

The following note is useful for this discussion: [Note 16](#)

1. Computing the SVD: A “Tall” Matrix Example

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}. \tag{1}$$

Here, we expect $U \in \mathbb{R}^{3 \times 3}$, $\Sigma \in \mathbb{R}^{3 \times 2}$, and $V \in \mathbb{R}^{2 \times 2}$ (recall that U and V must be square since they are orthonormal matrices).

In this problem, we will walk through the SVD algorithm, prove some important theorems about the SVD matrices and column/null spaces, and consider an alternate way to approach the SVD.

- (a) In this part, we will walk through Algorithm 7 in [Note 16](#). This algorithm applies for a general matrix $A \in \mathbb{R}^{m \times n}$.
- i. **Find $r := \text{rank}(A)$. Compute $A^\top A$ and diagonalize it using the spectral theorem (i.e. find V and Λ).**
 - ii. **Unpack $V := [V_r \ V_{n-r}]$ and unpack $\Lambda := \begin{bmatrix} \Lambda_r & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (n-r)} \end{bmatrix}$.**
 - iii. **Find $\Sigma_r := \Lambda_r^{1/2}$ and then find $\Sigma := \begin{bmatrix} \Sigma_r & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$.**
 - iv. **Find $U_r := AV_r \Sigma_r^{-1}$, where $U_r \in \mathbb{R}^{3 \times 1}$ and then extend the basis defined by columns of U_r to find $U \in \mathbb{R}^{3 \times 3}$.**
(HINT: How can we extend a basis, and why is that needed here?)
 - v. **Use the previous parts to write the full SVD of A .**
 - vi. **Use the Jupyter notebook to run the code cell that calls `numpy.linalg.svd` on A . What is the result? Does it match our result above?**

- (b) We now want to create the SVD of A^\top . Rather than repeating all of the steps in the algorithm, feel free to use the Jupyter notebook for this subpart (which defines a `numpy.linalg.svd` command). **What are the relationships between the matrices composing A and the matrices composing A^\top ?**

- (c) **Show, for a general matrix $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = r$ and $A = U\Sigma V^\top$, that $\text{Null}(A) = \text{Col}(V_{n-r})$. Then, find a basis for the null space of A in eq. (1).** (HINT: How do we show two sets are equal? Try and use that approach here. Consider the outer product summation form for the SVD. Also, consider using the rank-nullity theorem that $\dim \text{Col}(A) + \dim \text{Null}(A) = n$.)

(d) **Show, for a general matrix $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = r$ and $A = U\Sigma V^\top$, that $\text{Col}(A) = \text{Col}(U_r)$. Then, find a basis for the range (or column space) of A .**

(e) **(PRACTICE) Show, for a general matrix $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = r$ and $A = U\Sigma V^\top$, that $\text{Null}(A^\top) = \text{Col}(U_{m-r})$ and $\text{Col}(A^\top) = \text{Col}(V_r)$. Then show:**

- i. $\dim \text{Col}(A) + \dim \text{Null}(A^\top) = n$,
- ii. **and $\text{Col}(A)$ and $\text{Null}(A^\top)$ are orthogonal.**

(f) Suppose A was a wide matrix. Instead of finding $A^\top A$, we may want to find the SVD by computing AA^\top . The original Algorithm 7 from [Note 16](#), in its entirety, is shown below:

Algorithm 1 Constructing the SVD

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1: function FULLSVD( $A \in \mathbb{R}^{m \times n}$ )
2:    $r := \text{RANK}(A)$ 
3:    $(V, \Lambda) := \text{DIAGONALIZE}(A^\top A)$  ▷ Sorted so that  $\Lambda_{11} \geq \dots \geq \Lambda_{nn}$ 
4:   Unpack  $V := \begin{bmatrix} V_r & V_{n-r} \end{bmatrix}$ 
5:   Unpack  $\Lambda := \begin{bmatrix} \Lambda_r & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (n-r)} \end{bmatrix}$ 
6:    $\Sigma_r := \Lambda_r^{1/2}$ 
7:   Pack  $\Sigma := \begin{bmatrix} \Sigma_r & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$ 
8:    $U_r := AV_r \Sigma_r^{-1}$ 
9:    $U := \text{EXTENDBASIS}(U_r, \mathbb{R}^m)$ 
10:  return  $(U, \Sigma, V)$ 
11: end function

```

Write a modified version of Algorithm 7 where you solve for the SVD of A using AA^\top instead of $A^\top A$. (HINT: Consider replacing every instance of “ A ” in $A^\top A$ with “ A^\top ”. What happens? How can we use the result from the 1.b part?)

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