Homework 1

This homework is due on Monday, June 27 at 11:59 pm PT

NOTE: All other homeworks will generally be due on Sunday. Additionally, the first two non-administrative problems were derived from the EECS16A Fall 2020 Final Exam. This part of the homework will help you review 16A material, so it will benefit you to attempt the problems without looking up the solutions.

1. Administrivia

(a) **Please take the following policy quiz and attach a screenshot of your score**: link to quiz. The goal is to ensure that everyone is familiar with the course policies, which you can read about here.

If you have a problem accessing the quiz, try using your UC Berkeley account.

(b) We highly recommend joining a study group in order to foster a sense of community in the course and learn from others. EECS 16B is a pretty fast-paced course, and you can benefit quite a bit from your peers' perspectives on the material.

Please fill out this form regardless of if you'd like a study group or not. If you opt in, within a few weeks, you should get an email informing you of the group you have been matched with. It is respectful and professional behavior to follow up with your group members – we hope you stay true to your word!

Special shoutout to Prof. Ranade's group formation research team for making this possible!

For your answer, please provide a screenshot showing you have completed the form.

(c) We will use a variety of online tools and websites this semester. Please watch the following video tutorials about how to use them.

- Gradescope:
  - Submitting an online assignment. (This is especially pertinent for the Lab Sim and Discussion Checkoff portion of the course)
  - Submitting PDF homework.
  - Viewing feedback and requesting regrades.
- Office Hour Queue.

If you have a problem accessing any of these videos, try viewing them using your UC Berkeley account.

After watching the videos, please write down for your answer to this problem that you understand how the tools work; if you have any questions about the videos, please post them on the corresponding Piazza thread for this problem.

2. Least Squares for Robotics

Robots rely on sensors for understanding their environment and navigating in the real world. These sensors must be calibrated to ensure accurate measurements, which we explore in this problem.

(a) Your robot is equipped with two forward-facing sensors – a radar and camera.
However, the sensors are placed with an offset (i.e. a gap) of ℓ in meters (m), as depicted in Fig. 1, and you want to find its value. The radar returns a range ρ in meters (m) and heading angle θ in radians (rad) with respect to the object. In contrast, the camera only returns an angle, ϕ in radians (rad), with respect to the object.

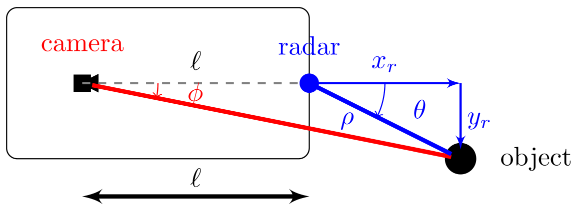

Figure 1: Sensor Placement and Offset ℓ.

These relationships are summarized by the following sensor model, where x_r and y_r are the Cartesian coordinates of the object with respect to the radar:

$$x_r = \rho\cos(\theta)\,, \tag{1}$$

$$y_r = \rho\sin(\theta)\,, \tag{2}$$

$$\tan(\phi) = \frac{y_r}{x_r + \ell}\,. \tag{3}$$

Assuming $\phi \neq 0$, use equations (1), (2), (3) to express ℓ in terms of ρ, θ, and ϕ.

Solution: From the sensor model, we have:

$$(x_r + \ell)\tan(\phi) = y_r$$
$$\Rightarrow \ell = \frac{y_r}{\tan(\phi)} - x_r$$
$$\Rightarrow \ell = \rho\left(\frac{\sin(\theta)}{\tan(\phi)} - \cos(\theta)\right).$$

Note: We stipulate that $\phi \neq 0$ since otherwise division by $\tan(\phi)$ would not be well-defined. When $\phi = 0$, the object would be located right in front of both the radar and camera, and any positive value of ℓ would solve the system of equations. This explanation is not required for full credit.

(b) Often it is difficult to precisely identify the value of ℓ. To learn the value of ℓ you decide to take a series of measurements. In particular, you take N measurements and get the equations:

$$a\,\ell + e_i = b_i$$

for $1 \leq i \leq N$. Here $a \neq 0$ is a fixed and known constant. Each b_i represents your i^{th} measurement and e_i represents the error in your measurement. While you know all of the b_i values, you do not know the error values e_i.

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We can write this equation in a vector format as:

$$\mathbf{A}\ell + \vec{e} = \vec{b},$$

where $\mathbf{A} = \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}$, $\vec{e} = \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}$.

In this simple 1-D case, the least squares solution is a scaled version of the average of $\{b_i\}_{i=1}^N$. **Find the best estimate for ℓ, denoted as $\hat{\ell}$, using least squares. Simplify your expression and express $\hat{\ell}$ in terms of a, b_i, and N. Your answer may not include any vector notation.**

Note: A is a vector and not a matrix.

Solution: $\hat{\ell}$ is given by the least square solution:

$$\begin{aligned} \hat{\ell} &= (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b} \\ &= (Na^2)^{-1} a \sum_{i=1}^{N} b_i \\ &= \frac{\sum_{i=1}^{N} b_i}{aN}. \end{aligned}$$

(c) Now we turn to the task of controlling the robot's velocity and acceleration, which is a key requirement for navigation.

We use the following model for the robot, which describes how the velocity and acceleration of the robot changes with timestep k:

$$\begin{bmatrix} v[k+1] \\ a[k+1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v[k] \\ a[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} j[k],$$

where

- k is the timestep;
- $v[k]$ is the velocity state at timestep k;
- $a[k]$ is the acceleration state at timestep k;
- $j[k]$ is the jerk (derivative of acceleration) control input at timestep k.

We start at a known initial state $\begin{bmatrix} v[0] \\ a[0] \end{bmatrix}$, and we want to find $j[0]$ to set $\begin{bmatrix} v[1] \\ a[1] \end{bmatrix}$ as close to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as possible. For this, we minimize:

$$E = \left\| \begin{bmatrix} v[1] \\ a[1] \end{bmatrix} \right\|^2.$$

Find the best estimate for the optimal choice of jerk, $\hat{j}[0]$, by using least squares method to minimize E. Express your solution in terms of $v[0]$ and $a[0]$. Show your work.

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Hint: Rewrite E in terms of $j[0]$ and other relevant terms.

Solution:

Starting from the hint, we try to rewrite the cost E. Applying the dynamics model, we find that:

$$\begin{aligned} E &= \left\| \begin{bmatrix} v[0] + a[0] \\ a[0] + j[0] \end{bmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} j[0] - \begin{bmatrix} -v[0] - a[0] \\ -a[0] \end{bmatrix} \right\|^2 \\ &= \left\| \mathbf{A} j[0] - \vec{b} \right\|^2. \end{aligned}$$

Therefore, $j\hat{[0]}$ is given by the least square solution:

$$\begin{aligned} j\hat{[0]} &= (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b} \\ &= (1)^{-1} \times \ (-a[0]) \\ &= -a[0]. \end{aligned}$$

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3. Ultrasound Sensing with Op-Amps

The transresistance amplifier is often used to convert a current from a sensor to a voltage. In this problem we will use it to build an ultrasound sensor! When an ultrasonic wave hits our sensor, it generates a current, i_{ultra}. Whenever no ultrasonic wave hits our sensor zero current is generated, so $i_{\text{ultra}} = 0$.

Note: An **ideal op-amp** is used in all subparts of this question. You can also assume that $V_{\text{DD}} = -V_{\text{SS}}$.

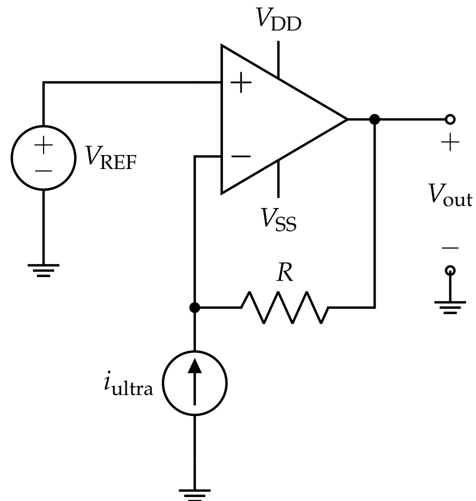

Figure 2: Transresistance sensor circuit

(a) **Calculate the output voltage, V_{out}, of the transresistance sensor circuit shown in Fig. 2, as a function of the reference voltage, V_{REF}, the sensor input current, i_{ultra}, and the resistor, R, when an ultrasonic wave hits the sensor.** Clearly show all your work and justify your answer. Writing only the final expression will not be given full credit.

Solution: Since we have an ideal op-amp in a negative feedback circuit, we can first say that $u_- = u_+ = V_{\text{REF}}$, where u_+ and u_- are the voltages at the positive and negative input terminals of the op-amp respectively. Next the current i_{ultra} must flow right though R due to KCL and the golden rules (no current can flow into the op-amp). Thus we establish

$$V_{\text{out}} = u_- - i_{\text{ultra}} R = u_+ - i_{\text{ultra}} R = V_{\text{REF}} - i_{\text{ultra}} R. \qquad \square$$

(b) Assume that the amplitude of the ultrasonic wave hitting the sensor is such that the current i_{ultra} fluctuates from a minimum value of $i_{\text{min}} = 1 \cdot 10^{-6}$A, to a maximum value of $i_{\text{max}} = 2 \cdot 10^{-6}$A. Also assume that the reference voltage is set to $V_{\text{REF}} = 1$V. In this case, calculate the following:

i. **The maximum value of the resistor, R, so that the output voltage, V_{out}, does not drop below 0V.** Clearly show all your work.

ii. Assuming you picked $R = 250 \cdot 10^3\Omega$ (which may or may not be the correct answer to part (i)), **calculate the maximum value of the output voltage, V_{out}.** Clearly show all your work.

Solution: From part (a) we identified the output voltage of our sensor circuit $V_{\text{out}} = V_{\text{REF}} - i_{\text{ultra}} R$.

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i. The worst-case scenario (in which the output voltage is most reduced) occurs for $i_{\text{ultra}} = i_{\max} = 2 \cdot 10^{-6}$A. If we include $V_{\text{REF}} = 1$V and set our nonnegative condition on V_{out}, we identify the restriction on R:

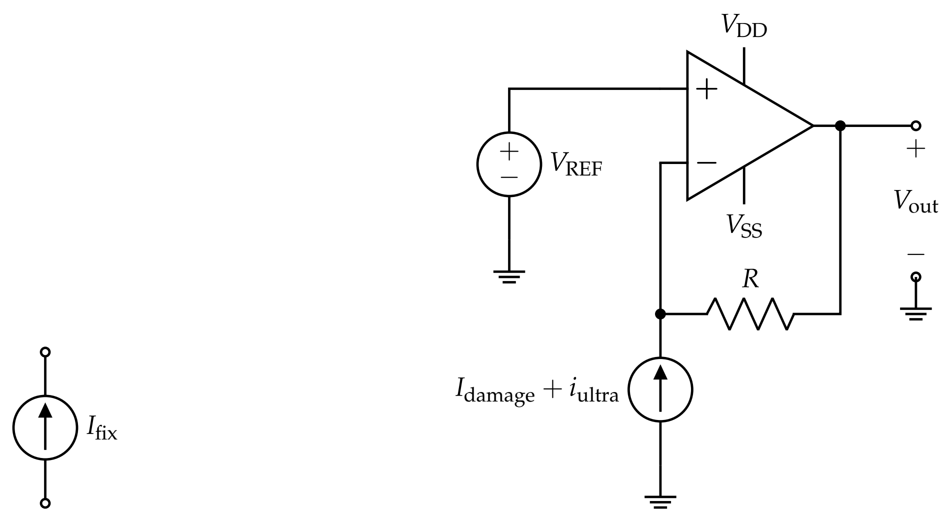
$$V_{\text{out}} = V_{\text{REF}} - i_{\max} R \geq 0 \quad \Longrightarrow \quad R \leq \frac{V_{\text{REF}}}{i_{\max}} = \frac{1V}{2 \cdot 10^{-6} A} = 500,000\,\Omega. \quad \square$$

ii. Based on our voltage formula, the highest V_{out} scenario now occurs for the low current condition $i_{\text{ultra}} = i_{\min} = 1 \cdot 10^{-6}$A. From this point we now substitute into the V_{out} formula:

$$V_{\text{out}} = V_{\text{REF}} - i_{\min}\,(250 \cdot 10^3 \Omega) = 1 - (1 \cdot 10^{-6} A)(250 \cdot 10^3 \Omega) = 1 - 0.25 = 0.75V. \quad \square$$

(c) Unfortunately, after a few hours of successful ultrasound sensing, our sensor got damaged. It now constantly generates a huge background current, I_{damage}. So when an ultrasonic wave hits it, the sensor produces $I_{\text{damage}} + i_{\text{ultra}}$, as shown in Fig 3(b). When no ultrasonic wave hits it, the sensor produces just I_{damage}. However, the huge background current causes our output to constantly be $V_{\text{out}} = V_{\text{SS}}$, so we are not able to tell whether an ultrasonic wave is present or not.

We would like to fix this in our circuit by canceling the background current and retaining only the useful signal. For this purpose we are going to use a current source, I_{fix}, shown in Fig. 3(a), whose value we can choose. **How would you connect this current source in your circuit and what value would you pick for it?** Redraw the entire circuit with the new current source, I_{fix}, added and give the value of I_{fix} in terms of $I_{\text{damage}}, i_{\text{ultra}}, R, V_{\text{REF}}$. **Explain how your design works.**



((a)) Constant current source I_{fix}. **((b))** Damaged transresistance amplifier circuit.

Figure 3: Circuits detailing the transresistance amplifier design, including the background signal I_{damage}.

Solution: We want to have only i_{ultra} flow through R. To achieve this we will insert the correcting current source in parallel with the input source $i_{\text{ultra}} + I_{\text{damage}}$ and set it at $I_{\text{fix}} = I_{\text{damage}}$ in opposite polarity, so that KCL gives:

$$I_{\text{damage}} + i_{\text{ultra}} = I_{\text{damage}} + I_R \quad \Longrightarrow \quad I_R = i_{\text{ultra}} \quad \square$$

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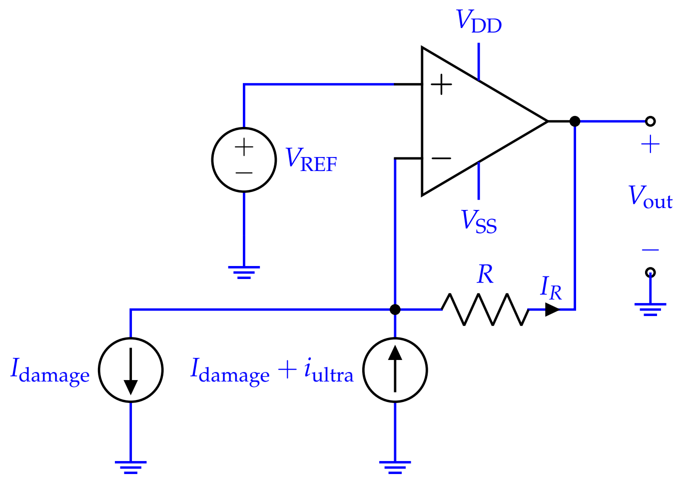

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4. Complex Numbers

Recall that a complex number $z \in \mathbb{C}$ is a number that can be expressed in the form

$$z = x + jy \tag{4}$$

where $x, y \in \mathbb{R}$ and $j^2 = -1$. This is known as the Cartesian form of a complex number. We call x the real part of z and denote it $\text{Re}\{z\} = x$. We call y the imaginary part of z and denote it $\text{Im}\{z\} = y$.

Complex numbers can be visualized as vectors on the complex plane, as in Figure 4.

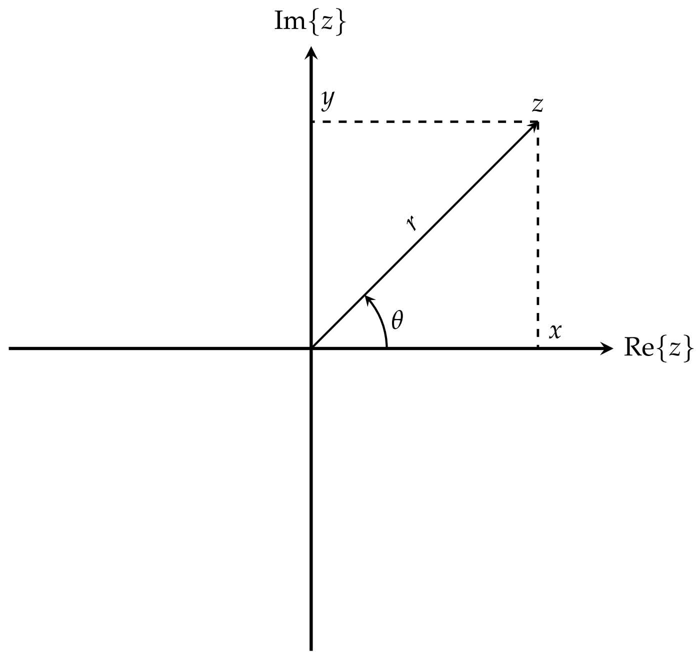

Figure 4: Complex plane

In this question, we will derive the polar form of a complex number and use this form to make some interesting conclusions.

(a) **Write an expression for the length of the vector z (as in Figure 4) in terms of x and y.** This is the magnitude of a complex number and is denoted by $|z|$ or r.

(HINT: Use the Pythagorean theorem.)

Solution:

$$r = \sqrt{x^2 + y^2} = |z| \tag{5}$$

(b) **Write expressions for x and y in terms of r and θ.**

Solution:

$$x = r\cos(\theta) \text{ and } y = r\sin(\theta) \tag{6}$$

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(c) **Substitute for x and y in Equation 4.** Use Euler's identity[1] $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ to **conclude that**

$$z = re^{j\theta}. \tag{7}$$

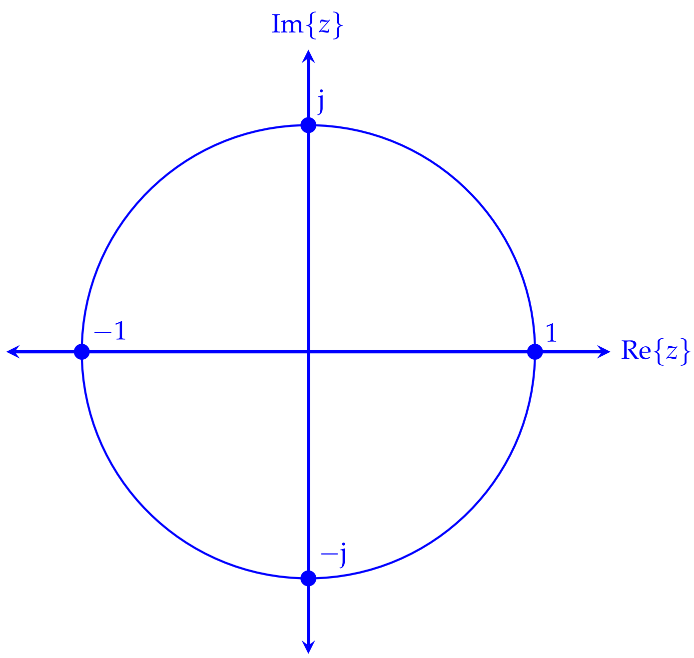
Solution:

$$z = r\cos(\theta) + jr\sin(\theta) \tag{8}$$
$$= r(\cos(\theta) + j\sin(\theta)) \tag{9}$$
$$= re^{j\theta} \tag{10}$$

(d) In the complex plane, **sketch the set of all the complex numbers such that $|z| = 1$. What are the z values where the sketched figure intersects the real axis and the imaginary axis?**

Solution:

The set of all points on the complex plane the same distance from the origin is a circle. The circle we want is the unit circle, since the distance here is $|z| = 1$.



We have labeled the intersections with the real and imaginary axes above. Going counterclockwise: $1, j, -1, -j$. These happen to be the fourth roots of unity, a fact that will become important closer to the end of the course.

(e) Assume $z = re^{j\theta}$. **Show that** $\overline{z} = re^{-j\theta}$. Recall that the complex conjugate of a complex number $z = x + jy$ is $\overline{z} = x - jy$.

Solution:

$$\overline{z} = \overline{(r(\cos(\theta) + j\sin(\theta)))} \tag{11}$$

[1] also known as de Moivre's Theorem.

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$$= r(\cos(\theta) - \mathrm{j}\sin(\theta)) \tag{12}$$
$$= r(\cos(-\theta) + \mathrm{j}\sin(-\theta)) \tag{13}$$
$$= r\mathrm{e}^{-\mathrm{j}\theta} \tag{14}$$

Here, we used the facts that $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$ from trigonometry. You can also see these facts if you remember the Taylor series for cos and sin. Remember that the one for cos just has even powers of θ while the one for sin just has the odd ones.

(f) **Show (by direct calculation) that**

$$r^2 = z\bar{z}. \tag{15}$$

Solution:

$$z\bar{z} = r\mathrm{e}^{\mathrm{j}\theta} r\mathrm{e}^{-\mathrm{j}\theta} = r^2\mathrm{e}^{\mathrm{j}\theta-\mathrm{j}\theta} = r^2\mathrm{e}^0 = r^2. \tag{16}$$

You can also do this in Cartesian coordinates:

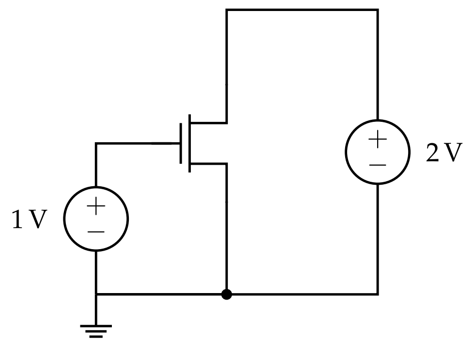
$$\begin{aligned} z\bar{z} &= r(\cos(\theta) + \mathrm{j}\sin(\theta))r(\cos(\theta) - \mathrm{j}\sin(\theta)) \\ &= r^2(\cos^2(\theta) - (j\sin(\theta))^2) \\ &= r^2(\cos^2(\theta) - j^2\sin^2(\theta)) \\ &= r^2(\cos^2(\theta) + \sin^2(\theta)) \\ &= r^2. \end{aligned}$$

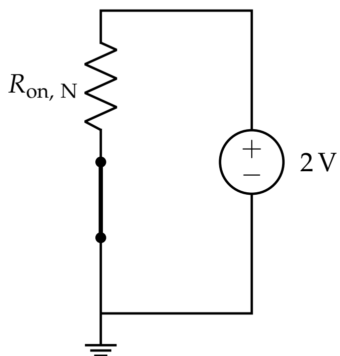
5. Transistor Behavior

Unlocked by Lectures 1 and 2

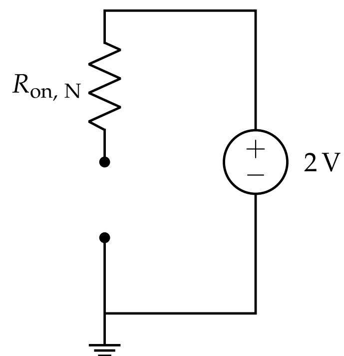
For all NMOS devices in this problem, $V_{\text{tn}} = 0.5\,\text{V}$. For all PMOS devices in this problem, $|V_{\text{tp}}| = 0.6\,\text{V}$. **Note: For this problem, we are also using the resistor-switch model for a transistor.**

(a) Which is the equivalent circuit as seen from the voltage source on the right-hand side of the circuit? **Fill in the correct bubble.**





Circuit A

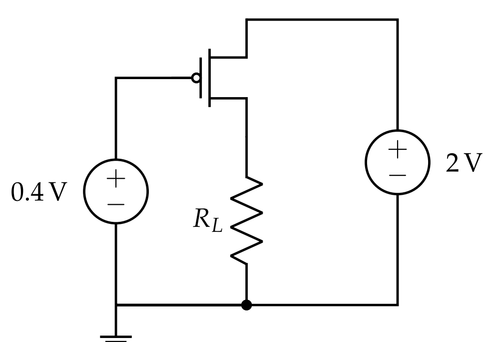


Circuit B

	A	**B**
Equivalent Circuit	◯	◯

Solution: For the NMOS, $V_{GS} = 1\,\text{V} > V_{\text{tn}} = 0.5\,\text{V}$, so the NMOS transistor is on. Thus circuit A is equivalent.

(b) Which is the equivalent circuit as seen from the voltage source on the right-hand side of the circuit? **Fill in the correct bubble.**



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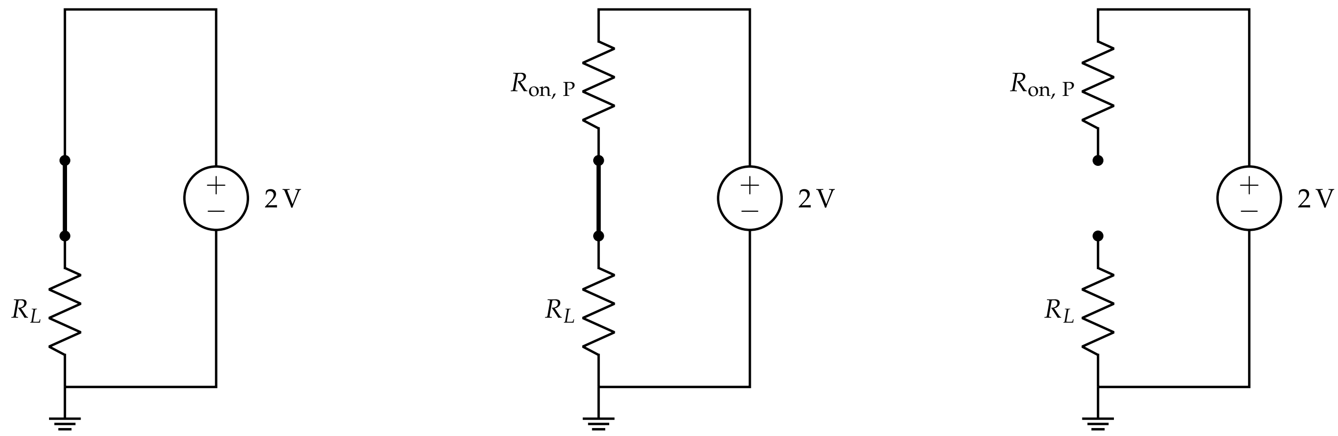
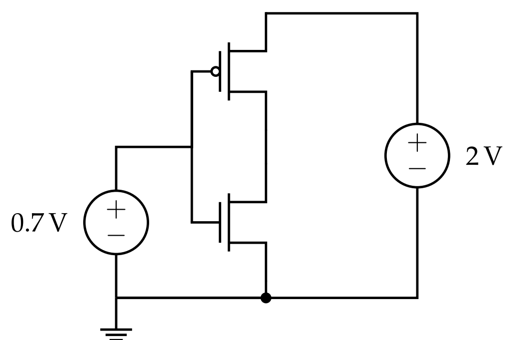


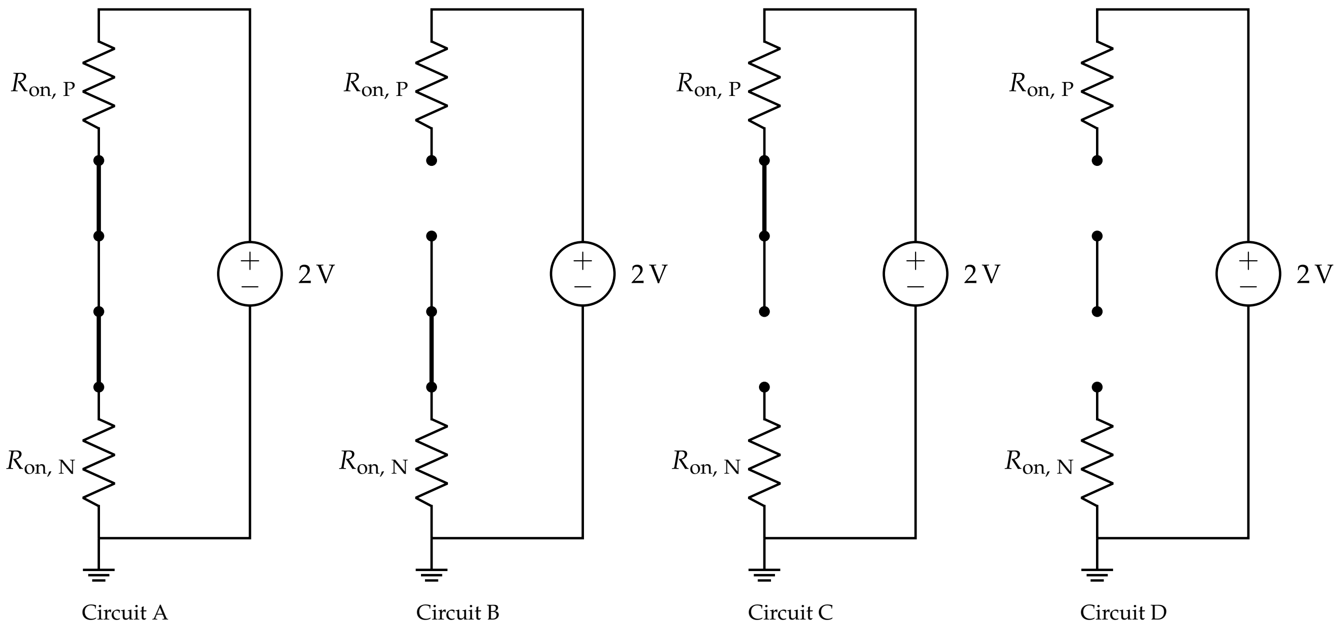
Circuit A Circuit B Circuit C

	A	B	C
Equivalent Circuit	◯	◯	◯

Solution: For the PMOS transistor, $|V_{\text{GS}}| = 1.6\,\text{V} > |V_{\text{tp}}| = 0.6\,\text{V}$, so the PMOS transistor is on. Thus circuit B is equivalent.

(c) Which is the equivalent circuit as seen from the voltage source on the right-hand side of the circuit? **Fill in the correct bubble.**





	A	B	C	D
Equivalent Circuit	◯	◯	◯	◯

Solution: For the PMOS transistor, $|V_{GS}| = 1.3\,V > |V_{tp}| = 0.6\,V$, so the PMOS transistor is on. For the NMOS transistor, $V_{GS} = 0.7\,V > V_{tn} = 0.5\,V$, so the NMOS transistor is on.

Note that in this case, both transistors are on.

Thus circuit A is equivalent.

Aside: In digital logic, it is usually undesirable to have this state in your system for several reasons. First, the output voltage of the inverter (the voltage at the shared drain of the NMOS and PMOS) will not be either 0 or V_{DD}, which means the output voltage is not at 'true' binary value. In addition, we now have a direct current path through the NMOS and PMOS transistors from VDD to ground. This will burn a lot of power! In reality, all inverters briefly transition through this state where both NMOS and PMOS are on when the inputs change from 1 to 0 or 0 to 1.

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6. Existence and uniqueness of solutions to differential equations

Unlocked by Lecture 2

When doing circuits or systems analysis, we sometimes model our system via a differential equation, and would often like to solve it to get the system trajectory. To this end, we would like to verify that a solution to our differential equation exists and is unique, so that our model is physically meaningful. There is a general approach to doing this, which is demonstrated in this problem.

We would like to show that there is a unique function $x\colon \mathbb{R} \to \mathbb{R}$ which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x(t) \tag{17}$$

$$x(0) = x_0. \tag{18}$$

In order to do this, we will first verify that a solution x_d exists. To show that x_d is the unique solution, we will take an arbitrary solution y and show that $x_d(t) = y(t)$ for every t.

(a) First, let us show that a solution to our differential equation exists. **Verify that $x_d(t) := x_0 e^{\alpha t}$ satisfies eq. (17) and eq. (18).**

Solution: We first verify eq. (17).

$$\frac{\mathrm{d}}{\mathrm{d}t}x_d(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x_0 e^{\alpha t}\right) \tag{19}$$

$$= x_0 \frac{\mathrm{d}}{\mathrm{d}t} e^{\alpha t} \tag{20}$$

$$= x_0 \cdot \alpha e^{\alpha t} \tag{21}$$

$$= \alpha \cdot x_0 e^{\alpha t} \tag{22}$$

$$= \alpha x_d(t). \tag{23}$$

Now we verify eq. (18).

$$x_d(0) = x_0 e^{\alpha \cdot 0} \tag{24}$$

$$= x_0 e^0 \tag{25}$$

$$= x_0. \tag{26}$$

(b) Now, let us show that our solution is unique. As mentioned before, suppose $y\colon \mathbb{R} \to \mathbb{R}$ also satisfies eq. (17) and eq. (18).

We want to show that $y(t) = x_d(t)$ for all t. Our strategy is to show that $\frac{y(t)}{x_d(t)} = 1$ for all t.

However, this particular differential equation poses a problem: if $x_0 = 0$, then $x_d(t) = 0$ for all t, so that the quotient is not well-defined. To patch this method, we would like to avoid using any function with x_0 in the denominator. One way we can do this is consider a modification of the quotient $\frac{y(t)}{x_d(t)} = \frac{y(t)}{x_0 e^{\alpha t}}$; in particular, we consider the function $z(t) := \frac{y(t)}{e^{\alpha t}}$.

Show that $z(t) = x_0$ for all t, and explain why this means that $y(t) = x_d(t)$ for all t.

(HINT: Show first that $z(0) = x_0$ and then that $\frac{\mathrm{d}}{\mathrm{d}t}z(t) = 0$. Argue that these two facts imply that $z(t) = x_0$ for all t. Then show that this implies $y(t) = x_d(t)$ for all t.)

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(HINT: Remember that we said y is any solution to eq. (17) and eq. (18), so we only know these properties of y. If you need something about y to be true, see if you can show it from eq. (17) and eq. (18).)

(HINT: When taking $\frac{\mathrm{d}}{\mathrm{d}t}z(t)$, remember to use the quotient rule, along with what we know about y.)

Solution: The solution goes in four stages, as per the hint.

Step 1. We show that $z(0) = x_0$. Indeed, using eq. (18),

$$z(0) = \frac{y(0)}{\mathrm{e}^{\alpha \cdot 0}} = \frac{x_0}{\mathrm{e}^0} = \frac{x_0}{1} = x_0. \tag{27}$$

Step 2. We show that $\frac{\mathrm{d}}{\mathrm{d}t}z(t) = 0$. Indeed, using the quotient rule from calculus and eq. (17),

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{y(t)}{\mathrm{e}^{\alpha t}} \tag{28}$$

$$= \frac{\mathrm{e}^{\alpha t}\left(\frac{\mathrm{d}}{\mathrm{d}t}y(t)\right) - y(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\alpha t}\right)}{\mathrm{e}^{2\alpha t}} \tag{29}$$

$$= \frac{\mathrm{e}^{\alpha t}(\alpha y(t)) - y(t)(\alpha \mathrm{e}^{\alpha t})}{\mathrm{e}^{2\alpha t}} \tag{30}$$

$$= \frac{\alpha \mathrm{e}^{\alpha t}y(t) - \alpha \mathrm{e}^{\alpha t}y(t)}{\mathrm{e}^{2\alpha t}} \tag{31}$$

$$= \frac{0}{\mathrm{e}^{2\alpha t}} \tag{32}$$

$$= 0. \tag{33}$$

Step 3. We show that $z(t) = x_0$ for all t. Indeed, since $\frac{\mathrm{d}}{\mathrm{d}t}z(t) = 0$, we know that $z(t)$ is a constant. Since $z(0) = x_0$, this gives that $z(t)$ is the constant value x_0, and hence $z(t) = x_0$ for all t.

Step 4. We show that $y(t) = x_d(t)$ for all t. Indeed, since $z(t) = x_0$ and $z(t) = \frac{y(t)}{\mathrm{e}^{\alpha t}}$, we have $x_0 = \frac{y(t)}{\mathrm{e}^{\alpha t}}$. We multiply both sides by $\mathrm{e}^{\alpha t}$ to get $y(t) = x_0\mathrm{e}^{\alpha t}$. But this is just $x_d(t)$, so $y(t) = x_d(t)$ for all t.

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7. Uniqueness Counterexample

Unlocked by Lecture 2

This problem explores an example of a differential equation that does not have a unique solution. The purpose is to show that uniqueness cannot always be assumed.

Along the way, this problem will also show you a heuristic way to guess the solutions to differential equations that is often called "separation-of-variables." The advantage of the separation-of-variables technique is that it can often be helpful in systematically coming up with guesses for nonlinear differential equations. However, as with any technique for guessing, it is not a proof and the guess definitely needs to be checked and uniqueness verified before proceeding.

The idea of separation-of-variables is to pretend that $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{\mathrm{d}x}{\mathrm{d}t}$ is a ratio of quantities rather than what it is — a shorthand for taking the derivative of the function $x(\cdot)$ with respect to its single argument, and then writing the result in terms of the free variable "t" for that argument. This little bit of make-believe (sometimes called "an abuse of notation") allows one the freedom to do calculations.

To demonstrate, let's do this for a case where we know the correct solution: $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t)$. This is how a separation-of-variables approach would try to get a guess:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) \tag{34}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda x \tag{35}$$

$$\frac{\mathrm{d}x}{x} = \lambda\,\mathrm{d}t \qquad \text{separating variables to sides} \tag{36}$$

$$\int \frac{\mathrm{d}x}{x} = \int \lambda\,\mathrm{d}t \qquad \text{integrating both sides} \tag{37}$$

$$\ln(x) + C_1 = \lambda t + C_2 \tag{38}$$

$$x(t) = K\mathrm{e}^{\lambda t} \qquad \text{exponentiating both sides and folding constants} \tag{39}$$

With the above guess obtained, $x(t) = K\mathrm{e}^{\lambda t}$ can be plugged in and seen to solve the original differential equation. Then of course, a uniqueness proof is required, but you did that in the previous homework.

To see why this technique can cause trouble, we will consider the following nonlinear differential equation involving a third root[2].

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x^{\frac{1}{3}} \tag{40}$$

with the initial condition

$$x(0) = 0. \tag{41}$$

Let's apply separation-of-variables and see what happens:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x^{\frac{1}{3}} \tag{42}$$

[2]This type of differential equation can arise from a physical setting of a inverted pyramidal container that had $x(t)$ liters of water in it, where the rate of water being poured in is proportional to the height of the water $x^{\frac{1}{3}}$. This fractional power arises since volume is a cubic quantity while the water is being poured in at a rate governed by a one-dimensional quantity of length. Similar equations can arise in microfluidic dynamics.

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$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x^{\frac{1}{3}} \tag{43}$$

$$x^{-\frac{1}{3}}\,\mathrm{d}x = \alpha\,\mathrm{d}t \tag{44}$$

$$\int x^{-\frac{1}{3}}\,\mathrm{d}x = \int \alpha\,\mathrm{d}t \tag{45}$$

$$\frac{3}{2}x^{\frac{2}{3}} + C_1 = \alpha t + C_2 \tag{46}$$

$$x = \left(\frac{2}{3}\alpha t + C_3\right)^{\frac{3}{2}} \tag{47}$$

(a) Given our separation-of-variables based calculation, let us guess a solution of the form

$$x(t) = \left(\frac{2}{3}\alpha t + c\right)^{\frac{3}{2}} \tag{48}$$

Show that this is a solution to the differential equation (40), and find the c that satisfies the initial condition. *(HINT: You'll need to use the power rule and chain rule.)*

Solution: We check that this is a solution by differentiating the guess for $x(t)$, and showing that it satisfies (40).

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}x(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{2}{3}\alpha t + c\right)^{\frac{3}{2}}\right) \\
&= \frac{3}{2}\cdot\left(\frac{2}{3}\alpha t + c\right)^{\frac{1}{2}}\cdot\frac{2}{3}\alpha \\
&= \alpha\left(\frac{2}{3}\alpha t + c\right)^{\frac{1}{2}} \\
&= \alpha\left(\left(\frac{2}{3}\alpha t + c\right)^{\frac{3}{2}}\right)^{\frac{1}{3}} \\
&= \alpha x^{\frac{1}{3}}
\end{aligned}$$

We thus satisfy (40).

To solve for the initial condition, we evaluate

$$\begin{aligned}
0 = x(0) &= \left(\frac{2}{3}\alpha\cdot 0 + c\right)^{\frac{3}{2}} \\
&= \left(\frac{2}{3}\cdot c\right)^{\frac{3}{2}} \\
\longrightarrow c &= 0
\end{aligned}$$

This is the only solution to the found form that satisfies the initial condition.

The equation $x(t) = \left(\frac{2}{3}\alpha t\right)^{\frac{3}{2}}$ thus satisfies the differential equation and the initial value condition.

(b) Let us guess a second solution:

$$x(t) = 0 \tag{49}$$

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Show that this new guess also satisfies (40), and the initial condition ($x(0) = 0$). [3]

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{\mathrm{d}}{\mathrm{d}t}0 = 0 = \alpha \cdot 0^{\frac{1}{3}} = \alpha \cdot x(t) \tag{51}$$

Furthermore, $x(0) = 0$.

So this second solution also satisfies the differential equation and the initial value condition.

(c) A known (not by you yet, but by the mathematical community) sufficient condition for the uniqueness of solutions to differential equations of the form $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = f(x(t))$ is that the function $f(x)$ be continuously differentiable (i.e. $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$ is a continuous function of x) with a bounded derivative $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$ at the initial condition $x(0)$ and everywhere that the solution $x(t)$ purports to go. (You will understand the importance of this condition and where it comes from better when we are in Module 2 of 16B. We are not going to prove it.)

Does this differential equation problem satisfy this condition that would let us trust guessing and checking?

Solution: No. To see this, consider our differential equation (40) at $t = 0$. In this case, $f(x) = \alpha x^{\frac{1}{3}}$.

We see that $\frac{\mathrm{d}}{\mathrm{d}x}f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha x^{\frac{1}{3}}\right) = \frac{\alpha}{3}x^{-\frac{2}{3}} = \frac{\alpha}{3x^{\frac{2}{3}}}$.

Note that at $x(t = 0) = 0$, our initial condition, the derivative becomes discontinuous in an unbounded fashion. It blows up. To be precise, if we take the limit

$$\lim_{x\to 0^+} \frac{\mathrm{d}}{\mathrm{d}x}f(x) = \lim_{x\to 0^+} \frac{\alpha}{3x^{\frac{2}{3}}} = +\infty.$$

Thus this differential equation does not satisfy the uniqueness condition that we have provided.

(d) The separation-of-variables technique may involve steps that may not agree with the initial condition. **Explain why (44) might be a bit problematic.** *(HINT: When is it not permissible to divide both sides of an equation by the same thing?)*

Solution: The problem with (44) is that there is $x^{-\frac{1}{3}}$ on the left hand side. If $x = 0$, this results in division by 0. As $x(0) = 0$ is the initial condition, this unbounded point becomes a problem for the separation-of-variables technique.

(e) **Write an example of a differential equation that satisfies this uniqueness condition, and explain why.**

Solution: The simplest example would be $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t)$ that we introduced earlier. Setting $f(x) = \lambda x(t)$, the derivative $\frac{\mathrm{d}}{\mathrm{d}x}f(x) = \lambda$ is continuous and bounded.

[3] Indeed, any solution of the form

$$x(t) = \begin{cases} 0, & \text{if } t < t_0 \\ \left(\frac{2}{3}\alpha\,(t - t_0)\right)^{\frac{3}{2}}, & \text{if } t \geq t_0 \end{cases} \tag{50}$$

also satisfies (40) and the initial condition $x(0) = 0$, for any $t_0 > 0$, which concludes that (40) has infinitely many solutions. We leave the verification of this solution for those who are interested.

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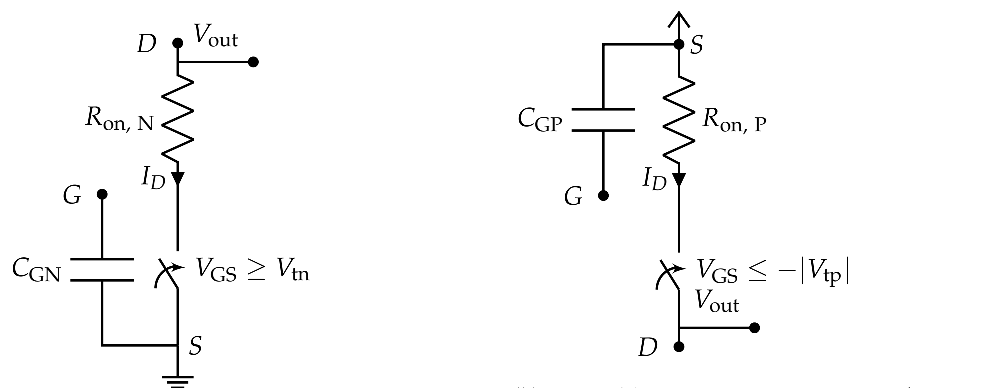
8. Transistor Switch Model

Unlocked by Lectures 2 and 3

We can improve our resistor-switch model of the transistor by adding in a gate capacitance. In this model, the gate capacitances C_{GN} and C_{GP} represent the lumped physical capacitance present on the gate node of all transistor devices. This capacitance is important as it determines the delay of a transistor logic chain.



(a) NMOS Transistor Resistor-switch-capacitor model

(b) PMOS Transistor Resistor-switch-capacitor model. Note we have drawn this so that it aligns with the inverter.

You have two CMOS inverters made from NMOS and PMOS devices. Both NMOS and PMOS devices have an "on resistance" of $R_{\text{on, N}} = R_{\text{on, P}} = 1\,\text{k}\Omega$, and each has a gate capacitance (input capacitance) of $C_{\text{GN}} = C_{\text{GP}} = 1$ fF (fF = femto-Farads = 1×10^{-15} F). We assume the "off resistance" (the resistance when the transistor is off) is infinite (i.e., the transistor acts as an open circuit when off). The supply voltage V_{DD} is 1V. The two inverters are connected in series, with the output of the first inverter driving the input of the second inverter (Figure 12).

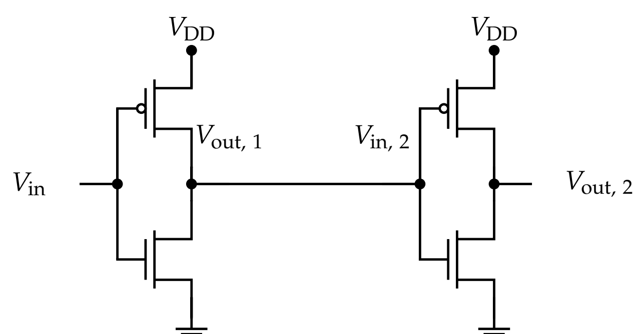


Figure 12: CMOS Inverter chain

(a) Assume the input to the first inverter has been low ($V_{\text{in}} = 0$ V) for a long time, and then switches at time $t = 0$ to high ($V_{\text{in}} = V_{\text{DD}}$).

Draw a simple RC circuit and write a differential equation describing the output voltage of the first inverter ($V_{\text{out, 1}}$) for time $t \geq 0$.

Don't forget that the second inverter is "loading" the output of the first inverter — you need to think about both of them.

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Solution: To analyze this circuit as an RC circuit we can recall the transistor switch model. Using this we can see that the first inverter's output appears as a resistor connected to V_{DD} when the input is low (NMOS off, PMOS on), or a resistor connected to ground when the input turns high (NMOS on, PMOS off).

Before $t = 0$, the input to the first inverter was low for a long time. This means that for $t < 0$, the output of the inverter ($V_{\text{out, 1}}$) had been held at V_{DD} for a long time.

At $t = 0$, the input goes high, which means that the input inverter's NMOS device turns on, connecting $V_{\text{out, 1}}$ to ground through a resistance of R_{on}.

The second inverter "loads" the output of the first inverter. From the notes in the problem, we can model the gates of the transistors as capacitors. These gates together form our capacitive load. The gate of the PMOS acts as a capacitor to V_{DD} and the gate of the NMOS acts as a capacitor to ground.

Using this we can draw the following RC circuit:

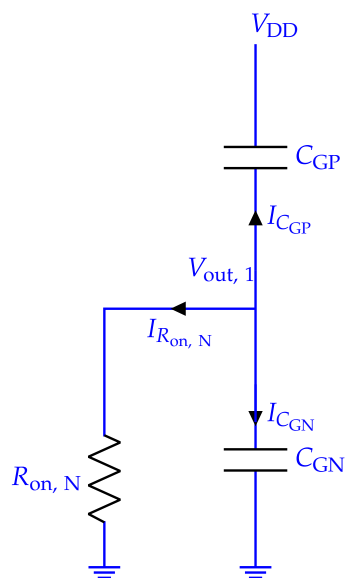

Figure 13: First inverter output at 0

To get the differential equation describing the output of the first inverter at time $t \geq 0$ let us first think about the behavior of the circuit at and after $t = 0$.

Before $t = 0$ we know that the output $V_{\text{out, 1}} = V_{DD}$. This means that C_{GN} is charged, while C_{GP} is not as there is no voltage difference across it.

At $t = 0$, when the input to the first inverter changes (input switches to high), the NMOS will turn on, discharging the $V_{\text{out, 1}}$ node. Thus $V_{\text{out, 1}}$ will eventually discharge to zero in steady state.

We know the voltage across C_{GP} is $V_{\text{out, 1}}(t) - V_{DD}$ and the voltage across C_{GN} is $V_{\text{out, 1}}(t)$. Using this information we can set up a differential equation to solve for $V_{\text{out}}(t)$.

Writing the expressions for the three branch currents yields:

$$I_{C_{GP}} = C_{GP}\frac{\mathrm{d}}{\mathrm{d}t}(V_{\text{out, 1}}(t) - V_{DD}) \tag{52}$$

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$$I_{C_{\text{GN}}} = C_{\text{GN}} \frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out, 1}}(t) \tag{53}$$

$$I_{R_{\text{on, N}}} = \frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}} \tag{54}$$

Writing KCL at the single node yields:

$$I_{C_{\text{GP}}} + I_{C_{\text{GN}}} + I_{R_{\text{on, N}}} = 0 \tag{55}$$

in other words:

$$I_{C_{\text{GP}}} + I_{C_{\text{GN}}} = -I_{R_{\text{on, N}}} \tag{56}$$

Expanding the branch currents with their expressions:

$$C_{\text{GP}} \frac{\mathrm{d}}{\mathrm{d}t}(V_{\text{out, 1}}(t) - V_{\text{DD}}) + C_{\text{GN}} \frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out, 1}}(t) = -\frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}} \tag{57}$$

$$C_{\text{GP}} \frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out, 1}}(t) + C_{\text{GN}} \frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out, 1}}(t) = -\frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}} \tag{58}$$

$$(C_{\text{GP}} + C_{\text{GN}}) \frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out, 1}}(t) = -\frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}} \tag{59}$$

Re-writing as a first-order differential equation for $V_{\text{out, 1}}$ yields:

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out, 1}}(t) = -\frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}(C_{\text{GP}} + C_{\text{GN}})} \tag{60}$$

(b) Given the initial conditions in part (a), **solve for $V_{\text{out, 1}}(t)$.**

Solution: We know that the solution to a differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out, 1}}(t) = -\frac{V_{\text{out, 1}}}{R_{\text{on, N}}(C_{\text{GP}} + C_{\text{GN}})} \tag{61}$$

is

$$V_{\text{out, 1}}(t) = k e^{-\frac{t}{R_{\text{on, N}}(C_{\text{GP}} + C_{\text{GN}})}} \tag{62}$$

Plugging in the initial condition $V_{\text{out, 1}}(0) = V_{\text{DD}}$ we find that $V_{\text{out, 1}}(t) = V_{\text{DD}} e^{-\frac{t}{R_{\text{on, N}}(C_{\text{GP}} + C_{\text{GN}})}}$.

(c) **Sketch the output voltage of the first inverter, showing clearly (1) the initial value, (2) the initial slope, (3) the asymptotic value, and (4) the time that it takes for the voltage to decay to roughly 1/3 of its initial value.**

Solution:

(1) We know that the output of our inverter started with the initial value V_{DD}.

(2) Since the differential equation tells us the change in value of $V_{\text{out, 1}}(t)$ at time t we can simply plug in $t = 0$ into our differential equation to get the initial slope:

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out, 1}}(t) = -\frac{V_{\text{out, 1}}(0)}{R_{\text{on, N}}(C_{\text{GP}} + C_{\text{GN}})} \tag{63}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out, 1}}(t) = -\frac{V_{\text{DD}}}{R_{\text{on, N}}(C_{\text{GP}} + C_{\text{GN}})} \tag{64}$$

Thus the initial slope is $-\frac{V_{\text{DD}}}{R_{\text{on, N}}(C_{\text{GP}} + C_{\text{GN}})}$.

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(3) Since the input to the inverter changed from low to high we know the output of the first inverter ($V_{\text{out, 1}}$) is going to go to 0 in steady state, as this node will be discharged by the first inverter's NMOS transistor.
Alternatively, we can find the asymptotic value by plugging in $t = \infty$ to the solution we found for $V_{\text{out, 1}}(t)$ to find $V_{\text{out, 1}} = V_{\text{DD}} e^{-\frac{\infty}{R_{\text{on, N}}(C_{\text{GP}}+C_{\text{GN}})}} = 0$.

(4) To approximate when the output will decay to $\frac{1}{3}$ its original value, we use the fact that $e^{-1} = \frac{1}{e} \approx \frac{1}{3}$. We thus want to find when $V_{\text{out, 1}} = V_{\text{DD}} e^{-1}$.
This will occur when the e term is raised to -1, which occurs when $t = R_{\text{on, N}}(C_{\text{GP}} + C_{\text{GN}}) = 2 \times 10^{-12}$s.
You should also give yourself full credit if you used $\frac{1}{3}$ itself and computed $-\ln(3)$, etc.

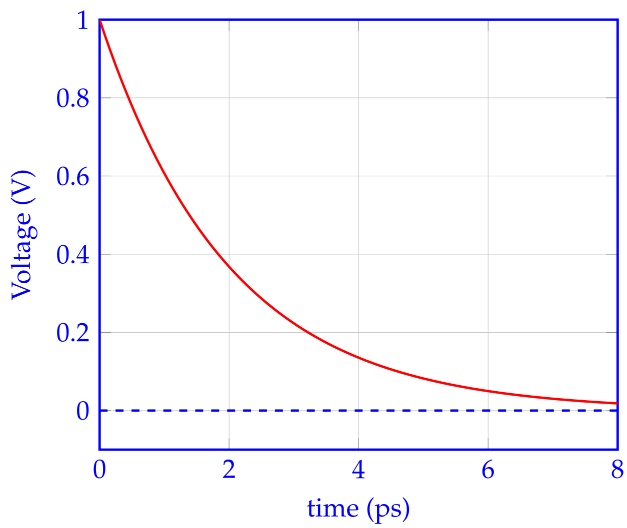

Figure 14

(d) A long time later, the input to the first inverter switches low again.
Solve for $V_{\text{out, 1}}(t)$.
Sketch the output voltage of the first inverter ($V_{\text{out, 1}}$), showing clearly (1) the initial value, (2) the initial slope, and (3) the asymptotic value.
Solution: We know that after a long time, the output of the first inverter has stabilized to 0. When the input switches low again, the input inverter's NMOS device turns off, while the input inverter's PMOS device turns on. This connects the $V_{\text{out, 1}}$ node to V_{DD}, as shown in Figure 15.

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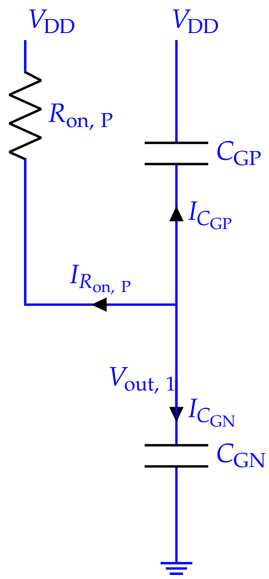

Figure 15: Inverter output at 1

To set up the differential equation, we apply KVL and KCL again:

$$I_{C_{GP}} = C_{GP}\frac{d}{dt}(V_{out,\,1}(t) - V_{DD}) \tag{65}$$

$$I_{C_{GN}} = C_{GN}\frac{d}{dt}V_{out,\,1}(t) \tag{66}$$

$$I_{R_{on,\,P}} = \frac{V_{out,\,1}(t) - V_{DD}}{R_{on,\,P}} \tag{67}$$

$$I_{C_{GP}} + I_{C_{GN}} = -I_{R_{on,\,P}} \tag{68}$$

$$C_{GP}\frac{d}{dt}(V_{out,\,1}(t) - V_{DD}) + C_{GN}\frac{d}{dt}V_{out,\,1}(t) = -\frac{V_{out,\,1}(t) - V_{DD}}{R_{on,\,P}} \tag{69}$$

$$C_{GP}\frac{d}{dt}V_{out,\,1}(t) + C_{GN}\frac{d}{dt}V_{out,\,1}(t) = -\frac{V_{out,\,1}(t) - V_{DD}}{R_{on,\,P}} \tag{70}$$

$$(C_{GP} + C_{GN})\frac{d}{dt}V_{out,\,1}(t) = -\frac{V_{out,\,1}(t) - V_{DD}}{R_{on,\,P}} \tag{71}$$

$$\frac{d}{dt}V_{out,\,1}(t) = -\frac{V_{out,\,1}(t) - V_{DD}}{R_{on,\,P}(C_{GP} + C_{GN})} \tag{72}$$

We will use substitution of variables:

$$x(t) = V_{out,\,1}(t) - V_{DD} \tag{73}$$

$$V_{out,\,1}(t) = x(t) + V_{DD} \tag{74}$$

$$\frac{d}{dt}x(t) = \frac{d}{dt}V_{out,\,1}(t) \tag{75}$$

Substituting in:

$$\frac{d}{dt}x(t) = -\frac{x}{R_{on,\,P}(C_{GP} + C_{GN})} \tag{76}$$

$$x(t) = Ae^{-\frac{t}{R_{on,\,P}(C_{GP}+C_{GN})}} \tag{77}$$

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Substituting again for $x(t)$:

$$V_{\text{out, 1}}(t) = V_{\text{DD}} + A\mathrm{e}^{-\frac{t}{R_{\text{on, P}}(C_{\text{GP}}+C_{\text{GN}})}} \tag{78}$$

Using the initial condition $V_{\text{out, 1}} = 0$ (as the input to the first inverter was high for a long time before switching low) implies $A = -V_{\text{DD}}$. Thus:

$$V_{\text{out, 1}}(t) = V_{\text{DD}}\left(1 - \mathrm{e}^{-\frac{t}{R_{\text{on, P}}(C_{\text{GP}}+C_{\text{GN}})}}\right) \tag{79}$$

(1) Because the input to the first inverter was high for a long time, we know the initial value of $V_{\text{out, 1}}(t) = 0$. This was the initial condition applied to the solution of the differential equation, above.

(2) To find the initial value of the slope we can plug in $t = 0$ to the above differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{\text{out, 1}}(t) = \frac{(V_{\text{DD}} - V_{\text{out, 1}}(0))}{R_{\text{on, P}}(C_{\text{GP}} + C_{\text{GN}})} \tag{80}$$

where $V_{\text{out, 1}}(0) = 0$. Thus our initial slope is $\frac{(V_{\text{DD}})}{R_{\text{on, P}}(C_{\text{GP}}+C_{\text{GN}})}$. Notice this slope is positive while the previous part had a negative slope.

(3) Since the input to the inverter changed from low to high and the input inverter's PMOS is now on, we know the output of the first inverter is going to go to V_{DD} in steady state.

(4) Alternatively, we can find the asymptotic value by plugging in $t = \infty$ to the solution we found for $V_{\text{out, 1}}(t)$ to find $V_{\text{out, 1}} = V_{\text{DD}}\left(1 - \mathrm{e}^{-\frac{\infty}{R_{\text{on, P}}(C_{\text{GP}}+C_{\text{GN}})}}\right) = V_{\text{DD}}(1-0) = V_{\text{DD}}$.

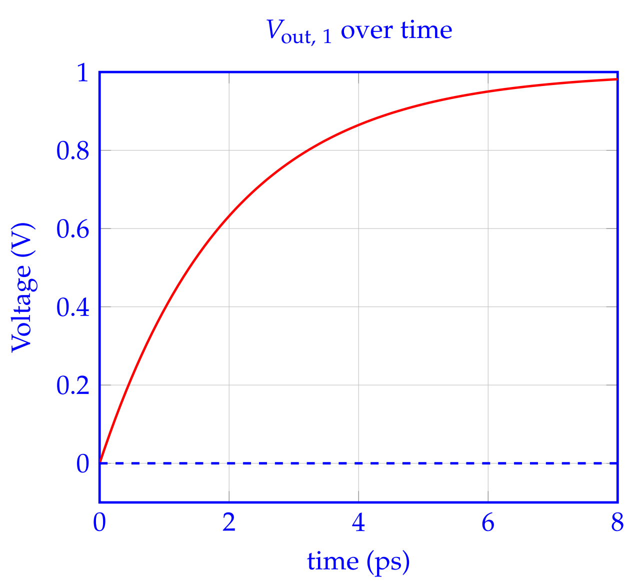

Figure 16

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9. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **Roughly how many total hours did you work on this homework?**
This is to evaluate if homework length is reasonable.

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