Discussion 5A

The following notes are useful for this discussion: Note 13.

1. Gram-Schmidt Algorithm

Let's apply Gram-Schmidt orthonormalization to a list of three linearly independent vectors $[\vec{s}_1, \vec{s}_2, \vec{s}_3]$.

(a) Let's say we had two collections of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\{\vec{w}_1, \ldots, \vec{w}_n\}$. **How can we prove that** $\operatorname{Span}(\{\vec{v}_1, \ldots, \vec{v}_n\}) = \operatorname{Span}(\{\vec{w}_1, \ldots, \vec{w}_n\})$**?**

(b) **Find unit vector** $\vec{q}_1$ **such that** $\operatorname{Span}(\{\vec{q}_1\}) = \operatorname{Span}(\{\vec{s}_1\})$**, where** $\vec{s}_1$ **is nonzero.**

(c) Let's say that we wanted to write

$$\vec{s}_2 = c_1\vec{q}_1 + \vec{z}_2 \tag{1}$$

where $c_1\vec{q}_1$ entirely represents the component of $\vec{s}_2$ in the direction of $\vec{q}_1$, and $\vec{z}_2$ represents the component of $\vec{s}_2$ that is distinctly *not* in the direction of $\vec{q}_1$ (i.e. $\vec{z}_2$ and $\vec{q}_1$ are orthogonal).

Given $\vec{q}_1$ from the previous step, **find** c_1 **as in eq. (1), and use** $\vec{z}_2$ **to find unit vector** $\vec{q}_2$ **such that** $\operatorname{Span}(\{\vec{q}_1, \vec{q}_2\}) = \operatorname{Span}(\{\vec{s}_1, \vec{s}_2\})$ **and** $\vec{q}_2$ **is orthogonal to** $\vec{q}_1$**. Show that** $\operatorname{Span}(\{\vec{q}_1, \vec{q}_2\}) = \operatorname{Span}(\{\vec{s}_1, \vec{s}_2\})$**.**

(d) **What would happen if $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ were *not* linearly independent, but rather $\vec{s}_1$ were a multiple of $\vec{s}_2$?**

(e) Now given $\vec{q}_1$ and $\vec{q}_2$ in parts 1.b and 1.c, **find $\vec{q}_3$ such that $\text{Span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}) = \text{Span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$, and $\vec{q}_3$ is orthogonal to both $\vec{q}_1$ and $\vec{q}_2$, and finally $\|\vec{q}_3\| = 1$.** You do not have to show that the two spans are equal.

(f) **(PRACTICE) Confirm that $\text{Span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}) = \text{Span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$.**

2. Orthonormal Matrices and Projections

A matrix A has orthonormal columns, $\vec{a}_i$, if they are:

- Orthogonal (ie. $\langle \vec{a}_i, \vec{a}_j \rangle = \vec{a}_j^\top \vec{a}_i = 0$ when $i \neq j$)
- Normalized (ie. vectors with length equal to 1, $\|\vec{a}_i\| = 1$). This implies that $\|\vec{a}_i\|_2 = \langle \vec{a}_i, \vec{a}_i \rangle = \vec{a}_i^\top \vec{a}_i = 1$.

(a) When $A \in \mathbb{R}^{n \times m}$ and $n \geq m$ (i.e. for tall matrices), **show that if the matrix is orthonormal, then $A^\top A = I_{m \times m}$.**

(b) Again, suppose $A \in \mathbb{R}^{n \times m}$ where $n \geq m$ is an orthonormal matrix. **Show that the projection of $\vec{y}$ onto the subspace spanned by the columns of A is now $AA^\top \vec{y}$.**

(c) **(PRACTICE) Show if $A \in \mathbb{R}^{n \times n}$ is an orthonormal matrix then the columns, $\vec{a}_i$, form a basis for $\mathbb{R}^n$.**

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