Discussion 4D

The following notes are useful for this discussion: Note 10, Note 11, Note 12

1. Eigenvalue Placement in Discrete Time

Recall that, for a discrete linear system to be stable, we require that all of the eigenvalues of A in $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$ must have magnitude less than 1.

Consider the following linear discrete time system

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] + \vec{w}[i] \tag{1}$$

(a) **Is the system given in eq. (1) stable?**

(b) We can attempt to stabilize the system by implementing closed loop feedback. That is, we choose our input $u[i]$ so that the system is stable. **If we were to use state feedback as in eq. (2), what is an equivalent representation for this system? Write your answer as $\vec{x}[i+1] = A_{\text{CL}}\vec{x}[i]$ for some matrix A_{CL}.**

$$u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] \tag{2}$$

HINT: If you're having trouble parsing the expression for $u[i]$, note that $\begin{bmatrix} f_1 & f_2 \end{bmatrix}$ is a row vector, while $\vec{x}[i]$ is column vector. What happens when we multiply a row vector with a column vector like this?)

(c) **Find the appropriate state feedback constants, f_1, f_2, that place the eigenvalues of the state space representation matrix at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.**

(d) **Is the system now stable in closed-loop, using the control feedback coefficients f_1, f_2 that we derived above?**

(e) Suppose that instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i]$ in eq. (1), we had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]$ as the way that the discrete-time control acted on the system. In other words, the system is as given in eq. (3). As before, we use $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i]$ to try and control the system.

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i] \tag{3}$$

Show that the resulting closed-loop state space matrix is

$$A_{\text{CL}} = \begin{bmatrix} f_1 & f_2 + 1 \\ f_1 + 2 & f_2 - 1 \end{bmatrix} \tag{4}$$

Is it possible to stabilize this system?

(f) **(PRACTICE)** Suppose you had a discrete, 2D, linear system with a real A matrix, and that you could modify both eigenvalues with feedback control (such as the system in eq. (1)). **Can you place the eigenvalues at complex conjugates, such that $\lambda_1 = a + \mathrm{j}b, \lambda_2 = a - \mathrm{j}b$ using only real feedback gains f_1, f_2? How about placing them at any arbitrary complex numbers, such that $\lambda_1 = a + \mathrm{j}b, \lambda_2 = c + \mathrm{j}d$?**

2. Uncontrollability

Recall that, for a n-dimensional, discrete-time linear system to be controllable, we require that the controllability matrix $\mathcal{C} = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \dots & AB & B \end{bmatrix}$ to be rank n.

Consider the following discrete-time system with the given initial state:

$$\vec{x}[i+1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i] \tag{5}$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

(a) **Is the system controllable?**

(b) **Show that we can write the** i**th state as**

$$\vec{x}[i] = \begin{bmatrix} 2^i \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \tag{7}$$

Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ **for some** ℓ**? If so, for what input sequence** $u[i]$ **up to** $i = \ell - 1$**?**

(c) **(PRACTICE) Is it possible to reach** $\vec{x}[\ell] = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ **for some** ℓ**? For what input sequence** $u[i]$ **for** $i = 0$ **to** $i = \ell - 1$**?**

HINT: Use the result for $\vec{x}[i]$ *from the previous part.*

(d) **Find the set of all $\vec{x}[2]$, given that you are free to choose any $u[0]$ and $u[1]$.**

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