Discussion 5A

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EECS16B Su22

Q1.

a) How can we prove $\underbrace{Span(\{\vec{v_1} \dots \vec{v_n}\})}_{V} = \underbrace{Span(\{\vec{w_1}, \dots, \vec{w_n}\})}_{W}$?

→ Argue that the spans V and W are subsets of one another

→ Prove spans are equal by showing

① $V \subseteq W$ i.e. $\forall \vec{v_i}, \vec{v_i} \in W$ $\Rightarrow V = W$

② $W \subseteq V$ i.e. $\forall \vec{w_i}, \vec{w_i} \in V$

b) $\vec{q_1}$ s.t. $\underbrace{span(\{\vec{q_1}\})}_{Q} = \underbrace{Span(\{\vec{s_1}\})}_{S}$

→ $\vec{q_1} = \frac{\vec{s_1}}{\|\vec{s_1}\|}$

① unit vector

$\|\vec{q_1}\| = 1$

② equal span (part (a)!)

→ $Q \subseteq S$ $S = span(\{\vec{s_1}\}) = \alpha \cdot \vec{s_1}$ $\alpha = \frac{1}{\|\vec{s_1}\|} \to \vec{q_1}$, $\therefore \vec{q_1} = \alpha \cdot \vec{s_1}$

→ $S \subseteq Q$ $Q = span(\{\vec{q_1}\}) = \beta \cdot \vec{q_1}$ $\beta = \|\vec{s_1}\| \to \vec{s_1}$, $\therefore \vec{s_1} = \beta \vec{q_1}$

c) Find $\vec{q_2}$

How $\vec{q_2}$ should look like

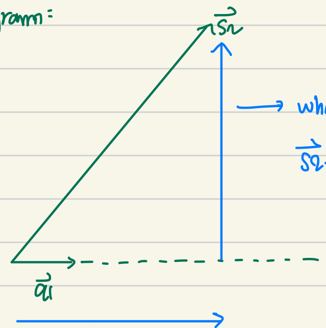
→ Recall:

$\{\vec{q_i}\}$ should be ① linearly independent → perpendicular to $\vec{q_1}$

② Normal → unit vector

③ Orthogonal

→ Diagram:



→ what is this vector?

$\vec{s_2} - \text{proj}_{\vec{q_1}}(\vec{s_2}) = \vec{z_2}$

$\text{proj}_{\vec{q}}(\vec{s_2})$

"projection of $\vec{s_2}$ onto $\vec{q}$"

→ Recall (from 16A):

$$\text{proj}_{\vec{q_1}}(\vec{s_2}) = \frac{\overbrace{(\vec{s_2}^T \vec{q_1})}^{\text{scalar}} \vec{q_1}}{\|\vec{q_1}\|_2^2}$$

↳ But since $\vec{q_1}$ is normal, $\|\vec{q_1}\|_2^2 = 1$

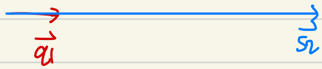
$$\Rightarrow \text{proj}_{\vec{q_1}}(\vec{s_2}) = (\vec{s_2}^T \vec{q_1})\vec{q_1}$$

⇒ ① $\vec{z_2} = \vec{s_2} - (\vec{s_2}^T \vec{q_1})\vec{q_1}$

② $\vec{q_2} = \dfrac{\vec{z_2}}{\|\vec{z_2}\|}$

d) What if $\{\vec{s_1}, \vec{s_2}, \vec{s_3}\}$ not linearly independent but rather $\vec{s_1} = \alpha\vec{s_2}$

→ Then if we draw $\underbrace{\vec{q_1}}_{\text{normalized } \vec{s_1}}$ + $\vec{s_2}$



→ Because $\vec{s_1}$ + $\vec{s_2}$ are in the same direction,

$$\underbrace{\vec{z_2}}_{\vec{q_2}\text{ before normalization}} = \vec{s_2} - \underbrace{(\vec{s_2}^T\vec{q_1})\vec{q_1}}_{\text{projection of } \vec{s_2} \text{ onto } \vec{q_1}} = \vec{s_2} - \vec{s_2} = 0 \Rightarrow \vec{q_2} = 0$$

e) Find $\vec{q_3}$

→ So far: ① $\vec{q_1} = \frac{\vec{s_1}}{\|\vec{s_1}\|}$

② $\vec{z_2} = \vec{s_2} - \text{proj}_{\vec{q_1}}(\vec{s_2}) \Rightarrow \vec{q_2} = \frac{\vec{z_2}}{\|\vec{z_2}\|}$

③ $\vec{z_3} = \vec{s_3} - \text{proj}_{\vec{q_1}}(\vec{s_3}) - \text{proj}_{\vec{q_2}}(\vec{s_3})$

$$= \vec{s_3} - (\vec{s_3}^T\vec{q_1})\vec{q_1} - (\vec{s_3}^T\vec{q_2})\vec{q_2}$$

$$\vec{q_3} = \frac{\vec{z_3}}{\|\vec{z_3}\|}$$

Q2.

a) $\underbrace{A}_{\text{tall matrix}} \in \mathbb{R}^{n\times m}$, $n \geq m$

Show if A orthogonal, $A^TA = I_{m\times m}$

→ Visualize & explicitly write down A^TA

$$A = \underbrace{\begin{bmatrix} | & | & & | \\ \vec{a_1} & \vec{a_2} & \cdots & \vec{a_m} \\ | & | & & | \end{bmatrix}}_{m} \Big\} n \qquad A^T = \underbrace{\begin{bmatrix} — \vec{a_1}^T — \\ \vdots \\ — \vec{a_m}^T — \end{bmatrix}}_{n} \Big\} m$$

$$A^TA = \begin{bmatrix} — \vec{a_1}^T — \\ \vdots \\ — \vec{a_m}^T — \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{a_1} & \cdots & \vec{a_m} \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} \vec{a_1}^T\vec{a_1} & \vec{a_1}^T\vec{a_2} & \cdots & \vec{a_1}^T\vec{a_m} \\ \vec{a_2}^T\vec{a_1} & \vec{a_2}^T\vec{a_2} & & \\ \vdots & & \ddots & \\ \vec{a_m}^T\vec{a_1} & & & \vec{a_m}^T\vec{a_m} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}}_{m} \Big\} m = I_{m\times m}$$

$\|\vec{a_i}\|_2 = 1$

b) $A \in \mathbb{R}^{n \times m}$, $n \geq m$

Show that projection of $\vec{y}$ onto subspace spanned by columns of A is $AA^T y$.

→ A is tall.

→ $\text{proj}_{col(A)}(\vec{y}) = A\hat{x}$ ← least squares sol'n

$$= \underbrace{A\underbrace{(A^TA)^{-1}A^T\vec{y}}_{\text{least squares}}}_{A\hat{x}}$$

→ 2a) $A^TA = I_{m \times m}$

→ $\underbrace{A_{n \times m} I_{m \times m}}_{} A_{m \times n} \vec{y}$

$A_{n \times m} A^T_{m \times n} \vec{y}$

$= AA^T y$