# Discussion 2A

For this discussion, Note 2 is helpful.

## 1. Differential Equations with Piecewise Constant Inputs

Working through this question will help you understand better differential equations with inputs. Along the way, we will also touch a bit on going from continuous-time (i.e. the real world) into a discrete-time view (i.e. what we can hope to see from computer programs).

(a) Consider the scalar system in eq. (1), where $\lambda \neq 0$.

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + bu(t). \tag{1}$$

Our goal is to solve this system (find an appropriate function $x(t)$) for general inputs $u(t)$. As a preview of the results we will derive in discussion and homework, all of this work helps guide us to much cleaner integral solution for $x(t)$ as follows:

$$x(t) = \mathrm{e}^{\lambda t} x_0 + b\mathrm{e}^{\lambda t} \int_0^t \mathrm{e}^{-\lambda \tau} u_c(\tau)\,\mathrm{d}\tau \tag{2}$$

We can also generalize this formula for an initial condition at some nonzero $t$:

$$x(t) = \mathrm{e}^{\lambda(t-t_0)} x_{t_0} + b \int_{t_0}^t \mathrm{e}^{\lambda(t-\tau)} u_c(\tau)\,\mathrm{d}\tau \tag{3}$$

This is the same result as at the end of Note 2!

To do this, we will start with a piecewise constant $u(t)$ which we will solve in this worksheet. This will be a natural extension of the kind of analysis we did in Discussion 1B, where we chained together 2 intervals' results to form a continuous curve.[1]

Suppose that our input $u(t)$ of interest is piecewise constant over durations of width $\Delta$:

$$u(t) = u(i\Delta) = u[i] \text{ if } t \in [i\Delta, (i+1)\Delta) \equiv i\Delta \leq t < (i+1)\Delta. \tag{4}$$

To stay consistent, we will use the notation

$$x_d[i] = x(i\Delta). \tag{5}$$

The square brackets are like array-indexing into a vector.

[1] In homework, we will see how to extend this to general $u(t)$ by taking limits in the style of Riemann integration to complete the process.

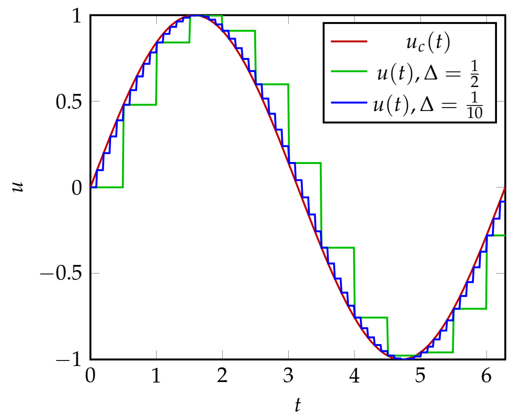


**Figure 1:** An example of a discrete input where the limit as the time-step $\Delta$ goes to 0 approaches a continuous function. The red line, the original signal $u_c(t) = \sin(t)$, is traced almost exactly by the blue line (small $\Delta$)) and not nearly as well by the green line (large $\Delta$).

The first step is to discover the system's behavior across a single time-step where the input stays constant. This is like how in discussion 1B, you solved single-interval systems for 2 initial conditions.

**Given that we know the value of $x(i\Delta) = x_d[i]$, compute $x_d[i+1] = x((i+1)\Delta)$.** This does involve some algebra!

For $t \in [i\Delta, (i+1)\Delta)$ (that is, $i\Delta \leq t < (i+1)\Delta$), the system is

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + bu[i]. \tag{6}$$

Also see Note 2.

If the general solution for any $i$ is difficult to start off with, consider the specific case of $i = 0$.

**Solution:** If $t \in [i\Delta, (i+1)\Delta)$, the differential equation takes the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + bu(t) = \lambda x(t) + bu[i]. \tag{7}$$

and the initial condition is $x(i\Delta) = x_d[i]$, which is some constant value (the value of the function at the start of the interval, based on the previous intervals).

**Deriving and Verifying the Change-Of-Variables**

We see that this is a nonhomogeneous differential equation, requiring a change-of-variables to solve. We want to transform this into a homogeneous differential equation and solve; then, we'll change variables back. We need to transform 2 quantities:

- the differential equation itself (same process as before)
- the initial condition (here, this is *not* at $t = 0$, but rather at $t = i\Delta$.)

Our past experience suggests that a change-of-variables of the form $\widetilde{x}(t) = x(t) + k$ will work, for some $k$. But, we can't immediately tell what $k$ value makes the new equation homogeneous, where $\frac{\mathrm{d}\widetilde{x}(t)}{\mathrm{d}t} = \lambda \widetilde{x}(t)$. To transform eq. (7), we first check if our change-of-variables satisfies the left-hand sides being equal (that is, does $\frac{\mathrm{d}\widetilde{x}(t)}{\mathrm{d}t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$?) We see that our current form does in fact work, since $k$ is a constant.

  

Next, we want the right-hand sides to match, and need to find the $k$ for which $\lambda\widetilde{x}(t) = \lambda x(t) + bu[i]$. Some algebra follows:

$$\lambda\widetilde{x}(t) = \lambda(x(t) + k) \tag{8}$$

$$= \lambda x(t) + \lambda k \qquad \text{[distributing]} \tag{9}$$

$$\lambda\widetilde{x}(t) = \lambda x(t) + bu[i] \qquad \text{[want LHS to equal]} \tag{10}$$

$$\implies \lambda x(t) + bu[i] = \lambda x(t) + \lambda k \tag{11}$$

$$\lambda k = bu[i] \tag{12}$$

$$k = \frac{bu[i]}{\lambda} \qquad . \tag{13}$$

So, we need the change of variables:

$$\widetilde{x}(t) = x(t) + \frac{bu[i]}{\lambda}. \tag{14}$$

**Solving the Homogeneous System in Changed Variables**

We have the homogeneous equation $\frac{\mathrm{d}\widetilde{x}(t)}{\mathrm{d}t} = \lambda\widetilde{x}(t)$ after transformation, but we need a new initial condition $\widetilde{x}(i\Delta)$. This is $\widetilde{x}(i\Delta) = x_d[i] + \frac{bu[i]}{\lambda}$. Now, we generate the following general form ($\gamma, A$ are constants):

$$\widetilde{x}(t) = A\mathrm{e}^{\gamma t} \tag{15}$$

We use the differential equation to see that

$$\frac{\mathrm{d}\widetilde{x}(t)}{\mathrm{d}t} = \gamma A\mathrm{e}^{\gamma t} \implies \gamma = \lambda. \tag{16}$$

What's $A$? We use the initial function value of $\widetilde{x}(i\Delta) = x_d[i] + \frac{bu[i]}{\lambda}$ (and some more algebra):

$$\widetilde{x}(i\Delta) = x_d[i] + \frac{bu[i]}{\lambda} \tag{17}$$

$$= A\mathrm{e}^{\lambda \cdot i\Delta} \qquad \text{[general form]} \tag{18}$$

$$\implies A = \mathrm{e}^{-\lambda \cdot i\Delta}\left(x_d[i] + \frac{bu[i]}{\lambda}\right) \tag{19}$$

The entire right-hand side is full of known constants. Combining and simplifying this info:

$$\widetilde{x}(t) = A\mathrm{e}^{\gamma t} \tag{20}$$

$$= \underbrace{\left(\mathrm{e}^{-\lambda \cdot i\Delta}\left(x_d[i] + \frac{bu[i]}{\lambda}\right)\right)}_{A} \mathrm{e}^{\overbrace{\lambda}^{\gamma} t} \tag{21}$$

$$= \mathrm{e}^{\lambda(t - i\Delta)}\left(x_d[i] + \frac{bu[i]}{\lambda}\right) \tag{22}$$

**Changing Variables Back and Solving the Original Equation**

$$x(t) = \widetilde{x}(t) - \frac{bu[i]}{\lambda} \qquad \text{[reverse change vars]} \tag{23}$$

$$= \mathrm{e}^{\lambda(t - i\Delta)}\left(x_d[i] + \frac{bu[i]}{\lambda}\right) - \frac{bu[i]}{\lambda} \tag{24}$$

 

$$= e^{\lambda(t-i\Delta)} x_d[i] + e^{\lambda(t-i\Delta)} \frac{bu[i]}{\lambda} - \frac{bu[i]}{\lambda} \quad (25)$$

$$= e^{\lambda(t-i\Delta)} x_d[i] + b\left(\frac{e^{\lambda(t-i\Delta)} - 1}{\lambda}\right) u[i] \quad (26)$$

And this is our solution for $x(t)$! We're almost there, the last step to answer the question is to evaluate this for $x((i+1)\Delta)$:

$$x_d[i+1] = x((i+1)\Delta) = e^{\lambda\Delta} x_d[i] + \frac{b(e^{\lambda\Delta} - 1)}{\lambda} u[i]. \quad (27)$$

(b) For ease of notation going forward, suppose that we want to use the following simpler form for the discrete-time model:

$$x_d[i+1] = \alpha x_d[i] + \beta u[i] \quad (28)$$

**Based on your result from part 1.a, what are $\alpha$ and $\beta$ in terms of $\lambda$, $b$, and $\Delta$?** You should be able to read the coefficients off from the equation you got before, but this may require some grouping of terms. We will use this again at the end of the worksheet.

**Solution:** We have:

$$x_d[i+1] = \underbrace{e^{\lambda\Delta}}_{\alpha} x_d[i] + \underbrace{\frac{b(e^{\lambda\Delta} - 1)}{\lambda}}_{\beta} u[i]. \quad (29)$$

(c) Now that we've found a one-step recurrence for $x_d[i+1]$ in terms of $x_d[i]$, we want to get an expression for $x_d[i]$ in terms of the original value $x(0) = x_d[0]$, and all the inputs $u$. This is so that we can eventually use this function for $x_d[i]$ to get a function for $x(t)$.

**Unroll the implicit recursion you derived in the previous part to write $x_d[i]$ as a sum that involves $x_d[0]$ and the $u[k]$ for $k = 0, 1, \ldots, i-1$.**

For this part, consider the simpler form of the discrete-time system as in part 1.b:

$$x_d[i+1] = \alpha x_d[i] + \beta u[i] \quad (30)$$

By "discrete-time system" here we are pointing to the fact that we understand this recursively in terms of discrete time steps instead of as a continuous waveform.

*(HINT: What is $x_d[1]$ in terms of $x_d[0]$? What is $x_d[2]$ in terms of (only) $x_d[0]$? What about $x_d[3]$? Can you find a pattern?)*

**Solution:** Let's look at the pattern, given that

$$x_d[i+1] = \alpha x_d[i] + \beta u[i]. \quad (31)$$

Starting from $i = 0$, we get

$$x_d[1] = \alpha x_d[0] + \beta u[0] \quad (32)$$

$$x_d[2] = \alpha x_d[1] + \beta u[1] = \alpha(\alpha x_d[0] + \beta u[0]) + \beta u[1] \quad (33)$$

$$= \alpha^2 x_d[0] + \beta(\alpha u[0] + u[1]) \quad (34)$$

$$x_d[3] = \alpha x_d[2] + \beta u[2] = \alpha\left(\alpha^2 x_d[0] + \beta(\alpha u[0] + u[1])\right) + \beta u[2] \quad (35)$$

$$= \alpha^3 x_d[0] + \beta\left(u[2] + \alpha u[1] + \alpha^2 u[0]\right) \quad (36)$$

The idea is to collect terms with all the $x_d$'s in one term and all the $u$'s in the other term. Again, this separates out the effect of the initial condition $x_d[0]$ and all the inputs $u[k]$.

So, given this pattern, we guess

$$x_d[i] = \alpha^i x_d[0] + \beta \sum_{k=0}^{i-1} \alpha^{i-1-k} u[k]. \tag{37}$$

Let's check that this works. The way we do this is compute $x_d[i+1]$ through this formula, and also from eq. (28), and check that they're equal.

$$\begin{aligned} x_d[i+1] = \alpha x_d[i] + bu[i] &= \alpha \left( a^i x_d[0] + \beta \sum_{k=0}^{i-1} \alpha^{i-1-k} u[k] \right) + \beta u[i] && (38) \\ &= \alpha^{i+1} x_d[0] + \beta \left( \sum_{k=0}^{i-1} \alpha^{i-k} u[k] \right) + \beta u[i] && (39) \\ &= \alpha^{i+1} x_d[0] + \beta \left( u[i] + \sum_{k=0}^{i-1} \alpha^{i-k} u[k] \right) && (40) \\ &= \alpha^{i+1} x_d[0] + \beta \sum_{k=0}^{i} \alpha^{i-k} u[k] && (41) \end{aligned}$$

This satisfies eq. (37), for $i+1$ and hence our guess was correct!

(d) **For a given time $t$ in continuous real time, what is the discrete $i$ interval that corresponds to it?**

*(HINT: $\lfloor c \rfloor$ is the largest integer smaller than c.)*

**Solution:** $i = \left\lfloor \frac{t}{\Delta} \right\rfloor$ is the discrete time index $i$ that corresponds to the time $t$ in real time, because it is the only $i$ satisfying $t \in [i\Delta, (i+1)\Delta)$.

(e) Here's the payoff for this discussion! Use the results of parts **1.**a and **1.**b to give an approximate expression for $x(t)$ for any $t$, in terms of $x_d[0] = x(0)$ and the inputs $u[j]$. You can assume that $\Delta$ is small enough that $x(t)$ does not change too much (is approximately constant) over an interval of length $\Delta$.

(*Hint*: The assumption we just made allows us to approximate $x(t) \approx x(\Delta \left\lfloor \frac{t}{\Delta} \right\rfloor) = x_d \left[ \left\lfloor \frac{t}{\Delta} \right\rfloor \right]$.)

**Solution:** Using the result derived in part **1.**d and the assumption,

$$x(t) \approx x\left(\Delta \left\lfloor \frac{t}{\Delta} \right\rfloor\right) = x_d \left[ \left\lfloor \frac{t}{\Delta} \right\rfloor \right]. \tag{42}$$

From part **1.**b,

$$\alpha = e^{\lambda\Delta} \qquad \text{and} \qquad \beta = \frac{b(e^{\lambda\Delta} - 1)}{\lambda}. \tag{43}$$

$$x(t) \approx \alpha^{\left\lfloor \frac{t}{\Delta} \right\rfloor} x_d[0] + \beta \sum_{j=0}^{\left\lfloor \frac{t}{\Delta} \right\rfloor - 1} \alpha^{\left\lfloor \frac{t}{\Delta} \right\rfloor - 1 - j} u[j] \tag{44}$$

Thus, overall

$$x(t) \approx \left(e^{\lambda\Delta}\right)^{\left\lfloor \frac{t}{\Delta} \right\rfloor} x_d[0] + \frac{b(e^{\lambda\Delta} - 1)}{\lambda} \sum_{j=0}^{\left\lfloor \frac{t}{\Delta} \right\rfloor - 1} (e^{\lambda\Delta})^{\left\lfloor \frac{t}{\Delta} \right\rfloor - 1 - j} u[j]. \tag{45}$$

 

We observe that the initial condition $x_d[0]$ has an exponential (in $t$) effect on $x(t)$, and inputs at the beginning have exponential (again in $t$) effect on $x(t)$, with the later inputs having an exponentially decaying effect on $x(t)$ relative to the earlier inputs. (It's exponentials all the way down.)

This problem brings together many different concepts and uses a lot of notation. As such, it may be difficult to fully comprehend everything the first time. *It is perfectly fine* to go back and spend more time on the problem until you completely understand it. Being able to quickly analyze complex mathematical problems like this is part of the vaunted "mathematical maturity" that this class helps you foster. As the semester continues, you will find that these kinds of problems will seem progressively easier, both to understand quickly and to solve. But it won't happen without practice. You'll get some of that practice on the next homework as we build from this discussion and give you a chance to exercise/review some calculus ideas like limits (of $\Delta \to 0$) and Riemann integration in this context.

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