# Discussion 6A

by Rebecca Won

EECS16B Su22

# Q1.

columns are linearly dependent

$$\underset{3\times 2}{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \underset{3\times 3}{U}\overset{3\times 2}{\Sigma}\underset{2\times 2}{V^T}, \quad \text{rank}(A) = 1$$

a) (i) $A^TA = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$

$\det(A-\lambda I) = 0$

$$\det\begin{pmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{pmatrix} = (9-\lambda)^2 - 81 = \lambda^2 - 18\lambda = \lambda(\lambda-18) = 0$$

$\therefore \lambda_1 = 18, \ \lambda_2 = 0$

<u>corresponding eigenvectors</u>

$\lambda_1 = 18 \rightarrow \begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad -9v_1 - 9v_2 = 0 \quad v_1 = -v_2 \rightarrow \vec{v_1} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$\lambda_2 = 0 \qquad \rightarrow \vec{v_2} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

(ii) $\Lambda = \begin{bmatrix} 18 & 0 \\ 0 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

unpack $\Lambda$

i.e $\Lambda_r$

$\Lambda_r = [18]$ $\quad \hookrightarrow$ diagonal matrix of $\sqrt{\lambda_i}$ for $i=1$ to $r$

$r = 1 \Rightarrow$ only have 1 entry i.e. $\lambda_1$

square root all the entries

(iii) $\Sigma_r = \Lambda_r^{1/2} = [\sqrt{18}]$

$$\Sigma = \underset{3\times 2}{\begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

fill in 0s such that dimensionality matches w/ A

(iv) So far, we found

$$\begin{cases} V_r = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} & V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ \Sigma_r = [\sqrt{18}] & \Sigma = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{cases}$$

→ Find $U_r$

$$A = U_r \Sigma_r V_r^T \Rightarrow AV_r = U_r \Sigma_r V_r^T V_r = U_r \Sigma_r$$

$$\Rightarrow U_r = AV_r \Sigma_r^{-1} = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

But for full SVD: $\underset{3\times 3}{U} \rightarrow$ Gram Schmidt!

$$\begin{bmatrix} -1/3 & 1 & 0 & 0 \\ 2/3 & 0 & 1 & 0 \\ -2/3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[G.S.]{} \begin{bmatrix} -1/3 & \sqrt{8}/3 & 0 \\ 2/3 & 1/3\sqrt{2} & 1/\sqrt{2} \\ -2/3 & -1/3\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = U$$

(v) • Full SVD $= U\Sigma V^T = \begin{bmatrix} -1/3 & \sqrt{8}/3 & 0 \\ 2/3 & 1/3\sqrt{2} & 1/\sqrt{2} \\ -2/3 & -1/3\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \left( \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \right)^T$

• Compact SVD $= U_r \Sigma_r V_r^T = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix} [\sqrt{18}] \left( \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right)^T$

b) SVD of $A^T$

$3\times2$ $A = U\Sigma V^T$

$2\times3$ $A^T = (U\Sigma V^T)^T = V\Sigma^T U^T$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{18} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1/3 & \sqrt{8}/3 & 0 \\ 2/3 & 1/3\sqrt{2} & 1/\sqrt{2} \\ -2/3 & -1/3\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

$2\times2$    $2\times3$    $3\times3$

$\rightarrow$ SVD dimensionality ch. to match the dimensionality of $A^T$

c) WTS: $Null(A) = Col(V_{n-r})$

We can write $A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$ (outer product form)

$Null(A)$ means $A\vec{x}=0$, $\vec{x}\neq 0$

$$A\vec{x} = \sum_{i=1}^{r} \sigma_i \vec{u_i} \vec{v_i}^T \vec{x}$$

What vector $\vec{x}$ makes $\vec{v_i}^T\vec{x}=0$?

$\rightarrow$ orthogonal vector to $\vec{v_i}$

$\Rightarrow$ $\vec{x}$ must be orthogonal to $\vec{v_i}$ for $i=1\cdots r$ (sum from 1 to $r$)

$\vec{x} \perp Col(V_r)$

$\vec{x} \in Col(V_{n-r})$

$\hookrightarrow$ $V$ is orthonormal $V = [V_r \; V_{n-r}]$ orthogonal to $V_r$!

$\Rightarrow$ $Null(A) = Col(V_{n-r})$

d) $Col(A) = Col(U_r)$

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

$Col(A)$ means that for a given nonzero vector $\vec{b}$, $A\vec{x} = \vec{b}$

$$\therefore A\vec{x} = \sum_{i=1}^{r} \sigma_i \vec{u}_i \underbrace{\vec{v}_i^T \vec{x}}_{\neq 0}$$

scalar

$$= \sum_{i=1}^{r} (\vec{v}_i^T x)\, \sigma_i \vec{u}_i$$

$$= \sum_{i=1}^{r} \alpha \vec{u}_i \rightarrow \text{linear combination of } \vec{u}_1, \dots, \vec{u}_r$$

$\therefore A\vec{x} = \vec{b} \in col(U_r)$

$\therefore Col(A) = col(U_r)$

f) $AA^T$

$r = rank(A)$

$(U, \Lambda) = diagonalize(AA^T)$

Unpack $U = \begin{bmatrix} U_r & U_{n-r} \end{bmatrix}$

Unpack $\Lambda = \begin{bmatrix} \Lambda_r & 0 \\ 0 & 0 \end{bmatrix}$

$$\Sigma_r = \Lambda_r^{1/2}$$

Pack $\Sigma$

$V_r = A^T U_r \Sigma_r^{-1}$

$V = extendbasis(V_r, \mathbb{R}^n)$

return $(U, \Sigma, V)$