Discussion 6B

The following notes are useful for this discussion: Note 16

1. Geometric Interpretation of the SVD

In this exercise, we explore the geometric interpretation of matrix transformations and how this connects to the SVD. We consider how a real 2×2 matrix acts on the unit circle, transforming it into an ellipse. It turns out that the principal semi-axes of the resulting ellipse are related to the singular values of the matrix, as well as the vectors in the SVD.

(a) Consider the real 2×2 matrix

$$A = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}. \tag{1}$$

Also consider the unit circle in $\mathbb{R}^2$,

$$S = \left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \middle| 0 \leq \theta < 2\pi \right\}. \tag{2}$$

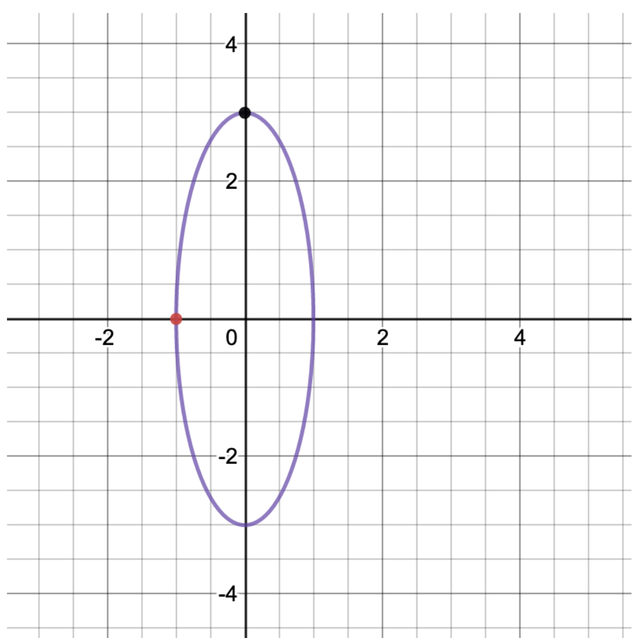
Plot the transformed circle, AS, on the $\mathbb{R}^2$ plane.

$$AS = \left\{ A \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \middle| 0 \leq \theta < 2\pi \right\}. \tag{3}$$

Solution:

$$AS = \left\{ \begin{bmatrix} -\sin\theta \\ 3\cos\theta \end{bmatrix} \middle| 0 \leq \theta < 2\pi \right\}. \tag{4}$$

The plot should be the ellipse centered at the origin that passes through the points $(0,3)$, $(0,-3)$, $(-1,0)$, $(1,0)$.



(b) Now let's consider how this transformation looks in the lens of the SVD. The SVD for matrix A is:

$$A = U\Sigma V^\top = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{5}$$

$$A\vec{x} = U\Sigma V^\top \vec{x} = U\left(\Sigma\left(V^\top \vec{x}\right)\right). \tag{6}$$

Let's start by examining the effects of each of these matrices one at a time, right to left, in the same order that they would be applied to a vector $\vec{x}$. **What does the unit circle look like after being transformed by just $V^\top$? Plot $S_1 = V^\top S$ on the $\mathbb{R}^2$ plane. Geometrically speaking, what does $V^\top$ do to any given $\vec{x}$?**

Solution: $V^\top$, being an orthonormal matrix can only rotate or reflect a vector $\vec{x}$. In particular, it applies a rotation or reflection such that the vectors $\vec{v}_i$ in the standard basis are transformed to the elementary vectors $\vec{e}_i$ in the V basis. Note that this matrix cannot do any scaling.

See jupyter notebook for plots.

(c) **What does the unit circle look like after being transformed by $\Sigma V^\top$? Plot $S_2 = \Sigma V^\top S$ on the $\mathbb{R}^2$ plane. Geometrically speaking, what is the Σ matrix doing to any given $V^\top \vec{x}$?**

Solution: The matrix Σ scales vectors that have been transformed into the V basis. In terms of the SVD, it scales the components of $\vec{x}$ in the direction of $\vec{v}_1$ by σ_1, the components in the direction of $\vec{v}_2$ by σ_2, and so on for larger matrices. All of the scaling done by the original matrix A is captured by the Σ matrix.

See jupyter notebook for plots.

(d) **What does the unit circle look like after being transformed by $U\Sigma V^\top$? Plot $S_3 = U\Sigma V^\top S$ on the $\mathbb{R}^2$ plane. Geometrically speaking, what is the U matrix doing to any given $\Sigma V^\top \vec{x}$?**

Solution: U is an orthonormal matrix similar to $V^\top$, and as such can only apply a rotation or reflection to a vector. In the context of the SVD, U rotates or reflects the scaled vectors $\Sigma V^\top \vec{x}$ to their final locations.

See jupyter notebook for plots.

(e) Consider the columns of the matrices U, V from the SVD of A in part (b), and treat them as vectors in $\mathbb{R}^2$. Let $U = (\vec{u}_1\ \vec{u}_2)$, $V = (\vec{v}_1\ \vec{v}_2)$. Let σ_1, σ_2 be the singular values of A, where $\sigma_1 \geq \sigma_2$. **In your plot of AS, draw the vectors $\sigma_1 \vec{u}_1$ and $\sigma_2 \vec{u}_2$ from the origin. What do these vectors correspond to geometrically?**

Solution: $\sigma_1 \vec{u}_1 = (0, -3)$ corresponds to the semi-major axis of the ellipse, while $\sigma_2 \vec{u}_2 = (-1, 0)$ corresponds to the semi-minor axis.

See jupyter notebook for plots.

(f) **Repeat parts (b-e) for the following matrices, and note down any interesting things you notice.**

i. A 3D matrix, $X = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

ii. A rotation matrix, $A_1 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

iii. A diagonal matrix, $A_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

iv. A symmetric matrix, $A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

v. A matrix with non-trivial nullspace, $A_4 = \begin{bmatrix} 4 & 2 \\ -2 & -1 \end{bmatrix}$.

vi. An arbitrary matrix, $A_5 = \begin{bmatrix} 1.6 & 2.4 \\ -0.4 & -1 \end{bmatrix}$.

Solution: See jupyter notebook for plots.

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