Discussion 3D

The following notes are useful for this discussion: Note 7 on Transfer Function Plots and Note 8 on Bode Plots.

1. **Plotting and Combining Transfer Functions**

   Recall that any transfer function can be written in polar form as

   $$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)} \tag{1}$$

   where $|H(j\omega)|$ and $\angle H(j\omega)$ are real functions of ω giving the magnitude and phase of the transfer function, respectively. To see how transfer functions combine, consider two $H_1(j\omega)$ and $H_2(j\omega)$.

   $$H_1(j\omega) = |H_1(j\omega)|e^{j\angle H_1(j\omega)} \tag{2}$$

   $$H_2(j\omega) = |H_2(j\omega)|e^{j\angle H_2(j\omega)} \tag{3}$$

   $$H_1(j\omega) \cdot H_2(j\omega) = |H_1|e^{j\angle H_1}|H_2|e^{j\angle H_2} = |H_1||H_2|e^{j(\angle H_1 + \angle H_2)} \tag{4}$$

   $$\frac{H_1(j\omega)}{H_2(j\omega)} = \frac{|H_1|e^{j\angle H_1}}{|H_2|e^{j\angle H_2}} = \frac{|H_1|}{|H_2|}e^{j(\angle H_1 - \angle H_2)} \tag{5}$$

   As you can see, magnitudes of transfer functions multiply and divide while the phases add and subtract.

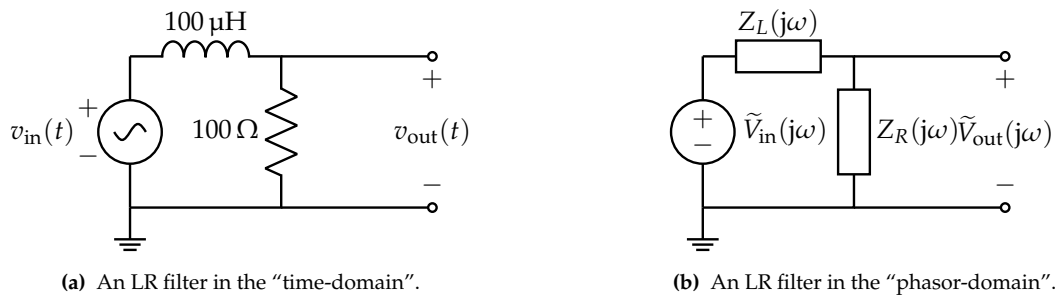   In this problem we will plot the transfer function of fig. 1a.

   (a) An LR filter in the "time-domain".   (b) An LR filter in the "phasor-domain".

   Figure 1: Circuit schematic of LR filter in both domains.

   (a) First, **solve for** $H(j\omega)$. Then, **write expressions for** $|H(j\omega)|$ **and** $\angle H(j\omega)$. For now, you can keep it in terms of R and L.

(b) **What is the cutoff frequency for this circuit? Mark it on the log-log plots of part item 1.c with a vertical line**. Note that the values of the circuit elements are given in fig. 3a.

Recall that a transfer function of the form $H(\mathrm{j}\omega) = \frac{k}{1+\frac{\mathrm{j}\omega}{\omega_c}}$ is defined to have a cutoff frequency of ω_c. This is because a signal with an input frequency of ω_c exactly will have its magnitude attenuated by a factor of $\frac{1}{\sqrt{2}}$, which is a conventionally convenient number.

(c) **Sketch plots of the magnitude and phase of this transfer function**. We have provided a table with the transfer function evaluated at a few representative points around the cutoff frequency to help you plot the transfer function by hand. You can join these points with a curve to arrive at a reasonable estimation of the transfer function.

ω	10^4	10^5	10^6	10^7	10^8
$\lvert H(\mathrm{j}\omega)\rvert$	1.00	0.995	0.707	0.100	0.01
$\angle H(\mathrm{j}\omega)$	$-0.6°$	$-6°$	$-45°$	$-84°$	$-89°$
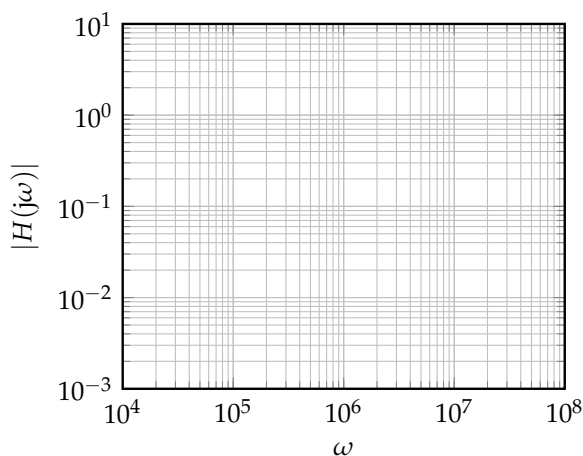
Plot of $\lvert H(\mathrm{j}\omega)\rvert$ (for you to draw).   Plot of $\angle H(\mathrm{j}\omega)$ (for you to draw).
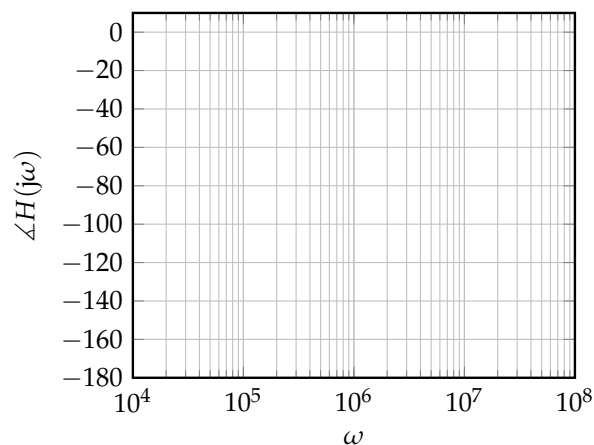
(d) Now suppose we want to compose the filter from fig. 3a with the filter from earlier (fig. 1a). You may recognize the first filter from the previous discussion. Use $R = 1\,\text{k}\Omega$ and $C = 1\,\mu\text{F}$ for the RC filter. We can compose two circuits by connecting the output of the first circuit into the second circuit, through a unity gain buffer. For this problem, the transfer function of the LR filter from this worksheet fig. 1a is H_1, and the transfer function of the other RC filter is H_2. The transfer function of the composed circuit is:

$$H(\mathrm{j}\omega) = H_1(\mathrm{j}\omega) \cdot H_2(\mathrm{j}\omega) \tag{6}$$
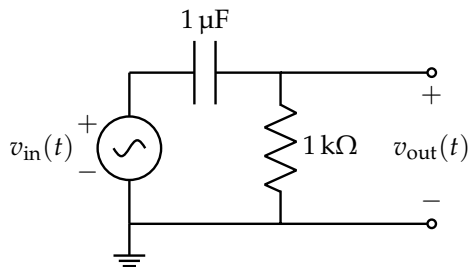
(a) An RC high-pass filter in the "time-domain".
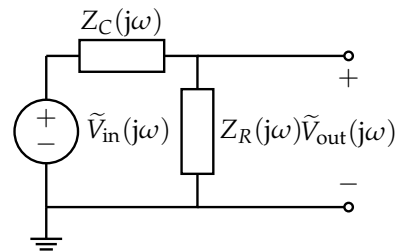
(b) An RC high-pass filter in the "phasor-domain".

i. **Draw this circuit**.

ii. **Plot the magnitude of the composed circuit**. In fig. 4 is a log-log plot with the magnitudes of $|H_1(\mathrm{j}\omega)|$ and $|H_2(\mathrm{j}\omega)|$ drawn to assist you.

iii. **Plot the phase of the composed circuit**. In fig. 5 is a semi-log plot with the phases $\angle H_1(\mathrm{j}\omega)$ and $\angle H_2(\mathrm{j}\omega)$ drawn to assist you.
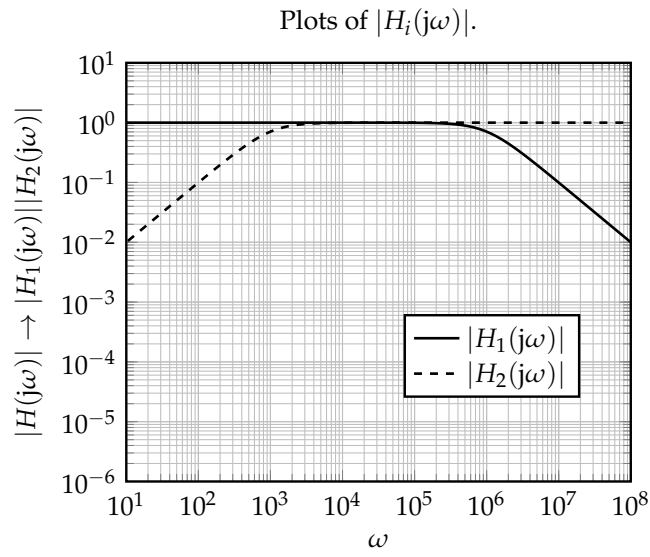
Plots of $|H_i(j\omega)|$.

Figure 4: Log-log plot template for magnitude.
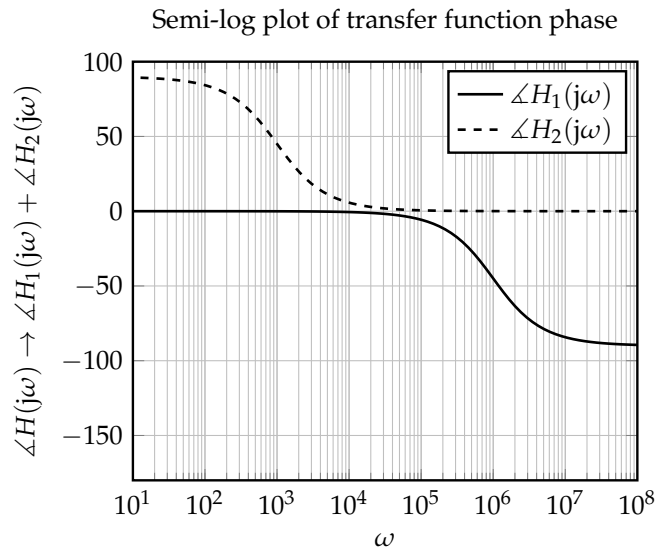
Semi-log plot of transfer function phase

Figure 5: Plot template for phase.

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2. Bode Plots (straight-line approximations) and filters

Our eventual goal is to construct Bode plots of the following circuit, with $L = 100\,\mu\text{H}$, $C = 1\,\mu\text{F}$, $R_1 = 100\,\Omega$, and $R_2 = 1\,\text{k}\Omega$:
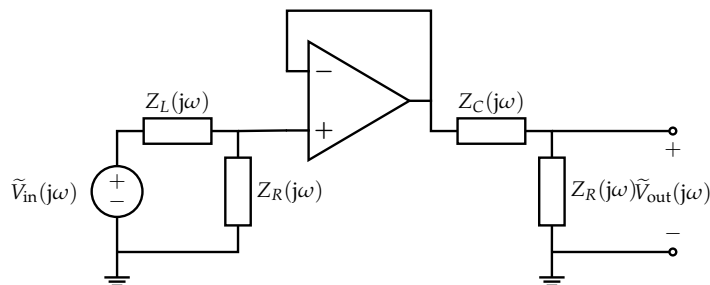

Figure 6

To do this we will leverage the fact that Bode plots can be composed in systematic ways.

Before we dive into the problem, let's consider a modification of the *magnitude* plot that will help us work with multiple magnitude plots at once. Namely, instead of plotting $|H(j\omega)|$ vs. ω where the y-axis is on a *logarithmic* scale, we plot $20\log_{10}(|H(j\omega)|)$ vs. ω instead, and now the y-axis is on a *linear* scale.

Why would we want to do this? Well, when combining magnitude transfer functions, we end up multiplying them. But we really want to add two plots *graphically* for simplicity, not multiply them, so we will just plot and add the logarithms. (The constant multiple 20 is there for convention reasons, related to decibels.)
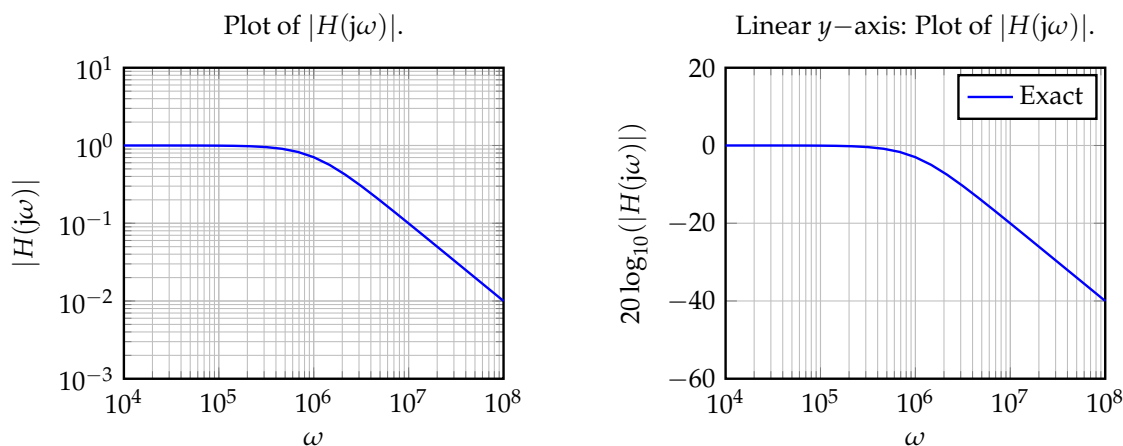
Notice that we do not need to do this for the *phase* plots, since their y axes are naturally in linear scale, and combining plots can already be done by addition. Now we are ready to begin working on the problems.
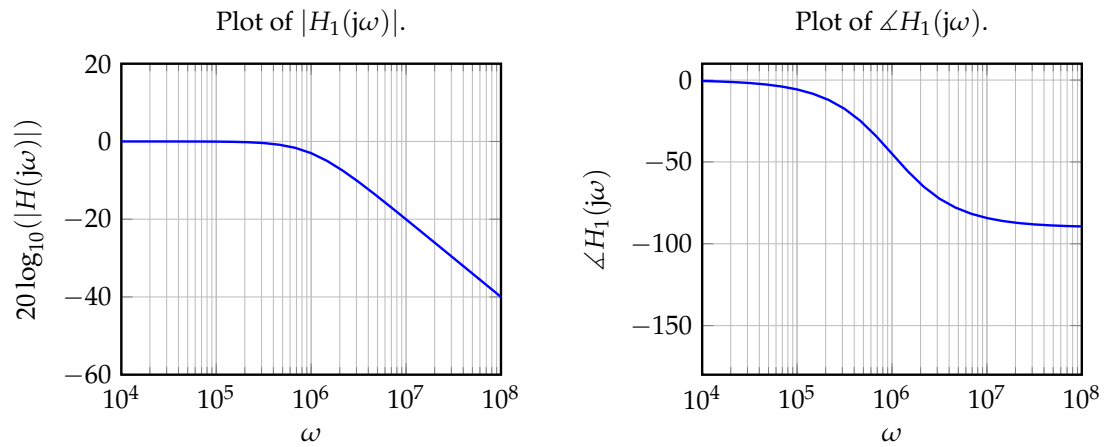
(a) Consider the first half of this circuit:
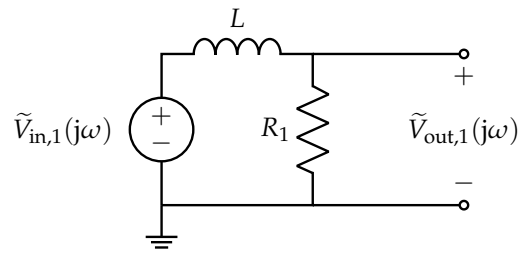
We learned in the previous discussion that the transfer function is given by

$$H_1(j\omega) = \frac{\widetilde{V}_{\text{out},1}}{\widetilde{V}_{\text{in},1}} = \frac{1}{1 + j\omega\frac{L}{R_1}} \tag{7}$$

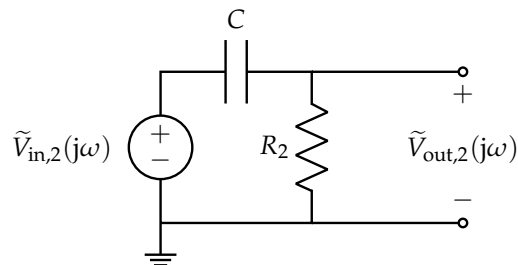and the cutoff frequency $\omega_{c,1}$ is given by

$$\omega_{c,1} = \frac{R_1}{L} = \frac{100\,\Omega}{100\,\mu\text{H}} = 1 \times 10^6\,\frac{\text{rad}}{\text{s}} \tag{8}$$

and plots of the transfer function are given by

Plot of $|H_1(j\omega)|$.

Plot of $\angle H_1(j\omega)$.

On these grids, **draw the Bode plots (piecewise linear approximations) for magnitude and phase**.
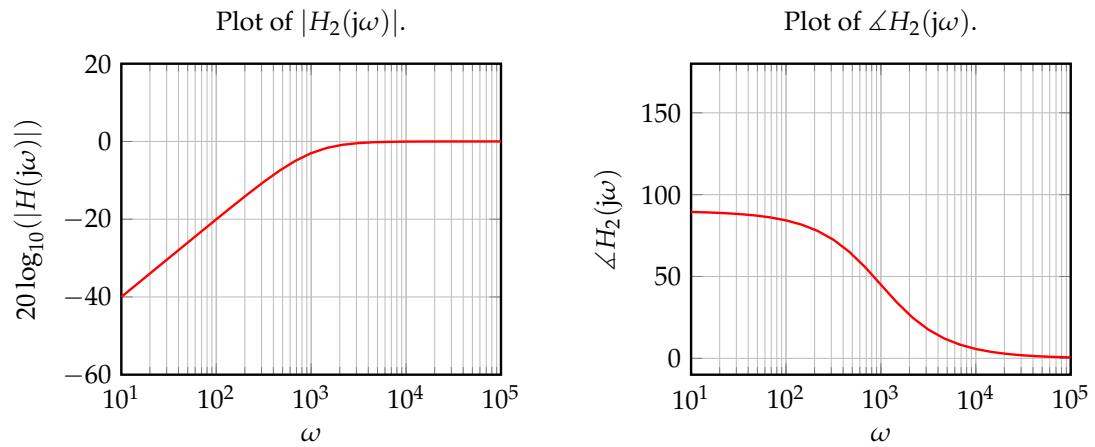
(b) Consider the second half of the circuit:

We learned in the previous discussion that the transfer function is given by

$$H_2(j\omega) = \frac{\widetilde{V}_{\text{out},2}}{\widetilde{V}_{\text{in},2}} = \frac{j\omega R_2 C}{1 + j\omega R_2 C} \tag{9}$$

and the cutoff frequency $\omega_{c,2}$ is given by

$$\omega_{c,2} = \frac{1}{R_2 C} = \frac{1}{(1\,\text{k}\Omega) \cdot (1\,\mu\text{F})} = 1 \times 10^3 \, \frac{\text{rad}}{\text{s}} \tag{10}$$

and plots of the transfer function are given by

Plot of $|H_2(\mathrm{j}\omega)|$.
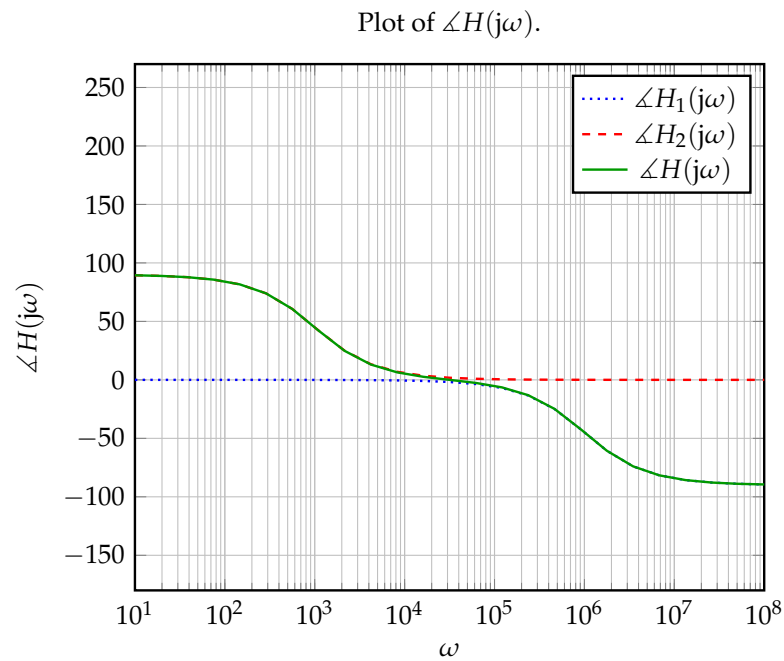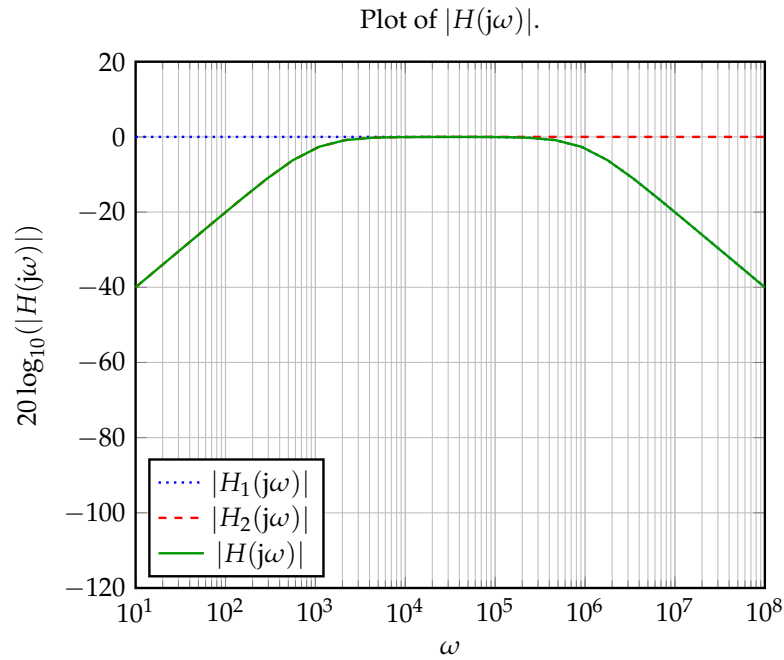
Plot of $\angle H_2(\mathrm{j}\omega)$.

On these grids, **draw the Bode plots (piecewise linear approximations) for magnitude and phase**.

(c) Now, we will put this circuit together. Recall the diagram in fig. :

We saw earlier in the discussion that the transfer function is

$$H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = H_1(\mathrm{j}\omega) H_2(\mathrm{j}\omega) \tag{11}$$

and the transfer function plots are given by

Plot of $|H(j\omega)|$.

Plot of $\angle H(j\omega)$.

Note that the green (solid) line overlaps parts of the red (dashed) and blue (dotted) lines. (On these grids, **draw the Bode plots (piecewise linear approximations) for magnitude and phase**.

Hint: Recall that

$$20\log_{10}(|H(j\omega)|) = 20\log_{10}(|H_1(j\omega)H_2(j\omega)|) = 20\log_{10}(|H_1(j\omega)||H_2(j\omega)|) \tag{12}$$

$$= 20\log_{10}(|H_1(j\omega)|) + 20\log_{10}(|H_2(\omega)|) \tag{13}$$

$$\angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega). \tag{14}$$

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