# Homework 2

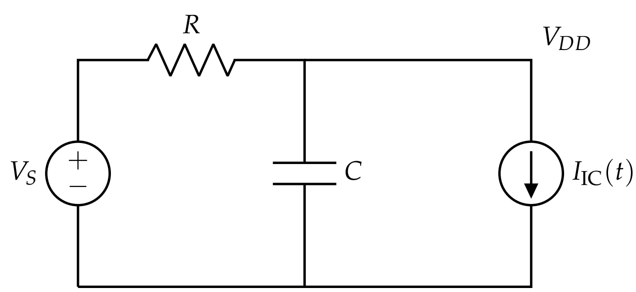
## This homework is due on Sunday, July 3rd at 11:59 pm PT

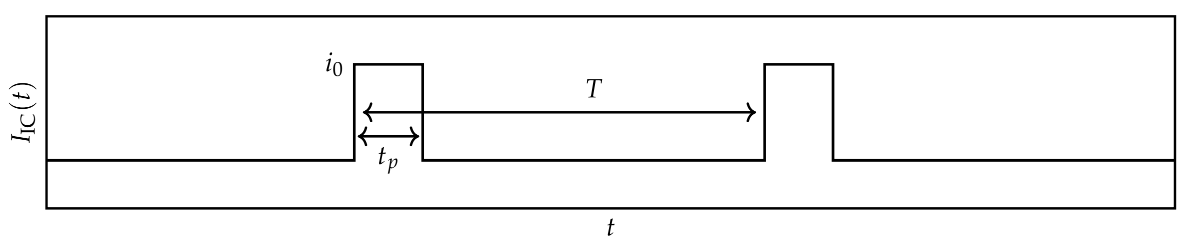
**1. IC Power Supply**

*Unlocked by Lectures 3 and 4*

Digital integrated circuits (ICs) often have very non-uniform current requirements which can cause voltage noise on the supply lines. If one IC is adding a lot of noise to the supply line, it can affect the performance of other ICs that use the same power supply, which can hinder performance of the entire device. For this reason, it is important to take measures to mitigate, or "smooth out", the power supply noise that each IC creates. A common way of doing this is to add a "supply capacitor" between each IC and the power supply. (If you look at a circuit board, and the supply capacitor is the small capacitor next to each IC.)

Here's a simple model for a power supply and digital circuit:




The current source is modeling the "spiky," non-uniform nature of digital circuit current consumption. 'The resistor represents the sum of the source resistance of the supply and any wiring resistance between the supply and the load.

The capacitor is added to minimize the noise on $V_{DD}$. Assume that $V_S = 3\,\text{V}$, $R = 1\,\Omega$, $i_0 = 1\,\text{A}$, $T = 11\,\text{ns}$, and $t_p = 1\,\text{ns}$.

(a) **Sketch the voltage $V_{DD}$ vs. time for two $T$ periods assuming that $C = 0$.**

(b) **Give expressions for and sketch the voltage $V_{DD}$ vs. time for two $T$ periods for each of three different capacitor values for $C$: $1\,\text{pF}$ , $1\,\text{nF}$, $1\,\mu\text{F}$. ($1\,\text{pF} = 1 \times 10^{-12}\,\text{F}$, $1\,\text{nF} = 1 \times 10^{-9}\,\text{F}$, $1\,\mu\text{F} = 1 \times 10^{-6}\,\text{F}$).** For this part, to find the initial condition for $V_{DD}$, feel free to assume that for a very long time, $I_{IC} = 0$.

(c) Launch the attached Jupyter notebook to interact with a simulated version of this IC power supply. Try to simulate the scenarios outlined in the previous parts. For one of these scenarios, keep the RC time constant fixed, but vary the relative value of $R$ vs. $C$ (e.g. compare $R = \texttt{1}, C = \texttt{2e-9}$ to the case where $R = \texttt{2}, C = \texttt{1e-9}$). **Is it better to have a lower $R$ or lower $C$ value for a fixed RC time constant when attempting to minimize supply noise? Give an intuitive explanation for why this might be the case.**

Be sure to play with the y limits on the graph as well as how long the simulation runs to best understand what is going on here.

**2. Simple Scalar Differential Equations Driven by an Input**

*Unlocked by Lecture 4*

In this question, we will show the existence and uniqueness of solutions to differential equations with inputs. In particular, we consider the scalar differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu(t) \tag{1}$$

$$x(0) = x_0 \tag{2}$$

where $u\colon \mathbb{R} \to \mathbb{R}$ is a known function of time. Feel free to assume $u$ is "nice" in the sense that it is integrable, continuous, and differentiable with bounded derivative – basically, let $u$ be nice enough that all the usual calculus theorems work.

(a) We will first demonstrate the existence of a solution to eqs. (1) and (2).

Define $x_d\colon \mathbb{R} \to \mathbb{R}$ by

$$x_d(t) := \mathrm{e}^{\lambda t}x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau\,. \tag{3}$$

**Show that $x_d$ satisfies eqs. (1) and (2).**

*(HINT: When showing that $x_d$ satisfies eq. (1), one possible approach to calculate the derivative of the integral term is to use the fundamental theorem of calculus and the product rule.)*

(b) Now, we will show that $x_d$ is the unique solution to eqs. (1) and (2).

Suppose that $y\colon \mathbb{R} \to \mathbb{R}$ also satisfies eqs. (1) and (2). **Show that $y(t) = x_d(t)$ for all $t$.**

*(HINT: This time, show that $z(t) := y(t) - x_d(t) = 0$ for all $t$. Do this by showing that $z(0) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}z(t) = \lambda z(t)$, then use the uniqueness theorem for homogeneous first-order linear differential equations from the last homework. Note that the specific form of $x_d(t)$ in eq. (3) is irrelevant for the solution and should not be used.)*

(c) In this part, we will calculate some values of $x_d$ for common values of $u$.

i. If $u(t) := u$ is a constant function, **what is $x_d(t)$?**

ii. If $u(t) := \mathrm{e}^{\alpha t}$ for some real number $\alpha \neq \lambda$, **what is $x_d(t)$?**

iii. If $u(t) := \mathrm{e}^{\lambda t}$, **what is $x_d(t)$?**

*NOTE*: Assume for simplicity that $\lambda \neq 0$.

**3. Tracking Terry**

*Unlocked by Lecture 5*

Terry is a mischievous child, and his mother is interested in tracking him.

(a) Terry texts his current location as a vector $\vec{x}_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, but there is a problem! These coordinates are *not* in the standard basis, but rather in the basis $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. That is to say that the first number 2 above is how many multiples of $\vec{v}_1$ to use and the second number 3 is how many multiples of $\vec{v}_2$ to use in computing his actual location. Here, both $\vec{v}_1$ and $\vec{v}_2$ are vectors in the standard basis.

**Let Terry's location in the standard basis be $\vec{x}$. Write $\vec{x}$ in terms of $\vec{v}_1$ and $\vec{v}_2$.**

(b) Terry's friend tells you that Terry's location in the standard basis is $\vec{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Using this along with the previous info that Terry's location in the $V$ basis is $\vec{x}_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, **is it possible to determine the basis vectors $\vec{v}_1, \vec{v}_2$ Terry is using. If it is impossible to do so, explain why.**
(HINT: How many unknowns do you have? How many equations?)

(c) Terry's basis vectors $\vec{v}_1, \vec{v}_2$ get leaked to his mom on accident, so she knows they are

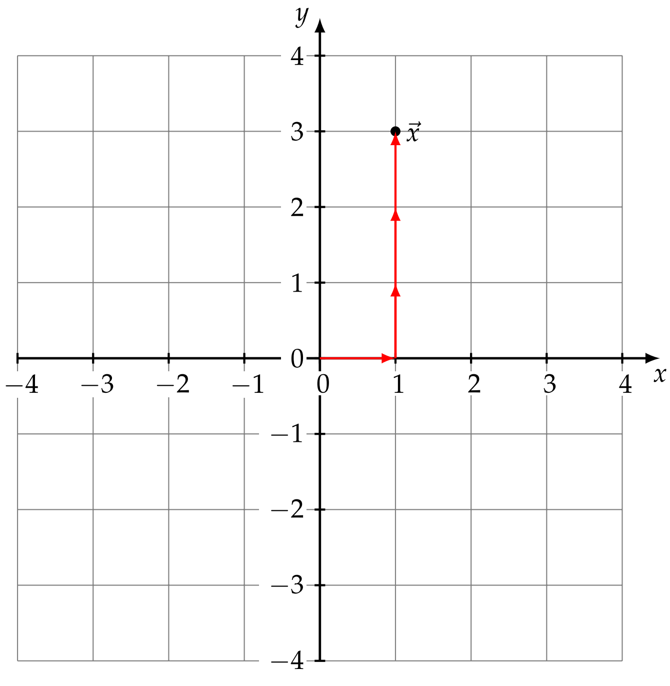
$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}. \tag{4}$$

To hide his location, Terry wants to switch to a new coordinate system with the basis vectors

$$\vec{p}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5}$$

In order to do this, he needs a way to convert coordinates from the $V$ basis to the $P$ basis. Thus, **find the matrix $T$ such that if $\vec{x}_v$ is a location expressed in $V$ coordinates and $\vec{x}_p$ is the same location expressed in $P$ coordinates, then $\vec{x}_p = T\vec{x}_v$.**

(d) Terry now wants to make a map and route to where he currently is, $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. **For both the $P$ and $V$ bases from part 3.c, illustrate the sum of scaled basis vectors that are necessary to go from the origin to $\vec{x}$.** An example is shown below when using the standard basis. This illustrates that the same location can be represented by many different coordinate systems/bases.


y
4
3
2
1
0
−1
−2
−3
−4
$\vec{x}$
−4 −3 −2 −1 0 1 2 3 4
x

**4. Eigenvectors and Diagonalization**

*Unlocked by Lecture 6*

(a) Let $A$ be an $n \times n$ matrix with $n$ linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define $V$ to be a matrix with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ as its columns, $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$.

**Show that $AV = V\Lambda$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a diagonal matrix with the eigenvalues of $A$ as its diagonal entries.**

(b) **Argue that $V$ is invertible, and therefore**

$$A = V\Lambda V^{-1}. \tag{6}$$

(Hint: Why is $V$ invertible? It is fine to cite the appropriate result from 16A.)

(c) **Write $\Lambda$ in terms of the matrices $A$, $V$, and $V^{-1}$.**

(d) A matrix $A$ is deemed diagonalizable if there exists a square matrix $U$ so that $A$ can be written in the form $A = UDU^{-1}$ for the choice of an appropriate diagonal matrix $D$.

**Show that the columns of $U$ must be eigenvectors of the matrix $A$, and that the entries of $D$ must be eigenvalues of $A$.**

(HINT: What does it mean to be an eigenvector? What is $U^{-1}U$? How does matrix multiplication work column-wise?)

The previous part shows that the *only* way to diagonalize $A$ is using its eigenvalues/eigenvectors. Now we will explore a payoff for diagonalizing $A$ – an operation that diagonalization makes *much* simpler.

(e) For a matrix $A$ and a positive integer $k$, we define the exponent to be

$$A^k = \underbrace{A \cdot A \cdot \cdots \cdot A \cdot A}_{k \text{ times}} \tag{7}$$

Let's assume that matrix $A$ is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ (i.e. the $n$ eigenvectors are all linearly independent).

**Show that $A^k$ has eigenvalues $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ and eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Conclude that $A^k$ is diagonalizable.**

**5. Vector Differential Equations**

*Unlocked by Lecture 6*

Note: it's recommended to finish Question 4 (Eigenvectors and Diagonalization) before this problem.

Consider a system of ordinary differential equations that can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) := \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\vec{x}(t) \tag{8}$$

where $x_1, x_2 : \mathbb{R} \to \mathbb{R}$ are scalar functions of time $t$, and $A \in \mathbb{R}^{2\times 2}$ is a $2 \times 2$ matrix with constant coefficients. We call eq. (8) a vector differential equation.

(a) Suppose we have a system of ordinary differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 7x_1 - 8x_2, \tag{9}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 4x_1 - 5x_2, \tag{10}$$

Here, we denote $x_1(t), x_2(t)$ as $x_1, x_2$ for notational simplicity.

**Find an appropriate matrix $A$ to write this system in the form of eq. (8). Compute the eigenvalues of $A$.** Denote the smaller and larger eigenvalues by $\lambda_1$ and $\lambda_2$ respectively, so that $\lambda_1 \leq \lambda_2$.

(b) **Compute the eigenvectors of $A$. Is A diagonalizable? Why?**

(c) We now transform our system (eq. (9) and eq. (10)) in $x$ coordinates, to new coordinates $z$ to simplify our system of differential equations. **What basis $V$ should we use so that in the new coordinates $\vec{z} = V^{-1}\vec{x}$, the $\Lambda$ matrix in the equation $\frac{\mathrm{d}\vec{z}(t)}{\mathrm{d}t} = \Lambda\vec{z}(t)$ is diagonal? Write out this new system in the $\vec{z}$ coordinates.**

(d) **Solve the new system in the $\vec{z}$ coordinates, using the initial conditions that $x_1(0) = 1, x_2(0) = -1$.**

(e) Now **convert your solution from the $\vec{z}$ coordinates back to the original $\vec{x}$ coordinates.** In other words, give us the functions $x_1(t)$ and $x_2(t)$.

(f) It turns out that we can actually turn all higher-order differential equations with constant coefficients into vector differential equations of the style of eq. (8).

Consider a second-order ordinary differential equation

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + a\frac{\mathrm{d}y(t)}{\mathrm{d}t} + by(t) = 0, \tag{11}$$

where $a, b \in \mathbb{R}$.

**Write this differential equation in the form of (eq. (8)), by choosing appropriate variables $x_1(t)$ and $x_2(t)$.**

*(HINT: Your original unknown function $y(t)$ has to be one of those variables. The heart of the question is to figure out what additional variable can you use so that you can express eq. (11) without having to take a second derivative, and instead just taking the first derivative of something. This is another manifestation of the larger thought pattern of "lifting.")*

(g) It turns out that all two-dimensional vector linear differential equations with distinct eigenvalues have a solution in the general form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_1 t} + c_1 e^{\lambda_2 t} \\ c_2 e^{\lambda_1 t} + c_3 e^{\lambda_2 t} \end{bmatrix} \tag{12}$$

where $c_0, c_1, c_2, c_3$ are constants, and $\lambda_1, \lambda_2$ are the eigenvalues of $A$ (this can be proven by just repeating the same steps in the previous parts and using the fact that distinct eigenvalues implies linearly independent eigenvectors). Thus, an alternate way of solving this type of differential equation in the future is to now use your knowledge that the solution is of this form and just solve for the constants $c_i$.

Now let $a = -1$ and $b = -2$ in eq. (11), i.e.

$$\frac{d^2 y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = 0, \tag{13}$$

**Verify that eq. (13) has a solution in the general form eq. (12). Solve eq. (13) with the initial conditions $y(0) = 1$, $\frac{dy}{dt}(0) = 1$, using this method.**

(HINT: You get two equations using the initial conditions above. How many unknowns are here?) (SECOND HINT: Given your specific choice of $x_1$ and $x_2$ in part (f), how many unknowns are there really?)

**6. Uniqueness justification for solutions to matrix/vector differential equations**

*Unlocked by Lectures 6 and 7*

In general, we have seen that we need to justify our methods of solving differential equations with a uniqueness proof. This is important as it allows us to trust our solution as being the only one for the problem at hand.

Consider matrix-vector differential equations of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{14}$$

with some initial condition $\vec{x}(0) = \vec{x}_0$.

All the uniqueness proofs that you have done for yourself have been concerned with scalar differential equations, and scalar differential equations driven by inputs. So, why can we trust the solutions that we are getting for such matrix-vector differential equations?

This question takes us part of the way to the answer.

(a) Suppose that the $n \times n$ matrix $A$ has $n$ distinct eigenvalues and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, so that the matrix $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix}$ has linearly independent columns.

**Find the diagonalized system corresponding to (14). Show that if you are given any valid solution for the original system (14), you can change coordinates to the eigenbasis and also get a valid solution for the diagonalized system. List the new initial conditions that are satisfied in the diagonalized system.**

(b) You have already proved the uniqueness of solutions for any scalar differential equation of the form $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t)$ with specified initial condition $x(0) = x_0$. How can you use this fact and the result of the previous part to argue that the solution must be unique for the matrix/vector differential equation?

*(HINT: (Start by assuming that you have two solutions to the original problem. Use the result of part a) to formulate solutions to the transformed, diagonal problem. Apply the uniqueness results for scalar ODEs to the solutions of the diagonalized problem. Finally conclude by considering what the invertibility of the transformation matrix V implies about the two solutions of the original problem?))*

We will see later in the course how the assumption we made on the eigenvectors of $A$ is not actually needed for this proof to hold. But for now, it is important to understand this case first.

**7. (PRACTICE) Solving the Differential Equation with Input**

*Unlocked by Lectures 3, 4, and Discussion 2A*

Recall that in Discussion 2A we tried to solve the differential equation with input:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu_c(t) \tag{15}$$

$$x(0) = x_0. \tag{16}$$

for some continuous input $u_c(t)$.

The general strategy we employ is:

- First we replace our continuous input $u_c(t)$ with an input $u(t)$ which is piecewise constant on the intervals $[i\Delta, (i+1)\Delta)$, that is,

$$u(t) = u(i\Delta) = u[i] \qquad t \in [i\Delta, (i+1)\Delta) \qquad i \in \{0, 1, 2, \ldots\} := \mathbb{N}. \tag{17}$$

  Using this assumption, in discussion we:

  – solved the differential equation on each interval $[i\Delta, (i+1)\Delta)$ and got a solution expressing $x(t)$ in terms of $x_d[i] := x(i\Delta)$ and $u[i]$, for $t \in [i\Delta, (i+1)\Delta)$;

  – arrived at a formula for $x_d[i+1]$ in terms of $x_d[i]$ and $u[i]$;

  – used this to get a formula for $x_d[i]$ in terms of $x_0$ and the inputs $u[0], u[1], \ldots, u[i-1]$;

  – approximated $x(t) \approx x_d[\lfloor \frac{t}{\Delta} \rfloor]$ to recover an approximate value for $x(t)$, that is,

$$x(t) \approx \left(\mathrm{e}^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} x_0 + b\frac{\mathrm{e}^{\lambda\Delta} - 1}{\lambda} \sum_{k=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} \left(\mathrm{e}^{\lambda\Delta}\right)^{(\lfloor \frac{t}{\Delta} \rfloor - 1) - k} u[k]. \tag{18}$$

- In this homework, we will take the limit $\Delta \to 0$. This transfers back from $u$ to $u_c$ – we saw in discussion that piecewise constant functions on very small intervals, i.e., our $u$, approximate general continuous functions $u_c$ arbitrarily well. Using Riemann sums and calculus, we will turn the sum into an integral and show that, if $u$ approximates $u_c$ as $\Delta \to 0$, then

$$x(t) = \mathrm{e}^{\lambda t} x_0 + b \int_0^t \mathrm{e}^{\lambda(t-\tau)} u_c(\tau)\, \mathrm{d}\tau\,. \tag{19}$$

(a) We first need to relate $u[i]$ to $u_c$. Suppose that the $u[i]$ is a sample of $u_c(t)$, namely,

$$u[i] = u_c(i\Delta). \tag{20}$$

To clarify where this fits in with the earlier notation:

- $u(t)$ is a piecewise constant function;
- $u[i]$ is the discrete input that constructs $u(t)$ based on eq. (17);
- and $u_c(t)$ is the underlying input $u[i]$ is sampled from based on eq. (20).

This is one good way to get a piecewise constant approximator of a continuous function.

**Substitute an appropriate value of $u_c$ for $u[k]$ in eq. (18) from the discussion.**

*NOTE*: Don't take any limits in this part of the problem; just do the substitution.

(b) To simplify our (discrete-time) eq. (18) so we can take $\Delta \to 0$, we would like to make some approximations which are valid for small $\Delta$.

**By using the following two estimates for small $\Delta$:**[1]

i. $\lfloor \frac{t}{\Delta} \rfloor \approx \frac{t}{\Delta}$;

ii. $\frac{e^{\lambda\Delta}-1}{\lambda} \approx \Delta$;[2]

**show that**

$$x(t) \approx e^{\lambda t} x_0 + b e^{-\lambda\Delta} \sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)} u_c(k\Delta)\Delta. \tag{21}$$

(c) **Take the limit of $x(t)$ as $\Delta \to 0$, and show that $x(t)$ is given by eq. (19).**

Recall that the definite integral is defined from Riemann sums as

$$\int_0^t f(\tau)\,d\tau = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(\tau_k^*)\Delta_k \tag{22}$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_n = t$, $\tau_k^* \in [\tau_k, \tau_{k+1}]$, and $\Delta_k = \tau_{k+1} - \tau_k$. The $\Delta_k$ is the length of the base of the rectangles and the $f(\tau_k^*)$ are the heights. As $n$ goes to infinity, the rectangles get skinnier and skinnier, but there are more and more of them.

*(HINT: Start with eq. (21) and take limits on both sides. What is n? What is $\tau_k$ and $\tau_k^*$? What is $\Delta_k$? What is f?)*

*(HINT: We chose the form of eq. (21) carefully; it turns out that $\Delta_k$ is one particular term involving $\Delta$ that goes to 0 as $\Delta \to 0$, and also that it is independent of k.)*

This problem brings together many different concepts and uses a lot of notation. As such, it may be difficult to fully comprehend everything the first time. Being able to grind through complex mathematical problems like this is part of the vaunted "mathematical maturity" that this class helps you foster. As the semester continues, you will find that these kinds of problems will seem progressively easier, both to understand quickly and to solve. But it won't happen without practice.

[1] Both these approximations become equalities in the limit $\Delta \to 0$.

[2] We can see this approximation using Taylor's theorem from calculus.

**8. Homework Process, Study Group, and Course Weekly Survey**

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

At the same time, we want to check-in weekly regarding Discussions, Lectures, Lab, and Office Hours and see how effective they have all been for you as students.

**Please fill out this survey link. For your submission, please attach a screenshot indicating that you have completed the survey this week.**

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