1. Translating System of Differential Equations from Continuous Time to Discrete Time

Oftentimes, we wish to apply controls model on a computer. However, modeling a continuous time system on a computer is a nontrivial problem. Hence, we turn to discretizing our controls problem. That is, we define a discretized state $\vec{x}_d[i]$ and a discretized input $\vec{u}_d[i]$ that we “sample” every $\Delta$ seconds. The notion of discretization is very similar to the approach covered in Discussion 2A.

(a) Consider the scalar system below:

$$\frac{dx(t)}{dt} = \lambda x(t) + bu(t). \tag{1}$$

where $x(t)$ is our state and $u(t)$ is our control input. Let $\lambda \neq 0$ be an arbitrary constant. Further suppose that our input $u(t)$ is piecewise constant, and that $x(t)$ is differentiable everywhere (and thus, continuous everywhere). That is, we define an interval $t \in [i\Delta, (i+1)\Delta]$ such that $u(t)$ is constant over this interval. Mathematically, we write this as

$$u(t) = u(i\Delta) = u_d[i] \text{ if } t \in [i\Delta, (i+1)\Delta]. \tag{2}$$

The now-discretized input $u_d[i]$ is the same as the original input where we only “observe” a change in $u(t)$ every $\Delta$ seconds. Similarly, for $x(t)$,

$$x(t) = x(i\Delta) = x_d[i] \tag{3}$$

Let’s revisit the solution for eq. (1), when we’re given the initial conditions at $t_0$, i.e we know the value of $x(t_0)$ and want to solve for $x(t)$ at any time $t \geq t_0$:

$$x(t) = e^{\lambda(t-t_0)}x(t_0) + b \int_{t_0}^{t} u(\theta) e^{\lambda(t-\theta)} \, d\theta \tag{4}$$

**Given that we start at $t = i\Delta$, where $x(t) = x_d[i]$ is known, and satisfy eq. (1), where do we end up at $x_d[i+1]$? (HINT): Think about the initial condition here. Where does our solution “start”?**

**Solution:** For $t \in [i\Delta, (i+1)\Delta)$, the differential equation takes the form

$$\frac{dx(t)}{dt} = \lambda x(t) + bu(t) = \lambda x(t) + bu_d[i] \tag{5}$$

where we choose our initial condition to be $x(i\Delta) = x_d[i]$, since this is a known quantity. We can solve this equation for $x(t)$ using the integral equation from eq. (4) and the fact that $u_d[i]$ is a constant value over this interval. In particular, we get the following form

$$x(t) = e^{\lambda(t-i\Delta)} \underbrace{x(i\Delta)}_{x_d[i]} + b \int_{i\Delta}^{t} \underbrace{u(i\Delta)}_{u_d[i]} e^{\lambda(t-\theta)} \, d\theta \tag{6}$$

$$= e^{\lambda(t-i\Delta)} x_d[i] + bu_d[i] \int_{i\Delta}^{t} e^{\lambda(t-\theta)} \, d\theta \tag{7}$$
Plugging in the timestep of interest, we set $t = (i + 1)\Delta$, to evaluate $x_d[i + 1]$ as

$$x_d[i + 1] = x((i + 1)\Delta)$$

$$= e^{\Lambda \Delta} x_d[i] + bu_d[i] \int_{i\Delta}^{(i+1)\Delta} e^{\lambda((i+1)\Delta - \theta)} \, d\theta$$

$$= e^{\Lambda \Delta} x_d[i] + bu_d[i] \frac{e^{\lambda \Delta} - e^{0}}{\lambda}$$

$$= e^{\Lambda \Delta} x_d[i] + bu_d[i] \frac{e^{\lambda \Delta} - 1}{\lambda}$$

which gives us the solution for $x_d[i + 1]$.

(b) Suppose we now have a continuous-time system of differential equations, that forms a vector differential equation. We express this with an input in vector form:

$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + \vec{b}u(t)$$

where $\vec{x}(t)$ is $n$-dimensional. Suppose further that the matrix $A$ has distinct and non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. We collect the eigenvectors together and form the matrix $V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$.

We now wish to find a matrix $A_d$ and a vector $\vec{b}_d$ such that

$$\vec{x}_d[i + 1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i]$$

where $\vec{x}_d[i] = \vec{x}(i\Delta)$.

Firstly, define terms

$$e^{\Lambda \Delta} = \begin{bmatrix} e^{\lambda_1 \Delta} & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & e^{\lambda_n \Delta} \end{bmatrix}$$

$$\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \frac{1}{\lambda_n} \end{bmatrix}$$

$$\vec{\tilde{u}}_d[i] = V^{-1} \vec{b}_d u_d[i]$$

Note that the term $e^{\Lambda \Delta}$ is just a label for our intents and purposes — this is not the same as applying $e^x$ to every element in the matrix $\Lambda$.

**Complete the following steps to derive a discretized system:**

i. Diagonalize the continuous time system using a change of variables (change of basis) to achieve a new system for $\vec{y}(t)$.

ii. Solve the diagonalized system. Remember that we only want a solution over the interval $t \in [i\Delta, (i + 1)\Delta)$. Use the value at $t = i\Delta$ as your initial condition.
iii. Discretize the diagonalized system to obtain $\vec{y}_d[i]$. Show that

$$\vec{y}_d[i + 1] = e^{\lambda_i \Delta} \vec{y}_d[i] + \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e^{\lambda_n \Delta} \end{bmatrix} \vec{u}_d[i]$$

Then, show that the matrix

$$\begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e^{\lambda_n \Delta - 1} \lambda_n \end{bmatrix}$$

can be compactly written as $\Lambda^{-1}(e^{\Lambda \Delta} - I)$.

iv. Undo the change of variables on the discretized diagonal system to get the discretized solution of the original system.

Solution:

i. First, following the hint, we notice that with a full set of distinct eigenvalues and corresponding eigenvectors, we can change coordinates so that $\vec{x}(t) = V\vec{y}(t)$ and $\vec{y}(t) = V^{-1}\vec{x}(t)$.

Using this transformation we diagonalize the system of differential equations, i.e

$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + \vec{b}u(t)$$

$$\implies \frac{dV\vec{y}(t)}{dt} = AV\vec{y}(t) + \vec{b}u(t)$$

$$\therefore \frac{d\vec{y}(t)}{dt} = V^{-1}AV\vec{y}(t) + V^{-1}\vec{b}u(t)$$

Note that using the basis of eigenvectors $V$, we’ve diagonalized $A$ to get $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots \\ 0 & \ldots & \ldots & \lambda_n \end{bmatrix}$

$$\therefore \frac{d\vec{y}(t)}{dt} = \Lambda\vec{y}(t) + V^{-1}\vec{b}u(t)$$

ii. Now, we can use the fact that we care about the solution for $\vec{y}(t)$ over the interval $t \in (i\Delta, (i + 1)\Delta]$, so $u(t)$ is a constant. Thus, we can write eq. (21) as follows:

$$\frac{d\vec{y}(t)}{dt} = \Lambda\vec{y}(t) + V^{-1}\vec{b}\overline{u}_d[i]$$

Notice that this system is diagonal (and hence we can write it as a system of $n$ differential equations). We can look at the $k$th differential equation. We will use the subscripting notation $(\vec{y}(t))_k$ and $\left(\overline{u}_d[i]\right)_k$ to denote the $k$th element of $\vec{y}(t)$ and $\overline{u}_d[i]$ respectively:

$$\frac{d(\vec{y}(t))_k}{dt} = \lambda_k(\vec{y}(t))_k + \left(\overline{u}_d[i]\right)_k$$
We can pattern match to the solution in eq. (7), setting \( \lambda \rightarrow \lambda_k, u_d[i] \rightarrow \left( \vec{u}_d[i] \right)_k, b \rightarrow 1, \) and \( x(t) \rightarrow (\vec{y}(t))_k \), to get

\[
(\vec{y}(t))_k = e^{\lambda_k(t-i\Delta)}(\vec{y}(i\Delta))_k + (\vec{u}_d[i])_k \int_{i\Delta}^t e^{\lambda_k(t-\theta)} \, d\theta 
\]  
(24)

for \( t \in (i\Delta, (i + 1)\Delta) \).

iii. Now, we want to find \( (\vec{y}_d[i+1])_k = (\vec{y}((i+1)\Delta))_k \), so we can plug in for \( t = (i + 1)\Delta \) in eq. (24) and we will get

\[
(\vec{y}_d[i + 1])_k = (\vec{y}((i+1)\Delta))_k = e^{\lambda_k\Delta}(\vec{y}(i\Delta))_k + \left( \frac{e^{\lambda_k\Delta} - 1}{\lambda_k} \right) (\vec{u}_d[i])_k 
\]  
(25)

Since we have a solution for the \( k \)th differential equation in the system, we can arrange all the differential equations in this matrix form as follows:

\[
\begin{bmatrix}
\vec{y}((i+1)\Delta) \\
\vec{y}_d[i+1]
\end{bmatrix} =
\begin{bmatrix}
e^{\lambda_1\Delta} & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & e^{\lambda_n\Delta} \\
e^{\lambda_1\Delta} & 0 & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
\vec{y}(i\Delta) \\
\vec{y}_d[i]
\end{bmatrix} +
\begin{bmatrix}
e^{\lambda_1\Delta} & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & e^{\lambda_n\Delta} \\
e^{\lambda_1\Delta} & 0 & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
e\lambda_{i+1} \\
\vec{u}_d[i]
\end{bmatrix} 
\]  
(26)

Using the notation in the hint, we can write the second matrix in eq. (26) as:

\[
\begin{bmatrix}
e^{\lambda_1\Delta} & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & e^{\lambda_n\Delta} \\
e^{\lambda_1\Delta} & 0 & \ldots & 0
\end{bmatrix} =
\begin{bmatrix}
\frac{e^{\lambda_1\Delta} - 1}{\lambda_1} & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & \frac{e^{\lambda_n\Delta} - 1}{\lambda_n} \\
\frac{e^{\lambda_1\Delta} - 1}{\lambda_1} & 0 & \ldots & 0
\end{bmatrix}
\]  
(27)

\[
= \Lambda^{-1}e^{\lambda\Delta} - \Lambda^{-1}I 
\]  
(28)

This gives us

\[
\vec{y}_d[i+1] = \vec{y}((i+1)\Delta) = e^{\lambda\Delta} \vec{y}(i\Delta) + \Lambda^{-1} \left( e^{\lambda\Delta} - I \right) \vec{u}_d[i] 
\]  
(31)

---

1In a matrix product, if both matrices are diagonal, the product is commutative.
iv. Recall that \( \bar{x}(t) = V \bar{y}(t) \) so \( \bar{x}_d[i] = \bar{x}(i\Delta) = V \bar{y}_d[i] \), and likewise, \( \bar{y}_d[i] = V^{-1} \bar{x}_d[i] \). Using this form in the simplification, we find that:

\[
\bar{x}_d[i + 1] = V \bar{y}_d[i + 1]
\]

\[
= V \left( e^{\Delta\Lambda} \bar{y}_d[i] + \Lambda^{-1} \left( e^{\Delta\Lambda} - I \right) \bar{u}_d[i] \right)
\]

\[
= \left( Ve^{\Delta\Lambda} V^{-1} \right) \bar{x}_d[i] + \left( V \Lambda^{-1} \left( e^{\Delta\Lambda} - I \right) \right) \bar{u}_d[i]
\]

Now, recall that our original goal was to write out \( A_d \) and \( \bar{b}_d \), and we can do that now with our expression. Re-substituting \( \bar{u}_d[i] = V^{-1} \bar{b}_u_d[i] \) we have:

\[
\bar{x}_d[i + 1] = \left( Ve^{\Delta\Lambda} V^{-1} \right) \bar{x}_d[i] + \left( V \Lambda^{-1} \left( e^{\Delta\Lambda} - I \right) \right) V^{-1} \bar{b}_u_d[i]
\]

\[
= \left( Ve^{\Delta\Lambda} V^{-1} \right) \bar{x}_d[i] + \left( V \Lambda^{-1} \left( e^{\Delta\Lambda} - I \right) V^{-1} \bar{b}_u_d \right) u_d[i]
\]

(c) Consider the discrete-time system

\[
\bar{x}_d[i + 1] = A_d \bar{x}_d[i] + \bar{b}_d u_d[i]
\]

Suppose that \( \bar{x}_d[0] = \bar{x}_0 \). Unroll the implicit recursion and show that the solution follows the form in eq. (38).

\[
\bar{x}_d[i] = A_d^i \bar{x}_d[0] + \left( \sum_{j=0}^{i-1} u_d[j] A_d^{i-1-j} \right) \bar{b}_d
\]

You may want to verify that this guess works by checking the form of \( \bar{x}_d[i + 1] \). You don’t need to worry about what \( A_d \) and \( \bar{b}_d \) actually are in terms of the original parameters.

(Hint: If we have a scalar difference equation, how would you solve the recurrence? Try writing \( \bar{x}_d[i] \) in terms of \( \bar{x}_d[0] \) for \( i = 1, 2, 3 \) and look for a pattern.)

**Solution**: Here, we derive the unrolled recursion and make a guess at the form of the solution in summation notation. Let’s look at the pattern starting with \( \bar{x}_d[1] \), given that \( \bar{x}_d[i + 1] = A_d \bar{x}_d[i] + \bar{b}_d u_d[i] \).

\[
\bar{x}_d[1] = A_d \bar{x}_d[0] + \bar{b}_d u_d[0]
\]

\[
\bar{x}_d[2] = A_d \bar{x}_d[1] + \bar{b}_d u_d[1]
\]

\[
= A_d (A_d \bar{x}_d[0] + \bar{b}_d u_d[0]) + \bar{b}_d u_d[1]
\]

\[
= A_d^2 \bar{x}_d[0] + A_d \bar{b}_d u_d[0] + \bar{b}_d u_d[1]
\]

\[
\bar{x}_d[3] = A_d \bar{x}_d[2] + \bar{b}_d u_d[2]
\]

\[
= A_d \left( A_d^2 \bar{x}_d[0] + A_d \bar{b}_d u_d[0] + \bar{b}_d u_d[1] \right) + \bar{b}_d u_d[2]
\]

\[
= A_d^3 \bar{x}_d[0] + A_d^2 \bar{b}_d u_d[0] + A_d \bar{b}_d u_d[1] + \bar{b}_d u_d[2]
\]

So, given this pattern, if we guess:

\[
\bar{x}_d[i] = A_d^i \bar{x}_d[0] + \left( \sum_{j=0}^{i-1} u_d[j] A_d^{i-1-j} \right) \bar{b}_d
\]
Then, let’s see what we get for $\bar{x}_{d}[i+1]$, and make sure our guess is correct:

$$\bar{x}_{d}[i+1] = A_d \bar{x}_{d}[i] + \bar{b}_d u_d[i]$$

(47)

$$= A_d \left( A_d^i \bar{x}_{d}[0] + \left( \sum_{j=0}^{i-1} u_d[j] A_d^{i-1-j} \right) \bar{b}_d \right) + \bar{b}_d u_d[i]$$

(48)

$$= A_d^{i+1} \bar{x}_{d}[0] + \left( \left( \sum_{j=0}^{i-1} u_d[j] A_d^{i-1-j} \right) + u_d[i] \right) \bar{b}_d$$

(49)

$$= A_d^{i+1} \bar{x}_{d}[0] + \left( \sum_{j=0}^{i} u_d[j] A_d^{i-j} \right) \bar{b}_d$$

(50)

This satisfies (46), for $i+1$ and hence our guess was correct!

**Contributors:**
- Anish Muthali.
- Neelesh Ramachandran.
- Druv Pai.
- Anant Sahai.
- Nikhil Shinde.
- Sanjit Batra.
- Aditya Arun.
- Kuan-Yun Lee.
- Kumar Krishna Agrawal.