Discussion 2D

The following notes are useful for this discussion: Note 3 and Note 4.

1. Introduction to Inductors

An inductor is a circuit element analogous to a capacitor; its voltage is proportional to the derivative of the current across it. That is:

$$V_L(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} \tag{1}$$

When first studying capacitors, we analyzed a circuit where a current source was directly attached to a capacitor. In Figure 1, we form the counterpart circuit for an inductor:

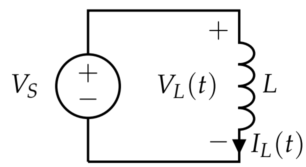


Figure 1: Inductor in series with a voltage source.

(a) **What is the current through an inductor as a function of time? If the inductance is $L = 3\,\mathrm{H}$, what is the current at $t = 6\,\mathrm{s}$?** Assume that the voltage source turns from $0\,\mathrm{V}$ to $5\,\mathrm{V}$ at time $t = 0\,\mathrm{s}$, and there's no current flowing in the circuit before the voltage source turns on, i.e $I_L(0) = 0\,\mathrm{A}$.

Solution: We proceed to analyze the given equation. Note that the voltage source is held at a constant value for $t \geq 0$, which allows us to express the derivative of current as a constant:

$$V_L(t) = L\frac{\mathrm{d}I_L}{\mathrm{d}t} \tag{2}$$

$$\frac{V_S}{L} = \frac{\mathrm{d}I_L}{\mathrm{d}t} \tag{3}$$

From here, we can see that the derivative of the current is a constant with respect to time! This immediately indicates that we have a linear relationship between current and time, with a slope set by the derivative. This means that the current in the inductor is given by

$$I_L(t) = I_L(0) + \frac{V_S}{L}t \tag{4}$$

This is exactly how we came up with the equation for the voltage across a capacitor in series with a constant current source. So, the current in the inductor keeps growing over time! Inductors store energy in their magnetic field, so the more time that this voltage source feeds the inductor, the higher the current, and the greater the stored energy.

Substituting in the specific values asked for, $I_L(6\,\mathrm{s}) = \frac{5\,\mathrm{V}}{3\,\mathrm{H}} \cdot 6\,\mathrm{s} = 10\,\mathrm{A}$.

(b) Now, we add some resistance in series with the inductor, as in Figure 2.

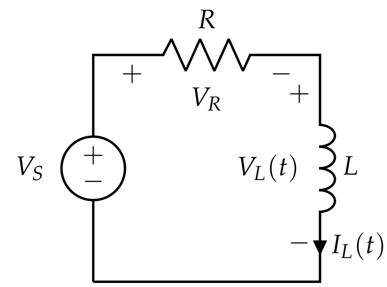


Figure 2: Inductor in series with a voltage source.

Solve for the current $I_L(t)$ and voltage $V_L(t)$ in the circuit over time, in terms of R, L, V_S, t. Note that $I_L(0) = 0\,\mathrm{A}$.

Solution: We begin by considering the voltage drop across the resistor, in terms of source voltage and inductor voltage. There's also only a single current in the circuit (the one we're solving for, $I(t)$):

$$V_R(t) = V_S - V_L(t) \tag{5}$$

$$R\,I_L(t) = V_S - L\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) = -\frac{R}{L}I_L(t) + \frac{V_S}{L} \tag{7}$$

We recognize this as a first-order differential equation! To solve this, we can employ one of two approaches:

Approach 1: Recall that the solution to

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu(t) \tag{8}$$

is

$$x(t) = x(0)\mathrm{e}^{\lambda t} + b\mathrm{e}^{\lambda t}\int_0^t \mathrm{e}^{-\lambda\tau}u(\tau)\,\mathrm{d}\tau \tag{9}$$

Pattern matching eq. (7) to eq. (8), we obtain $\lambda = -\frac{R}{L}$, $b = \frac{1}{L}$, $x(t) = I_L(t)$, $x(0) = I_L(0) = 0$, and $u(t) = V_S$, and from eq. (9) obtain

$$I_L(t) = \frac{1}{L}e^{-\frac{R}{L}t}\int_0^t \mathrm{e}^{\frac{R}{L}\tau}V_S\,\mathrm{d}\tau \tag{10}$$

$$= \frac{V_S}{R}\left(1 - \mathrm{e}^{-\frac{R}{L}t}\right) \tag{11}$$

Approach 2: First, we can rewrite eq. (7) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) = -\frac{R}{L}\left(I_L(t) - \frac{V_S}{R}\right) \tag{12}$$

We can convert this into a differential equation of the form $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t)$ with a change of variables. Here, we can choose $\widetilde{I}_L(t) = I_L(t) - \frac{V_S}{R}$, so our new differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{I}_L(t) = -\frac{R}{L}\widetilde{I}_L(t) \tag{13}$$

with initial condition $\widetilde{I}_L(0) = I_L(0) - \frac{V_S}{R} = -\frac{V_S}{R}$. Since we have an initial condition for $t = 0$, we know that the solution will be

$$\widetilde{I}_L(t) = \widetilde{I}_L(0)\mathrm{e}^{-\frac{R}{L}t} = -\frac{V_S}{R}\mathrm{e}^{-\frac{R}{L}t} \tag{14}$$

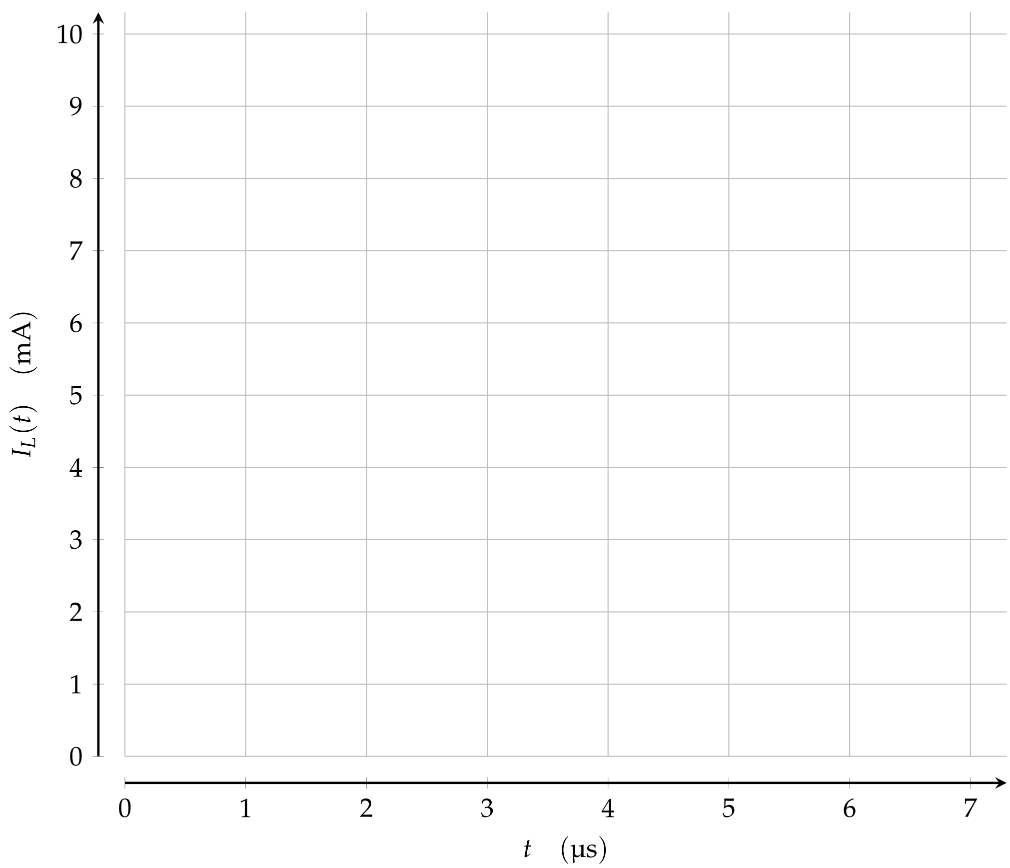
Hence,

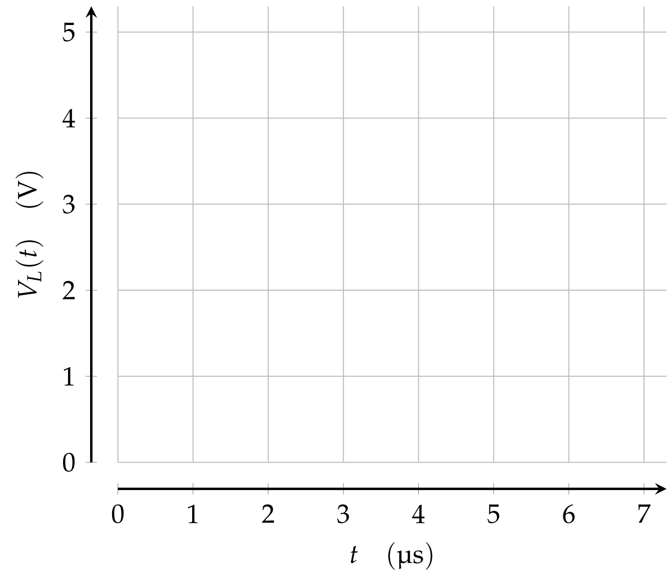
$$I_L(t) = \frac{V_S}{R}\left(1 - \mathrm{e}^{-\frac{R}{L}t}\right) \tag{15}$$

To find the voltage across the inductor, we can use eq. (1) and plug in for eq. (11) (equivalently, eq. (15)). This yields:

$$\begin{aligned} V_L(t) &= L\frac{\mathrm{d}}{\mathrm{d}t}I_L(t) && (16)\\ &= L\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{V_S}{R}\left(1 - \mathrm{e}^{-\frac{R}{L}t}\right)\right) && (17)\\ &= L\left(-\frac{V_S}{R}\right)\left(-\frac{R}{L}\right)\mathrm{e}^{-\frac{R}{L}t} && (18)\\ &= V_S\mathrm{e}^{-\frac{R}{L}t} && (19) \end{aligned}$$

(c) **Suppose** $R = 500\,\Omega, L = 1\,\mathrm{mH}, V_S = 5\,\mathrm{V}$**. Plot the current through and voltage across the inductor (**$I_L(t), V_L(t)$**), as these quantities evolve over time.**





Solution: The current begins at $0\,\mathrm{A}$ and over time, the inductor begins to look like a short. In the long-term, the current settles to $\frac{V_S}{R}\mathrm{A} = 10\,\mathrm{mA}$. The voltage begins at $V_S = 5\,\mathrm{V}$ because the inductor initially looks like an open circuit, and this voltage decreases exponentially over time down to zero.

The time constant governing both of these transient curves is $\tau = \frac{L}{R} = 2\,\mu\mathrm{s}$. Using this information, we can sketch the curves for current (Figure 3) and inductor voltage (Figure 4). Notice that it is perfectly fine for the voltage to be discontinuous, but the same is not true for the current.

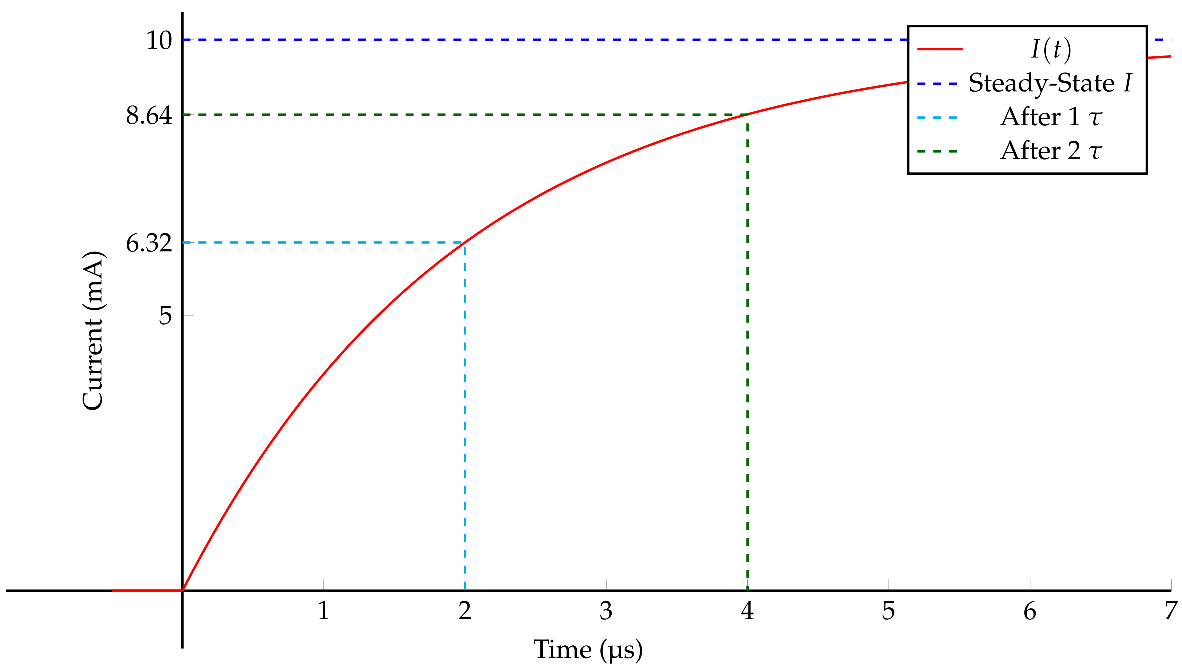


Figure 3: Transient Current in an RL circuit (with initial current $I(0) = 0\,\mathrm{A}$.)

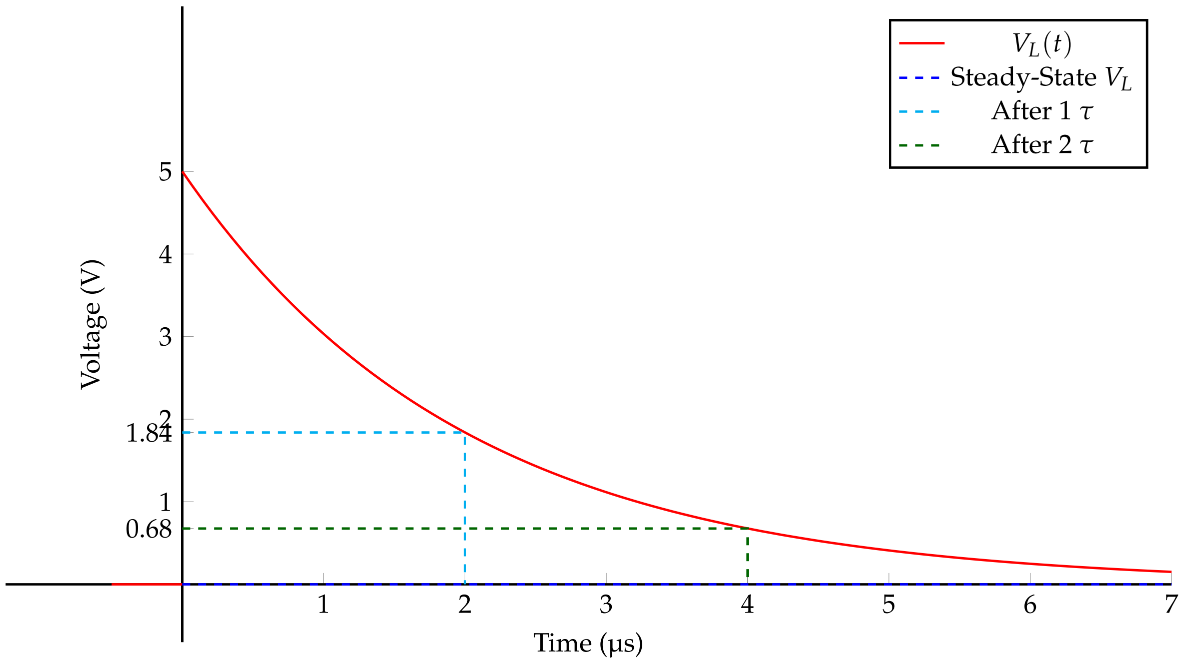


Figure 4: Transient Voltage across the inductor in an RL circuit (with initial current $I(0) = 0$ A.)

2. **Differential Equations with Complex Numbers**

   Recall the steps we take to solve a non-diagonal (coupled) system of differential equations. So far, we have dealt with matrices that have purely real eigenvalues. In this problem, we will apply the same principles to solve a system with complex eigenvalues.

   (a) Consider the following system:

   $$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \vec{z}(t) \tag{20}$$

   with the initial condition $\vec{z}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. **Solve the differential equation to find $\vec{z}(t)$.** We will have to perform a change of variables, since the system is not diagonal. It may help to recall the change of variables strategy in fig. 5. You may use the fact that $\begin{bmatrix} \mathrm{j} \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -\mathrm{j} \\ 1 \end{bmatrix}$ are eigenvectors of $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

$$\begin{tikzcd}
\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = A\vec{z}(t) \arrow[r, squiggly, "\text{Too difficult}"'] \arrow[d, "\vec{y}_\lambda(t)=V^{-1}\vec{z}(t)"'] & \vec{z}(t) = \ldots \\
\frac{\mathrm{d}}{\mathrm{d}t}\vec{y}_\lambda(t) = \underbrace{V^{-1}AV}_{\tilde{A}}\vec{y}_\lambda(t) \arrow[r, "\substack{\text{Solve a}\\ \text{diagonal system}}"'] & \vec{y}_\lambda(t) = \ldots \arrow[u, "\vec{z}(t)=V\vec{y}_\lambda(t)"']
\end{tikzcd}$$

Figure 5: Change of Variables Strategy

Solution: We can define

$$V = \begin{bmatrix} \mathrm{j} & -\mathrm{j} \\ 1 & 1 \end{bmatrix} \tag{21}$$

which we obtain by stacking the eigenvectors. We can also compute

$$V^{-1} = \begin{bmatrix} \mathrm{j} & -\mathrm{j} \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2\mathrm{j}} \begin{bmatrix} 1 & \mathrm{j} \\ -1 & \mathrm{j} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\mathrm{j} & 1 \\ \mathrm{j} & 1 \end{bmatrix} \tag{22}$$

Hence, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\vec{y}_\lambda(t) &= V^{-1}AV\vec{y}_\lambda(t) && (23)\\ &= \frac{1}{2}\begin{bmatrix} -\mathrm{j} & 1 \\ \mathrm{j} & 1 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \mathrm{j} & -\mathrm{j} \\ 1 & 1 \end{bmatrix}\vec{y}_\lambda(t) && (24)\\ &= \begin{bmatrix} 2\mathrm{j} & 0 \\ 0 & -2\mathrm{j} \end{bmatrix}\vec{y}_\lambda(t) && (25) \end{aligned}$$

Now, to find the initial condition for $\vec{y}_\lambda(t)$, we can compute

$$\vec{y}_\lambda(0) = V^{-1}\vec{z}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{26}$$

We can write this system of differential equations as

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\vec{y}_{\lambda_1}(t) = 2\mathrm{j}\vec{y}_{\lambda_1}(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\vec{y}_{\lambda_2}(t) = -2\mathrm{j}\vec{y}_{\lambda_2}(t) \end{cases} \tag{27}$$

with $\vec{y}_{\lambda_1}(0) = \vec{y}_{\lambda_2}(0) = 1$. Since we have an initial condition at $t = 0$, we know the solution will have the form

$$\begin{cases} \vec{y}_{\lambda_1}(t) = \vec{y}_{\lambda_1}(0)\mathrm{e}^{2\mathrm{j}t} = \mathrm{e}^{2\mathrm{j}t} \\ \vec{y}_{\lambda_2}(t) = \vec{y}_{\lambda_2}(0)\mathrm{e}^{-2\mathrm{j}t} = \mathrm{e}^{-2\mathrm{j}t} \end{cases} \tag{28}$$

which, written in vector form, is

$$\vec{y}_\lambda(t) = \begin{bmatrix} \mathrm{e}^{2\mathrm{j}t} \\ \mathrm{e}^{-2\mathrm{j}t} \end{bmatrix} \tag{29}$$

Undoing the change of variables to find $\vec{z}(t)$, we get

$$\vec{z}(t) = V\vec{y}_\lambda(t) = \begin{bmatrix} \mathrm{j}\mathrm{e}^{2\mathrm{j}t} - \mathrm{j}\mathrm{e}^{-2\mathrm{j}t} \\ \mathrm{e}^{2\mathrm{j}t} + \mathrm{e}^{-2\mathrm{j}t} \end{bmatrix} \tag{30}$$

We can simplify $\mathrm{j}\mathrm{e}^{2\mathrm{j}t} - \mathrm{j}\mathrm{e}^{-2\mathrm{j}t}$ as follows:

$$\begin{aligned} \mathrm{j}\mathrm{e}^{2\mathrm{j}t} - \mathrm{j}\mathrm{e}^{-2\mathrm{j}t} &= -\frac{\mathrm{e}^{2\mathrm{j}t} - \mathrm{e}^{-2\mathrm{j}t}}{\mathrm{j}} && (31)\\ &= -2\frac{\mathrm{e}^{2\mathrm{j}t} - \mathrm{e}^{-2\mathrm{j}t}}{2\mathrm{j}} && (32)\\ &= -2\sin(2t) && (33) \end{aligned}$$

Next, we can simplify $e^{2jt} + e^{-2jt}$ as follows:

$$e^{2jt} + e^{-2jt} = 2\frac{e^{2jt} + e^{-2jt}}{2} \tag{34}$$
$$= 2\cos(2t) \tag{35}$$

Thus,

$$\vec{z}(t) = \begin{bmatrix} -2\sin(2t) \\ 2\cos(2t) \end{bmatrix} \tag{36}$$

(b) Notice that the solution in part 2.a is purely real, even though we had complex eigenvalues in our system. Now, we will investigate why this is the case. Indeed, we can define

$$\vec{z}(t) = \begin{bmatrix} a & \overline{a} \\ b & \overline{b} \end{bmatrix} \vec{y}(t) \tag{37}$$

where $\vec{y}(t) = \begin{bmatrix} c_0 e^{\lambda t} \\ \overline{c_0} e^{\overline{\lambda} t} \end{bmatrix}$ and $a, b, c_0, \lambda \in \mathbb{C}$ are arbitrary (nonzero) constants. **Show $\vec{z}(t)$ will be purely real.** *HINT: First show $x + \overline{x}$ is real for $x \in \mathbb{C}$. Also, recall $\overline{(x \cdot y)} = \overline{x} \cdot \overline{y}$ for $x, y \in \mathbb{C}$.* **Use this result to conclude that the solution in part 2.a is real.**

Solution: We can write $\vec{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$. Notice that

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} a & \overline{a} \\ b & \overline{b} \end{bmatrix} \begin{bmatrix} c_0 e^{\lambda t} \\ \overline{c_0} e^{\overline{\lambda} t} \end{bmatrix} = \begin{bmatrix} a c_0 e^{\lambda t} + \overline{a c_0} e^{\overline{\lambda} t} \\ b c_0 e^{\lambda t} + \overline{b c_0} e^{\overline{\lambda} t} \end{bmatrix} \tag{38}$$

Approach 1:

Proof. In this approach, we will show that $x + \overline{x}$ is purely real, for $x \in \mathbb{C}$. Write $x = x_r + jx_i$ where $x_r, x_i \in \mathbb{R}$. This means $\overline{x} = x_r - jx_i$. Hence, $x + \overline{x} = x_r + jx_i + x_r - jx_i = 2x_r \in \mathbb{R}$.
Now, notice that $\overline{a c_0 e^{\lambda t}} = \overline{a c_0}\, \overline{e^{\lambda t}}$. Next, notice that $\overline{e^{\lambda t}} = e^{\overline{\lambda} t}$. Thus, we have $z_1(t) = x + \overline{x}$ where $x = a c_0 e^{\lambda t}$, which means $z_1(t)$ is real. We see the same for $z_2(t)$, where we now set $x = b c_0 e^{\lambda t}$. Hence, $z_2(t)$ is real and $\vec{z}(t)$ is real. □

Approach 2:

Proof. In this approach, we will not explicitly use the fact that $z + \overline{z}$ is purely real. Firstly, let $ac_0 = A = A_r + jA_i$, $bc_0 = B = B_r + jB_i$, and $\lambda = \lambda_r + j\lambda_i$, where $A_r = \text{Re}(A)$ (similarly for B_r and λ_r) and $A_i = \text{Im}(A)$ (similarly for B_i and λ_i). Then, recall the following identities:

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{39}$$
$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{40}$$
$$-j = \frac{1}{j} \tag{41}$$

We notice that $z_1(t)$ and $z_2(t)$ appear very similar (except for the arbitrary constant terms), so we will show that $z_1(t)$ is purely real and pattern match to show $z_2(t)$ is also real. Starting with $z_1(t)$:

$$\begin{aligned}
z_1(t) &= ac_0 e^{\lambda t} + \overline{ac_0} e^{\overline{\lambda} t} && (42)\\
&= A e^{\lambda t} + \overline{A} e^{\overline{\lambda} t} && (43)\\
&= (A_r + jA_i) e^{\lambda_r t + j\lambda_i t} + (A_r - jA_i) e^{\lambda_r t - j\lambda_i t} && (44)\\
&= e^{\lambda_r t}\left(A_r e^{j\lambda_i t} + A_r e^{-j\lambda_i t}\right) + e^{\lambda_r t}\left(jA_i e^{j\lambda_i t} - jA_i e^{-j\lambda_i t}\right) && (45)\\
&\underset{\substack{\uparrow\\ \text{eq. (41)}}}{=} 2A_r e^{\lambda_r t}\frac{e^{j\lambda_i t} + e^{j\lambda_i t}}{2} - A_i e^{\lambda_r t}\frac{e^{j\lambda_i t} - e^{-j\lambda_i t}}{j} && (46)\\
&\underset{\substack{\uparrow\\ \text{eq. (40)}}}{=} 2A_r \cos(\lambda_i t) - 2A_i e^{\lambda_r t}\frac{e^{j\lambda_i t} - e^{-j\lambda_i t}}{2j} && (47)\\
&\underset{\substack{\uparrow\\ \text{eq. (39)}}}{=} 2A_r \cos(\lambda_i t) - 2A_i e^{\lambda_r t}\sin(\lambda_i t) && (48)
\end{aligned}$$

which is real-valued. For $z_2(t)$, we can follow the exact same steps above, replacing A with B, A_r with B_r, A_i with B_i, and a with b. The final result we see is

$$z_2(t) = 2B_r e^{\lambda_r t}\cos(\lambda_i t) - 2B_i e^{\lambda_r t}\sin(\lambda_i t) \tag{49}$$

which is real-valued. Thus, $\vec{z}(t)$ is real. □

We can pattern match $\vec{y}(t)$ with $\vec{y}_\lambda(t)$ from part **2.**a and the $\vec{z}(t)$'s. Recall that $V = \begin{bmatrix} j & -j \\ 1 & 1 \end{bmatrix}$ which matches the form $\begin{bmatrix} a & \overline{a} \\ b & \overline{b} \end{bmatrix}$ with $a = j$ and $b = 1$. Hence, the solution from part **2.**a is real.

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