Discussion 1D

For this discussion, Note 1 is helpful for the differential equations, and Note j covers the complex numbers fundamentals.

1. RC Circuits: Solving the Differential Equations

Recall that in the last discussion, we were tasked with analyzing an example RC circuit (in 1) and using element equations (governing equations for resistors and capacitors) to formulate a differential equation. This equation describes the time-varying behavior of this circuit. Specifically, we had the following differential equation:

$$RC\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} + V_C(t) = V(t) \tag{1}$$

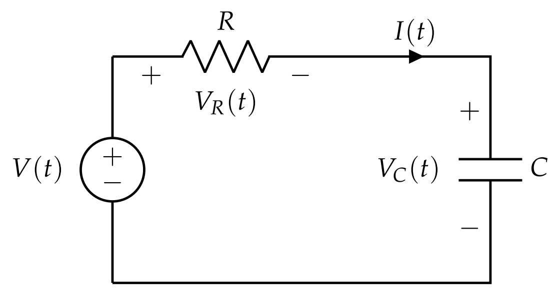


Figure 1: Sample RC Circuit

Our goal is to now solve this differential equation for the voltage across the capacitor, $V_C(t)$. Recall that, in the previous discussion and lecture, we covered two kinds of differential equations:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) \tag{2}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + u \tag{3}$$

where $\lambda, u \in \mathbb{R}$. Eq. (2) has a solution of the form

$$x(t) = A\mathrm{e}^{bt} \tag{4}$$

for some constants $A, b \in \mathbb{R}$ that we have to find. We can solve the differential equation in eq. (3) by performing a change of variables operation. This will yield a new differential equation that resembles eq. (2), and reversing the change of variables operation will give us the solution to eq. (3).

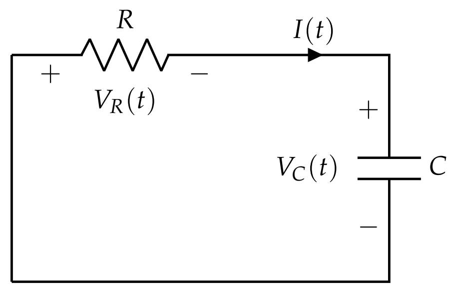


Figure 2: RC Circuit for part 1.a. Note that the voltage source has been turned off (0 V) for this subpart, and the initial voltage on the capacitor is V_{DD}.

(a) Let's suppose that at $t = 0$, the capacitor is charged to a voltage V_{DD} ($V_C(0) = V_{DD}$). Let's also assume that $V(t) = 0$ for all $t \geq 0$ (voltage source is turned off). **Solve the differential equation for $V_C(t)$ for $t \geq 0$.**

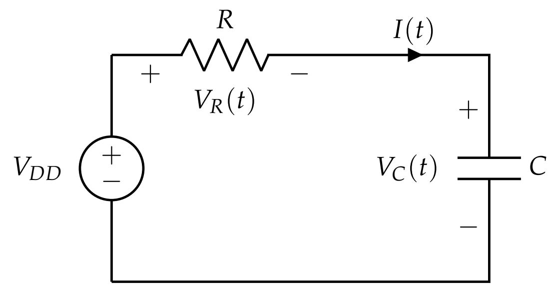


Figure 3: Circuit for part 1.b

(b) Now, let's suppose that we start with an uncharged capacitor $V_C(0) = 0$. We apply some constant voltage $V(t) = V_{DD}$ across the circuit for all $t \geq 0$. **Solve the differential equation for $V_C(t)$ for $t \geq 0$.**

(c) We now want to combine the principles from the previous two subparts to understand the voltage waveform when a switch occurs at some time t. Specifically, suppose that at $t = 0$, $V(t) = 0\,\text{V}$, $V_C(0) = V_{DD}$. Then, at some $t = t_{\text{switch}}$, the voltage source is turned on $V(t) = V_{DD}$ for $t \geq t_{\text{switch}}$. **Find the equation for the overall capacitor voltage as a function of time (for times before and after t_{switch}).**

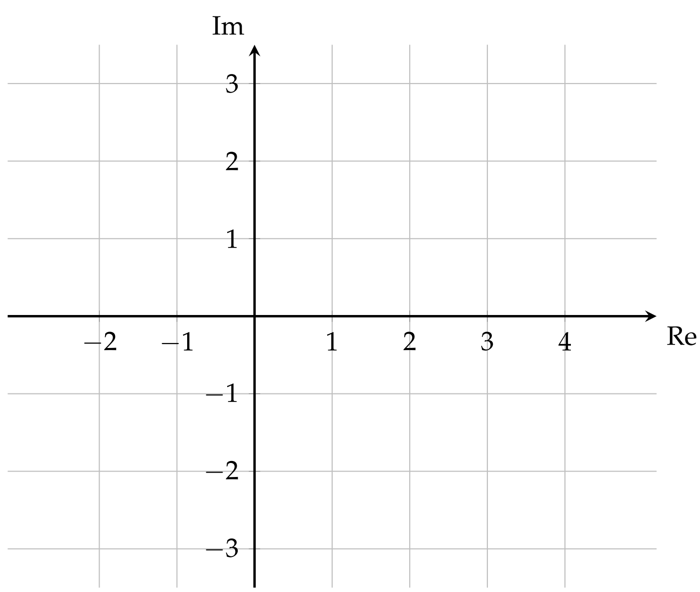
2. Complex Algebra (Review)

(a) **Express the following values in polar forms:** -1, j, $-$j, $(\mathrm{j})^{\frac{1}{2}}$**, and** $(-\mathrm{j})^{\frac{1}{2}}$**.** Recall $\mathrm{j}^2 = -1$, and the complex conjugate of a complex number is denoted with a bar over the variable. The complex conjugate is defined as follows: for a complex number $z = x + \mathrm{j}y$, the complex conjugate $\overline{z} = x - \mathrm{j}y$.

(b) Represent $\sin(\theta)$ and $\cos(\theta)$ using complex exponentials. (*Hint:* Use Euler's identity $e^{\mathrm{j}\theta} = \cos(\theta) + \mathrm{j}\sin(\theta)$.)

For the next parts, let $a = 1 - \mathrm{j}\sqrt{3}$ and $b = \sqrt{3} + \mathrm{j}$.

(c) Show the number a in complex plane, marking the distance from origin and angle with real axis.



(d) Show that multiplying a with j is equivalent to rotating the complex number by $\frac{\pi}{2}$ or $90°$ in the complex plane.

(e) **(Practice)** For complex number $z = x + \mathrm{j}y$ show that $|z| = \sqrt{z\overline{z}}$, where $\overline{z}$ is the complex conjugate of z.

(f) **(Practice)** Express a and b in polar form.

(g) **(Practice)** Find ab, $a\overline{b}$, $\frac{a}{b}$, $a + \overline{a}$, $a - \overline{a}$, $\overline{ab}$, $\overline{a}\overline{b}$, and $(b)^{\frac{1}{2}}$.

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