The following notes are useful for this discussion: Note 13.

1. Gram-Schmidt Algorithm

Let’s apply Gram-Schmidt orthonormalization to a list of three linearly independent vectors \([\vec{s}_1, \vec{s}_2, \vec{s}_3]\).

(a) Let’s say we had two collections of vectors \(\{\vec{v}_1, \ldots, \vec{v}_n\}\) and \(\{\vec{w}_1, \ldots, \vec{w}_n\}\). **How can we prove that** \(\text{Span}(\{\vec{v}_1, \ldots, \vec{v}_n\}) = \text{Span}(\{\vec{w}_1, \ldots, \vec{w}_n\})\)?

(b) **Find unit vector** \(\vec{q}_1\) **such that** \(\text{Span}(\{\vec{q}_1\}) = \text{Span}(\{\vec{s}_1\})\), where \(\vec{s}_1\) is nonzero.

(c) Let’s say that we wanted to write
\[
\vec{s}_2 = c_1 \vec{q}_1 + \vec{z}_2
\]
where \(c_1 \vec{q}_1\) entirely represents the component of \(\vec{s}_2\) in the direction of \(\vec{q}_1\), and \(\vec{z}_2\) represents the component of \(\vec{s}_2\) that is distinctly **not** in the direction of \(\vec{q}_1\) (i.e. \(\vec{z}_2\) and \(\vec{q}_1\) are orthogonal).

Given \(\vec{q}_1\) from the previous step, **find** \(c_1\) **as in eq. (1)**, and **use** \(\vec{z}_2\) **to find unit vector** \(\vec{q}_2\) **such that** \(\text{Span}(\{\vec{q}_1, \vec{q}_2\}) = \text{Span}(\{\vec{s}_1, \vec{s}_2\})\) and \(\vec{q}_2\) **is orthogonal to** \(\vec{q}_1\). **Show that** \(\text{Span}(\{\vec{q}_1, \vec{q}_2\}) = \text{Span}(\{\vec{s}_1, \vec{s}_2\})\).
(d) What would happen if \( \{ \vec{s}_1, \vec{s}_2, \vec{s}_3 \} \) were not linearly independent, but rather \( \vec{s}_1 \) were a multiple of \( \vec{s}_2 \)?

(e) Now given \( \vec{q}_1 \) and \( \vec{q}_2 \) in parts 1.b and 1.c, find \( \vec{q}_3 \) such that \( \text{Span}(\{ \vec{q}_1, \vec{q}_2, \vec{q}_3 \}) = \text{Span}(\{ \vec{s}_1, \vec{s}_2, \vec{s}_3 \}) \), and \( \vec{q}_3 \) is orthogonal to both \( \vec{q}_1 \) and \( \vec{q}_2 \), and finally \( \|\vec{q}_3\| = 1 \). You do not have to show that the two spans are equal.

(f) (PRACTICE) Confirm that \( \text{Span}(\{ \vec{q}_1, \vec{q}_2, \vec{q}_3 \}) = \text{Span}(\{ \vec{s}_1, \vec{s}_2, \vec{s}_3 \}) \).

2. Orthonormal Matrices and Projections

A matrix \( A \) has orthonormal columns, \( \vec{a}_j \), if they are:

- Orthogonal (ie. \( \langle \vec{a}_i, \vec{a}_j \rangle = \vec{a}_j^\top \vec{a}_i = 0 \) when \( i \neq j \))
- Normalized (ie. vectors with length equal to 1, \( \|\vec{a}_i\| = 1 \)). This implies that \( \|\vec{a}_i\|_2 = \langle \vec{a}_i, \vec{a}_i \rangle = \vec{a}_i^\top \vec{a}_i = 1 \).

(a) When \( A \in \mathbb{R}^{n \times m} \) and \( n \geq m \) (i.e. for tall matrices), show that if the matrix is orthonormal, then \( A^\top A = I_{m \times m} \).
(b) Again, suppose $A \in \mathbb{R}^{n \times m}$ where $n \geq m$ is an orthonormal matrix. Show that the projection of $\vec{y}$ onto the subspace spanned by the columns of $A$ is now $AA^\top \vec{y}$.

(c) (PRACTICE) Show if $A \in \mathbb{R}^{n \times n}$ is an orthonormal matrix then the columns, $\vec{a}_i$, form a basis for $\mathbb{R}^n$.

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