

The following notes are useful for this discussion: [Note 16](#)

1. Geometric Interpretation of the SVD

In this exercise, we explore the geometric interpretation of matrix transformations and how this connects to the SVD. We consider how a real 2×2 matrix acts on the unit circle, transforming it into an ellipse. It turns out that the principal semi-axes of the resulting ellipse are related to the singular values of the matrix, as well as the vectors in the SVD.

(a) Consider the real 2×2 matrix

$$A = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}. \quad (1)$$

Also consider the unit circle in \mathbb{R}^2 ,

$$S = \left\{ \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \mid 0 \leq \theta < 2\pi \right\}. \quad (2)$$

Plot the transformed circle, AS , on the \mathbb{R}^2 plane.

$$AS = \left\{ A \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \mid 0 \leq \theta < 2\pi \right\}. \quad (3)$$

(b) Now let's consider how this transformation looks in the lens of the SVD. The SVD for matrix A is:

$$A = U\Sigma V^T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (4)$$

$$A\vec{x} = U\Sigma V^T \vec{x} = U\left(\Sigma\left(V^T \vec{x}\right)\right). \quad (5)$$

Let's start by examining the effects of each of these matrices one at a time, right to left, in the same order that they would be applied to a vector \vec{x} . **What does the unit circle look like after being transformed by just V^T ? Plot $S_1 = V^T S$ on the \mathbb{R}^2 plane. Geometrically speaking, what does V^T do to any given \vec{x} ?**

- (c) **What does the unit circle look like after being transformed by ΣV^\top ? Plot $S_2 = \Sigma V^\top S$ on the \mathbb{R}^2 plane. Geometrically speaking, what is the Σ matrix doing to any given $V^\top \vec{x}$?**
- (d) **What does the unit circle look like after being transformed by $U\Sigma V^\top$? Plot $S_3 = U\Sigma V^\top S$ on the \mathbb{R}^2 plane. Geometrically speaking, what is the U matrix doing to any given $\Sigma V^\top \vec{x}$?**
- (e) **Consider the columns of the matrices U, V from the SVD of A in part (b), and treat them as vectors in \mathbb{R}^2 . Let $U = (\vec{u}_1 \ \vec{u}_2)$, $V = (\vec{v}_1 \ \vec{v}_2)$. Let σ_1, σ_2 be the singular values of A , where $\sigma_1 \geq \sigma_2$. In your plot of AS , draw the vectors $\sigma_1 \vec{u}_1$ and $\sigma_2 \vec{u}_2$ from the origin. What do these vectors correspond to geometrically?**

(f) Repeat parts (b-e) for the following matrices, and note down any interesting things you notice.

i. A 3D matrix, $X = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

ii. A rotation matrix, $A_1 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

iii. A diagonal matrix, $A_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

iv. A symmetric matrix, $A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

v. A matrix with non-trivial nullspace, $A_4 = \begin{bmatrix} 4 & 2 \\ -2 & -1 \end{bmatrix}$.

vi. An arbitrary matrix, $A_5 = \begin{bmatrix} 1.6 & 2.4 \\ -0.4 & -1 \end{bmatrix}$.

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