_d}$

$= A_d^3\vec{x_d}[0] + (A_d^2 u_d[0] + A_d u_d[1] + u_d[2])\vec{b_d}$

$\Rightarrow \vec{x_d}[i] = A_d^i\vec{x_d}[0] + \left(\sum_{j=0}^{i-1} u_d[j] A^{i-1-j}\right)\vec{b_d}$