Discussion 4B

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EECS16B Su22

Q1.

a) $x[i+1] = ax[i] + bu[i] + w\cancel{[i]}$ cannot predict, "small"

$\Rightarrow x[i+1] \simeq ax[i] + bu[i]$

Given we know $x[0] \cdots x[l]$ and $u[0] \cdots u[l-1]$

$$\begin{cases} x[1] = ax[0] + bu[0] \\ x[2] = ax[1] + bu[1] \\ \vdots \\ x[l] = ax[l-1] + bu[l-1] \end{cases}$$

$$\Rightarrow \underbrace{\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[l] \end{bmatrix}}_{\vec{s}} = \underbrace{\begin{bmatrix} x[0] & u[0] \\ x[1] & u[1] \\ \vdots & \vdots \\ x[l-1] & u[l-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{p}}$$

$\vec{s} = D\vec{p}$, $\vec{s}$ & D known

$\Rightarrow$ Least squares!

$\vec{p} = (D^T D)^{-1} D^T \vec{s}$

b) $x[i+1] = a x[i] + b_1 u_1[i] + b_2 u_2[i] + w\cancel{[i]}$

$\Rightarrow x[i+1] \approx a x[i] + b_1 u_1[i] + b_2 u_2[i]$

$$x[1] = a x[0] + b_1 u_1[0] + b_2 u_2[0]$$

$$\vdots$$

$$x[l] = a x[l-1] + b_1 u_1[l-1] + b_2 u_2[l-1]$$

$$\Rightarrow \underbrace{\begin{bmatrix} x[1] \\ \vdots \\ x[l-1] \end{bmatrix}}_{\vec{s}} = \underbrace{\begin{bmatrix} x[0] & u_1[0] & u_2[0] \\ \vdots & \vdots & \vdots \\ x[l-1] & u_1[l-1] & u_2[l-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a \\ b_1 \\ b_2 \end{bmatrix}}_{\vec{p}}$$

$\vec{u_1}$ $\vec{u_2}$

$$\Rightarrow \vec{p} = (D^T D)^{-1} D^T \vec{s}$$

c) $\vec{p} = \underline{(D^T D)^{-1}} D^T \vec{s}$

no guarantee that $D^T D$ is invertible

ex) If $\vec{u_1}$ & $\vec{u_2}$ linearly dependent on each other

$\Rightarrow$ D not in full rank

$\Rightarrow$ $D^T D$ not in full rank

$\Rightarrow$ $D^T D$ not invertible

d) $$\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i]$$

$$\Rightarrow \begin{cases} x_1[i+1] = a_{11}x_1[i] + a_{12}x_2[i] + b_1 u[i] \\ x_2[i+1] = a_{21}x_1[i] + a_{22}x_2[i] + b_2 u[i] \end{cases}$$

$$\begin{cases} x_1[1] = a_{11}x_1[0] + a_{12}x_2[0] + b_1 u[0] \\ x_2[1] = a_{21}x_1[0] + a_{22}x_2[0] + b_2 u[0] \end{cases}$$

$$\vdots$$

$$\begin{cases} x_1[l] = a_{11}x_1[l-1] + a_{12}x_2[l-1] + b_1 u[l-1] \\ x_2[l] = a_{21}x_1[l-1] + a_{22}x_2[l-1] + b_2 u[l-1] \end{cases}$$

① Stack vertically

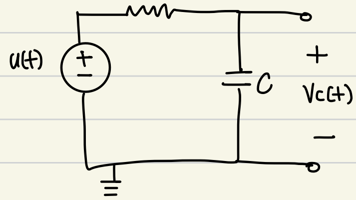
$$\begin{bmatrix} x_1[1] \\ x_2[1] \\ \vdots \\ x_1[l] \\ x_2[l] \end{bmatrix} = \begin{bmatrix} x_1[0] & x_2[0] & 0 & 0 & u[0] & 0 \\ 0 & 0 & x_1[0] & x_2[0] & 0 & u[0] \\ & & & \vdots & & \\ x_1[l-1] & x_2[l-1] & 0 & 0 & u[l-1] & 0 \\ 0 & 0 & x_1[l-1] & x_2[l-1] & 0 & u[l-1] \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \\ b_1 \\ b_2 \end{bmatrix}$$

② Stack horizontally

$$\begin{bmatrix} x_1[1] & x_2[1] \\ \vdots & \vdots \\ x_1[l] & x_2[l] \end{bmatrix} = \begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ \vdots & \vdots & \vdots \\ x_1[l-1] & x_2[l-1] & u[l-1] \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ b_1 & b_2 \end{bmatrix}$$

Q2.

a)



$$\frac{dV_c(t)}{dt} = -2V_c(t) + 2u(t)$$ → in the form $\frac{d}{dt}x(t) = \lambda x(t) + bu(t)$

Recall : general sol'n to $\frac{d}{dt}x(t) = \lambda x(t) + bu(t)$ is

$$x(t) = x_0 e^{\lambda t} + \int_0^t e^{\lambda(t-\theta)} bu(\theta)\,d\theta$$

→ $x(t) = V_c(t)$, $\lambda = -2$, $b = 2$

$$V_c(t) = V_c(0)e^{-2t} + \int_0^t e^{-2(t-\theta)}\, 2u(\theta)\, d\theta$$

$$= V_c(0)e^{-2t} + 2\int_0^t e^{-2(t-\theta)} u(\theta)\,d\theta$$

Goal: Show that $|V_c(t)| < \infty$

$$|V_c(t)| = \left| V_c(0)e^{-2t} + 2\int_0^t e^{-2(t-\theta)} u(\theta)\,d\theta \right|$$

$$\leq |V_c(0)e^{-2t}| + 2\left|\int_0^t e^{-2(t-\theta)} u(\theta)\,d\theta\right|$$ → triangular Inequality

$$\leq |V_c(0)\underbrace{e^{-2t}}| + 2\int_0^t |\underbrace{e^{-2(t-\theta)}} u(\theta)|\, d\theta$$ → hint

↳ bounded : $0 \leq e^{-2t} \leq 1$

$$= |V_c(0)|\, e^{-2t} + 2\int_0^t e^{-2(t-\theta)}\, |u(\theta)|\, d\theta$$

$$\Rightarrow |V_c(t)| \le |V_c(0)|e^{-2t} + 2\int_0^t e^{-2(t-\theta)} \underbrace{|u(\theta)|}_{\text{bounded input } |u(\theta)|<K<\infty} d\theta$$

$$= |V_c(0)|e^{-2t} + 2K\int_0^t e^{-2(t-\theta)} d\theta$$

$$= \underbrace{|V_c(0)|e^{-2t}}_{\text{bounded}} + \underbrace{2K(1-e^{-2t})}_{\text{bounded}} < \infty$$

c) $x[i+1] = 2x[i] + u[i]$, $x[0] = 0$

$x[1] = 2x[0] + u[0] = u[0]$

$x[2] = 2x[1] + u[1] = 2u[0] + u[1]$

$x[3] = 2x[2] + u[2]$

$= 2(2u[0] + u[1]) + u[2]$

$= 2^2u[0] + 2u[1] + u[2]$

$\vdots$

$$\Rightarrow x[i+1] = 2^i u[0] + \sum_{j=1}^{i-1} 2^{i-1-j} u[j]$$

as long as $u[0]$ is nonzero

$2^i u[0] \to \infty$ as $i \to \infty$

$\therefore u(t) = \begin{cases} 1 & i=0 \\ 0 & i>0 \end{cases}$ → will blow up the sys.