Homework 2

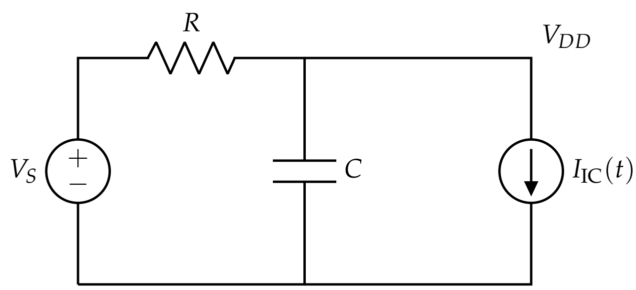
This homework is due on Sunday, July 3rd at 11:59 pm PT

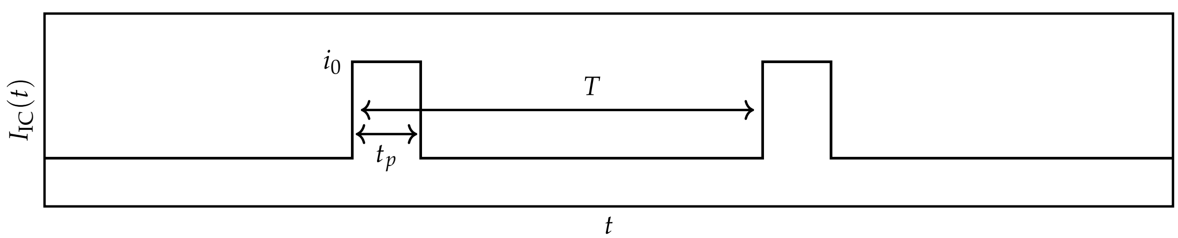
1. IC Power Supply

Unlocked by Lectures 3 and 4

Digital integrated circuits (ICs) often have very non-uniform current requirements which can cause voltage noise on the supply lines. If one IC is adding a lot of noise to the supply line, it can affect the performance of other ICs that use the same power supply, which can hinder performance of the entire device. For this reason, it is important to take measures to mitigate, or "smooth out", the power supply noise that each IC creates. A common way of doing this is to add a "supply capacitor" between each IC and the power supply. (If you look at a circuit board, and the supply capacitor is the small capacitor next to each IC.)

Here's a simple model for a power supply and digital circuit:

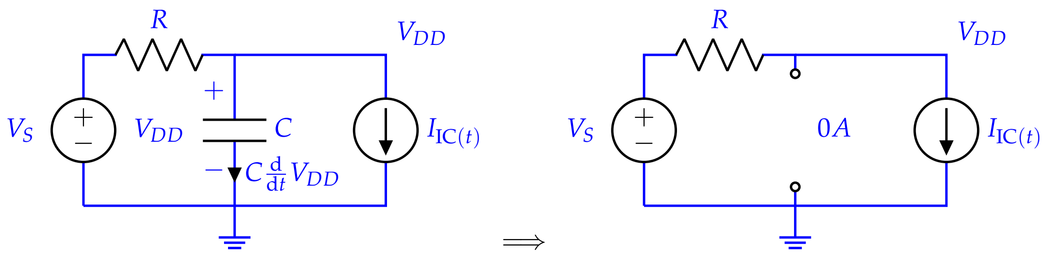




The current source is modeling the "spiky," non-uniform nature of digital circuit current consumption. 'The resistor represents the sum of the source resistance of the supply and any wiring resistance between the supply and the load.

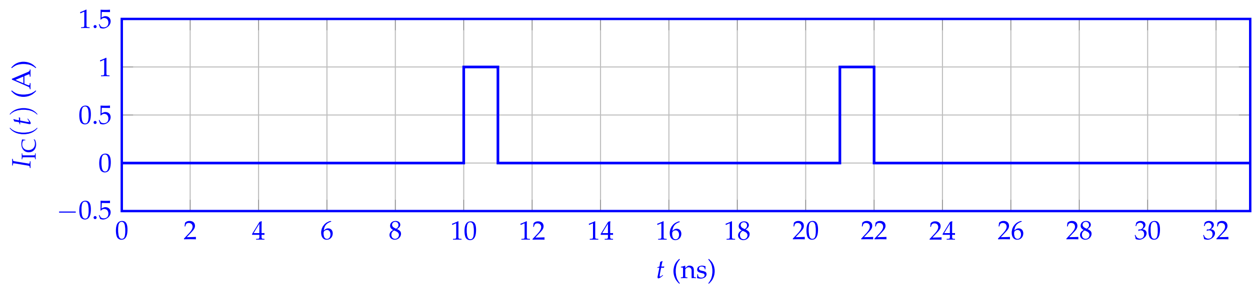
The capacitor is added to minimize the noise on V_{DD}. Assume that $V_S = 3\,\text{V}$, $R = 1\,\Omega$, $i_0 = 1\,\text{A}$, $T = 11\,\text{ns}$, and $t_p = 1\,\text{ns}$.

(a) **Sketch the voltage V_{DD} vs. time for two T periods assuming that $C = 0$.** **Solution:**

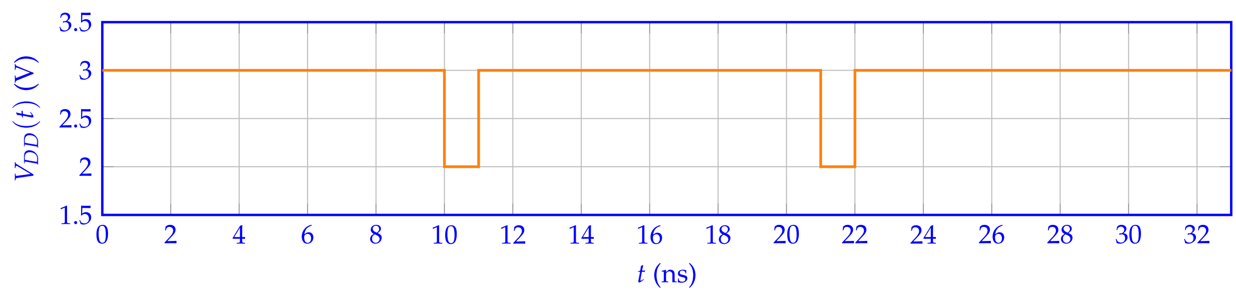


If $C = 0$, the capacitor behaves as an open circuit with $I_C = C\frac{d}{dt}V_{DD} = 0A$. This circuit will respond instantaneously to changes in the current; thus we may break this down into two different time segments, wherein the current source $I_{IC}(t)$ equals 0 (and thus $V_{DD} = V_S$ as no current flows through the resistor), and where the current source $I_{IC}(t)$ equals i_0 (and thus $V_{DD} = V_S - i_0 R = 2\,\text{V}$). These will follow the current source's flips precisely. With that in mind, your sketch should look something like this:

Transient waveform of current drawn by IC



Transient waveform of IC supply voltage



In the above waveform, we have the first current spike at $t = 10\,\text{ns}$. Yours doesn't have to align with that: for example, if you had the first current spike at $t = 0$, that's okay. However, what *does* matter is that you have the timing between the current spikes drawn correctly.

(b) **Give expressions for and sketch the voltage V_{DD} vs. time for two T periods for each of three different capacitor values for C: 1 pF , 1 nF, 1 µF. (1 pF = 1×10^{-12} F, 1 nF = 1×10^{-9} F, 1 µF = 1×10^{-6} F).** For this part, to find the initial condition for V_{DD}, feel free to assume that for a very long time, $I_{IC} = 0$.

Solution: Since the current through the source is a series of repeating pulses, where there is no current for a long time and then there is a constant current for a short time, we solve for $V_{DD}(t)$ by writing a piecewise constant function for the current flowing through the circuit, I_{IC}. Starting with KVL:

$$
\begin{aligned}
V_S &= V_R + V_{DD} && (1)\\
&= \left(I_{\text{IC}} + C\frac{\mathrm{d}}{\mathrm{d}t}V_{DD}\right)R + V_{DD} && (2)\\
\frac{\mathrm{d}}{\mathrm{d}t}V_{DD} &= \frac{1}{RC}(V_S - RI_{\text{IC}} - V_{DD}). && (3)
\end{aligned}
$$

At this point we can use substitution with $\widetilde{V} = V_S - RI_{\text{IC}} - V_{DD}$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{V} &= -\frac{\mathrm{d}}{\mathrm{d}t}V_{DD} && (4)\\
\frac{\mathrm{d}}{\mathrm{d}t}V_{DD} &= \frac{\widetilde{V}}{RC} && (5)\\
\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{V} &= -\frac{\widetilde{V}}{RC} && (6)\\
\widetilde{V}(t) &= A\mathrm{e}^{-\frac{t}{RC}} && (7)
\end{aligned}
$$

Substituting back to solve for $V_{DD}(t)$ we get the following general expression for the voltage during any one piecewise constant time interval:

$$
V_{DD}(t) = V_S - RI_{\text{IC}} - A\mathrm{e}^{-\frac{t}{RC}} \tag{8}
$$

From here, based on the value of the piecewise current I_{IC} and the initial conditions imposed by previous time segments we can simplify the V_{DD} expression and solve for A. At the start, we will assume a convenient initial condition which corresponds with the behavior of V_{DD} if $I_{\text{IC}} = 0$ for a long time before $t = 0$. In this case, $V_{DD} = V_S$ and there is zero current flowing through the circuit. Things get exciting once the first current pulse starts such that $I_{\text{IC}} = i_0$. The initial condition for this piecewise section is the final voltage V_{DD} from the previous section, V_S.

$$
\begin{aligned}
V_S &= V_S - Ri_0 - A\mathrm{e}^0 && (9)\\
A &= -Ri_0 && (10)\\
V_{DD}(t) &= V_S - Ri_0(1 - \mathrm{e}^{-\frac{t}{RC}}) && (11)
\end{aligned}
$$

You can compute the voltage at the end of the pulse by plugging in $t = t_p$ and the R and C values for your scenario. This voltage will serve as the initial condition for the next piecewise constant section. The process of simplifying the general piecewise differential equation and solving for A can be performed repeatedly to determine the shape of the plot for further pulses. Below shows the steps to compute $V_{DD}(t)$ over t for the piecewise function $I_{IC}(t)$.

$$
I_{IC}(t) = \begin{cases} 0\ A & t < 10ns \\ 1\ A & 10ns \le t < 11ns \\ 0\ A & 11ns \le t < 21ns \\ 1\ A & 21ns \le t < 22ns \\ 0\ A & 22ns \le t \end{cases} \tag{12}
$$

For $t < 10ns$:

$$V_{DD}(t) = V_S = 3V \tag{13}$$

For $10ns \leq t < 11ns$:

$$V_{DD}(t) = V_S - R \cdot I_{IC} - A_{10ns} e^{-(t-10ns)/(RC)}, \quad I_{IC} = 1\ A \tag{14}$$
$$V_{DD}(10ns) = 3V - 1V - A_{10ns} = 3V \tag{15}$$
$$A_{10ns} = -1V \tag{16}$$
$$V_{DD}(t) = 3V - 1V\left(1 - e^{-(t-10ns)/(RC)}\right) \tag{17}$$

For $11ns \leq t < 21ns$:

$$V_{DD}(t) = V_S - R \cdot I_{IC} - A_{11ns} e^{-(t-11ns)/(RC)}, \quad I_{IC} = 0\ A \tag{18}$$
$$V_{DD}(11ns) = 3V - A_{11ns} = 3V - 1V\left(1 - e^{-(11ns-10ns)/(RC)}\right) \tag{19}$$
$$A_{11ns} = 1V\left(1 - e^{-(1ns)/(RC)}\right) \tag{20}$$
$$V_{DD}(t) = 3V - A_{11ns} e^{-(t-11ns)/(RC)} \tag{21}$$

For $21ns \leq t < 22ns$:

$$V_{DD}(t) = V_S - R \cdot I_{IC} - A_{21ns} e^{-(t-21ns)/(RC)}, \quad I_{IC} = 1\ A \tag{22}$$
$$V_{DD}(21ns) = 3V - 1V - A_{21ns} = 3V - A_{11ns} e^{-(21ns-11ns)/(RC)} \tag{23}$$
$$A_{21ns} = -1V + A_{11ns} e^{-(10ns)/(RC)} \tag{24}$$
$$V_{DD}(t) = 2V - A_{21ns} e^{-(t-21ns)/(RC)} \tag{25}$$

For $22ns \leq t$:

$$V_{DD}(t) = V_S - R \cdot I_{IC} - A_{22ns} e^{-(t-22ns)/(RC)}, \quad I_{IC} = 0\ A \tag{26}$$
$$V_{DD}(22ns) = 3V - A_{22ns} = 2V - A_{21ns} e^{-(22ns-21ns)/(RC)} \tag{27}$$
$$A_{22ns} = 1V + A_{21ns} e^{-(1ns)/(RC)} \tag{28}$$
$$V_{DD}(t) = 3V - A_{22ns} e^{-(t-22ns)/(RC)} \tag{29}$$

By substituting the different values of RC, we have expressions for each of the different capacitances.

For $RC = 1\Omega \cdot 1\,\text{pF} = 1\,\text{ps}$, the A values can be found as:

$$A_{11ns} = 1V(1 - e^{-1ns/1ps}) = 1V(1 - e^{-1000}) \approx 1V \tag{30}$$
$$A_{21ns} = -1V + A_{11ns} e^{-10ns/1ps} \approx -1V \tag{31}$$
$$A_{22ns} = 1V + A_{21ns} e^{-1ns/1ps} \approx 1V \tag{32}$$

We can plug in these A values into the $V_{DD}(t)$ expressions we already found which will give the following result

$$V_{DD}(t) = \begin{cases} 3V & t < 10ns \\ 3V - 1V\left(1 - e^{-(t-10ns)/1ps}\right) & 10ns \leq t < 11ns \\ V_{DD}(t) = 3V - 1Ve^{-(t-11ns)/1ps} & 11ns \leq t < 21ns \\ V_{DD}(t) = 2V + 1Ve^{-(t-21ns)/1ps} & 21ns \leq t < 22ns \\ V_{DD}(t) = 3V - 1Ve^{-(t-22ns)/1ps} & 22ns \leq t \end{cases} \tag{33}$$

Similarly for $RC = 1\Omega \cdot 1\,\text{nF} = 1\,\text{ns}$:

$$A_{11ns} = 1V(1 - e^{-1ns/1ns}) = 1V(1 - e^{-1}) \approx 0.632V \tag{34}$$

$$A_{21ns} = -1V + 0.632e^{-10ns/1ns} \approx -0.999V \tag{35}$$

$$A_{22ns} = 1V + A_{21ns}e^{-1ns/1ns} \approx 0.632V \tag{36}$$

$$V_{DD}(t) = \begin{cases} 3V & t < 10ns \\ 3V - 1V\left(1 - e^{-(t-10ns)/1ns}\right) & 10ns \leq t < 11ns \\ V_{DD}(t) = 3V - 0.632Ve^{-(t-11ns)/1ns} & 11ns \leq t < 21ns \\ V_{DD}(t) = 2V + 0.999Ve^{-(t-21ns)/1ns} & 21ns \leq t < 22ns \\ V_{DD}(t) = 3V - 0.632Ve^{-(t-22ns)/1ns} & 22ns \leq t \end{cases} \tag{37}$$

For $RC = 1\Omega \cdot 1\,\mu\text{F} = 1\,\mu\text{s}$:

$$A_{11ns} = 1V(1 - e^{-1ns/1\mu s}) = 1V(1 - e^{-0.001}) \approx 0.001V \tag{38}$$

$$A_{21ns} = -1V + 0.001e^{-10ns/1\mu s} \approx -0.999V \tag{39}$$

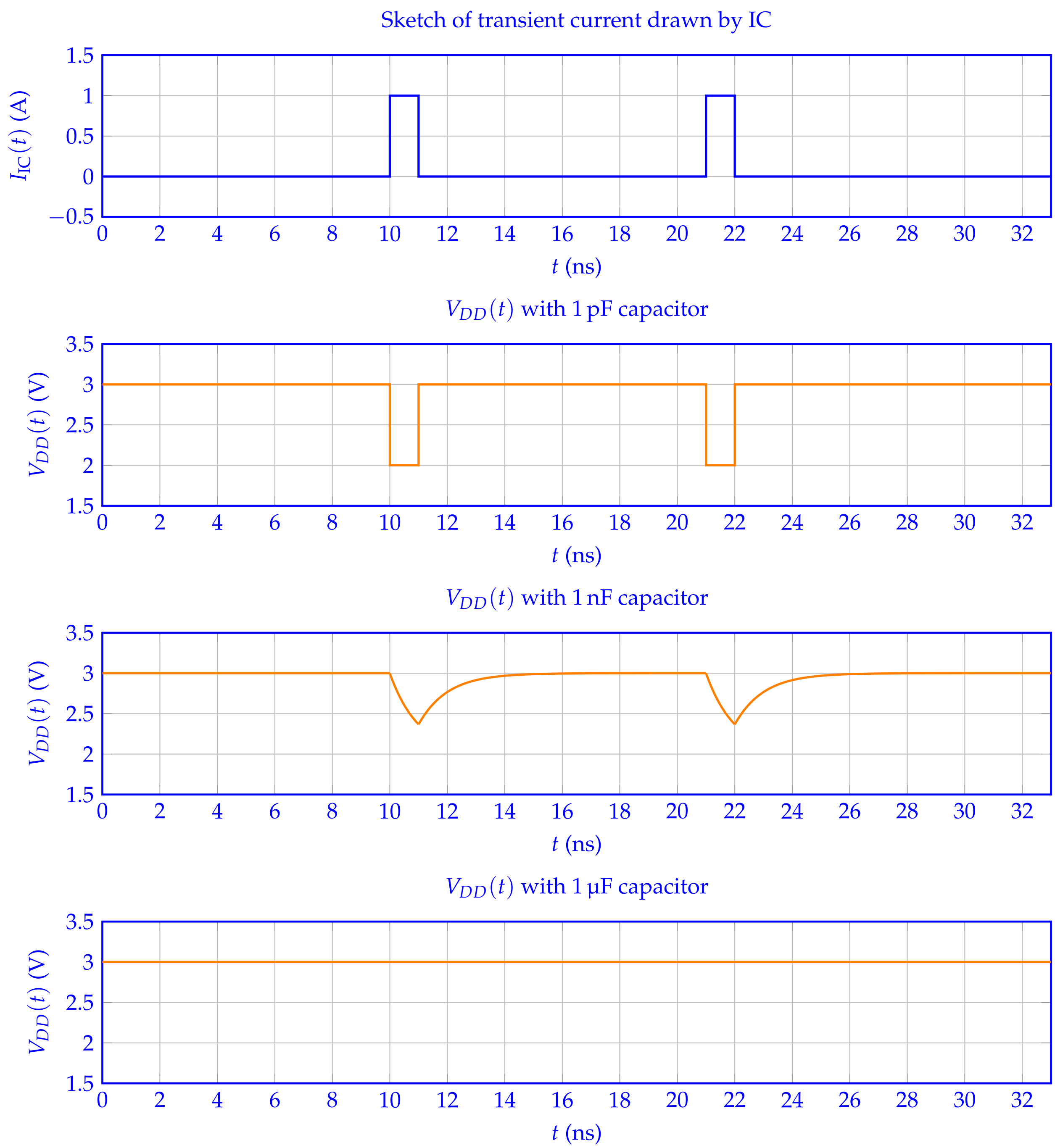
$$A_{22ns} = 1V + A_{21ns}e^{-1ns/1\mu s} \approx 0.002V \tag{40}$$

$$V_{DD}(t) = \begin{cases} 3V & t < 10ns \\ 3V - 1V\left(1 - e^{-(t-10ns)/1\mu s}\right) & 10ns \leq t < 11ns \\ V_{DD}(t) = 3V - 0.001Ve^{-(t-11ns)/1\mu s} & 11ns \leq t < 21ns \\ V_{DD}(t) = 2V + 0.999Ve^{-(t-21ns)/1\mu s} & 21ns \leq t < 22ns \\ V_{DD}(t) = 3V - 0.002Ve^{-(t-22ns)/1\mu s} & 22ns \leq t \end{cases} \tag{41}$$

In general: each of the three curves will tend towards a final value $V_{DD} = V_S$, growing exponentially slower towards this goal as time progresses. However, on each time interval t_p, the current source will start drawing charge from both V_S–whose current decreases as time proceeds–and C–whose charge, and therefore whose potential to contribute voltage, tends to increase with time. With each t_p, V_{DD} decreases nonlinearly, as there are both exponential and linear factors contributing to the rise and fall of the voltage.

With a lower capacitance, we will see the capacitor charge and discharge faster with time–this means that V_{DD} will fluctuate more/change more drastically on t_p; as you increase capacitance, this fluctuation is less evident, as C has more charge to pull from, and will thus be less affected by the change of charge incurred by the current source.

The final sketches should look something like this:



If you have the plots and the expressions and work for $V_{DD}(t)$, but not the verbose explanation regarding the plots, you still deserve full credit - this is here for your understanding.

The idea here is to see the effect that the capacitors have on $V_{DD}(t)$ when viewed at the time scale of the current spikes.

- the 1 pF capacitor causes the RC circuit to have a time constant of $\tau = RC = 1$ ps, that is 1picosecond $= 10^{-12}$ seconds, and the effect that this has on $V_{DD}(t)$ is invisible at the nanosecond time scale. For this reason, we can conclude that the 1 pF capacitor would not be adequate to mitigate the noise that the IC will put on the power supply.

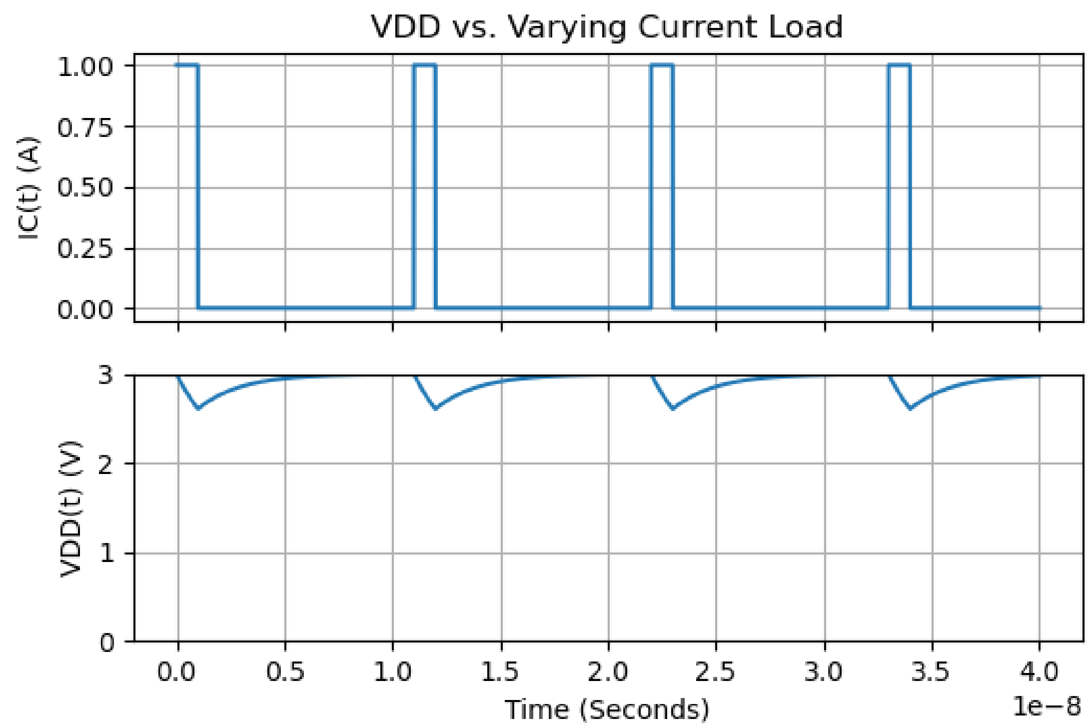
- the 1 nF capacitor causes the RC circuit to have a time constant of $\tau = RC = 1\,\text{ns}$. This is a long enough time scale that the effect on $V_{DD}(t)$ will be visible. At the end of the 1 ns current spike, $V_{DD}(t)$ will have dropped from 3 V to $2 + \exp(-1) \approx 2.37\,\text{V}$. This means that the 1 nF capacitor is actually reducing the power supply noise a little bit, but not much.
- the 1 µF capacitor causes the RC circuit to have a time constant of $\tau = RC = 1\,\mu\text{s}$. This time constant is 1000 times longer than the duration of the current spike. At the end of the current spike, $V_{DD}(t)$ will have dropped by only 1 mV, so at the scale at which these sketches are drawn, there is no visible change. The 1 µF capacitor has almost totally removed the power supply noise.

(c) Launch the attached Jupyter notebook to interact with a simulated version of this IC power supply. Try to simulate the scenarios outlined in the previous parts. For one of these scenarios, keep the RC time constant fixed, but vary the relative value of R vs. C (e.g. compare $R = \texttt{1}, C = \texttt{2e-9}$ to the case where $R = \texttt{2}, C = \texttt{1e-9}$). **Is it better to have a lower R or lower C value for a fixed RC time constant when attempting to minimize supply noise? Give an intuitive explanation for why this might be the case.**

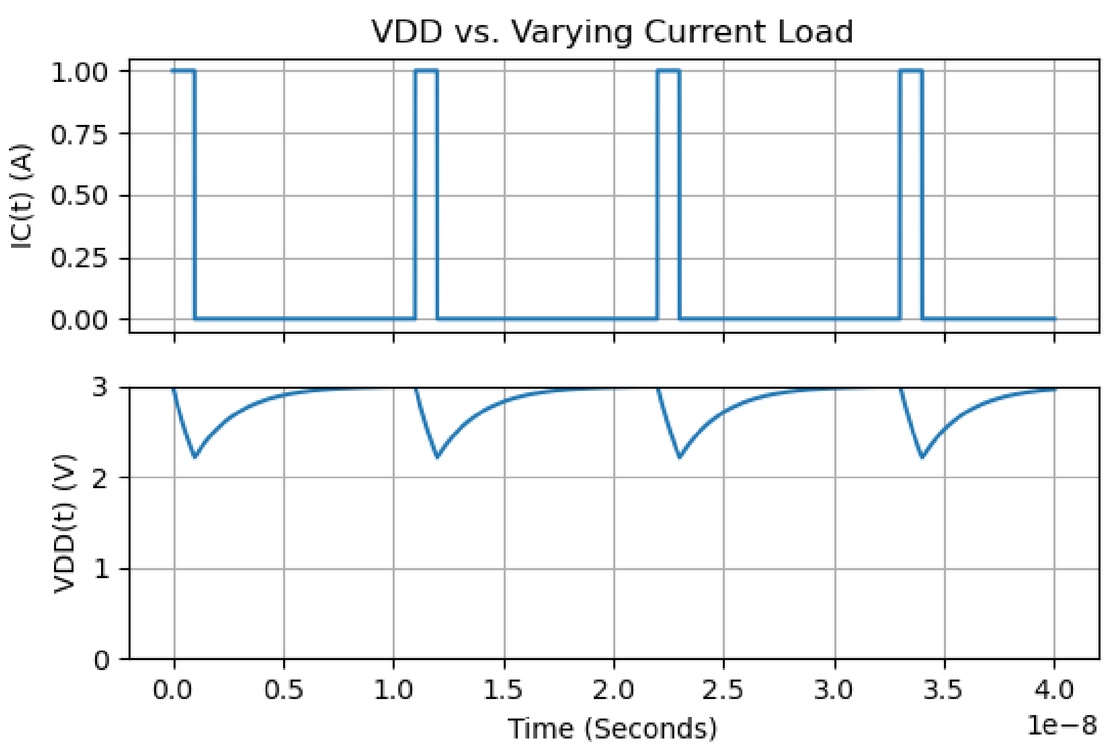
Be sure to play with the y limits on the graph as well as how long the simulation runs to best understand what is going on here.

Solution: A lower resistance and higher capacitance leads to smaller variation in the supply voltage with each current spike. One intuitive way to see this is to think about where the charge comes from whenenver the current source turns on. The charge comes from the capacitor and from the voltage source through the resistor. By $Q = CV$ for a constant amount of charge drawn, a larger capacitor results in lower voltage change. By $V = IR$ for a constant amount of current drawn through the resistor, a larger resistor leads to a larger voltage drop.

In the case where $R = \texttt{1}$ and $C = \texttt{2e-9}$, we get the following plot:



In the case where $R = 2$ and $C =$ `1e-9`, we get the following plot:



Notice that the shape of the V_{DD} curves is the same because the RC constant is the same. However they drop to different voltages by the end of each pulse.

2. Simple Scalar Differential Equations Driven by an Input

Unlocked by Lecture 4

In this question, we will show the existence and uniqueness of solutions to differential equations with inputs. In particular, we consider the scalar differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu(t) \tag{42}$$

$$x(0) = x_0 \tag{43}$$

where $u\colon \mathbb{R} \to \mathbb{R}$ is a known function of time. Feel free to assume u is "nice" in the sense that it is integrable, continuous, and differentiable with bounded derivative – basically, let u be nice enough that all the usual calculus theorems work.

(a) We will first demonstrate the existence of a solution to eqs. (42) and (43).

Define $x_d\colon \mathbb{R} \to \mathbb{R}$ by

$$x_d(t) := \mathrm{e}^{\lambda t}x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau\,. \tag{44}$$

Show that x_d satisfies eqs. (42) and (43).

(HINT: When showing that x_d satisfies eq. (42), one possible approach to calculate the derivative of the integral term is to use the fundamental theorem of calculus and the product rule.)

Solution: We first show that x_d satisfies eq. (42). Using the fundamental theorem of calculus and the product rule, we can calculate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}x_d(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\lambda t}x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau\right) && (45)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\lambda t}x_0\right) + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau && (46)\\
&= x_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda t}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\lambda t}\int_0^t \mathrm{e}^{-\lambda\tau}bu(\tau)\,\mathrm{d}\tau\right) && (47)\\
&= x_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda t}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\lambda t}\int_0^t \mathrm{e}^{-\lambda\tau}bu(\tau)\,\mathrm{d}\tau\right) && (48)\\
&= x_0\left(\lambda\mathrm{e}^{\lambda t}\right) + \left\{\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda t}\right)\left(\int_0^t \mathrm{e}^{-\lambda\tau}bu(\tau)\,\mathrm{d}\tau\right) + \left(\mathrm{e}^{\lambda t}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{-\lambda\tau}bu(\tau)\,\mathrm{d}\tau\right)\right\} && (49)\\
&= \lambda x_0\mathrm{e}^{\lambda t} + \lambda\mathrm{e}^{\lambda t}\int_0^t \mathrm{e}^{-\lambda\tau}bu(\tau)\,\mathrm{d}\tau + \mathrm{e}^{\lambda t}\left(\mathrm{e}^{-\lambda\tau}bu(\tau)\Big|_{\tau=t}\right) && (50)\\
&= \lambda\left(x_0\mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t}\int_0^t \mathrm{e}^{-\lambda\tau}bu(\tau)\,\mathrm{d}\tau\right) + \mathrm{e}^{\lambda t}\left(\mathrm{e}^{-\lambda t}bu(t)\right) && (51)\\
&= \lambda x_d(t) + bu(t) && (52)
\end{aligned}$$

so x_d satisfies eq. (42).

Alternatively, we could have used the Leibniz rule (not in scope) to get the terms in the square brackets:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}x_d(t) &= \lambda\mathrm{e}^{\lambda t}x_0 + \left[\mathrm{e}^{\lambda(t-t)}bu(t)\cdot 1 - \mathrm{e}^{\lambda(t-0)}\cdot 0 + \lambda\int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau\right] && (53)\\
&= \lambda\left[\mathrm{e}^{\lambda t}x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)}bu(\tau)\,\mathrm{d}\tau\right] + bu(t) && (54)
\end{aligned}$$

$$= \lambda x_d(t) + bu(t) \tag{55}$$

which again shows that x_d satisfies eq. (42).

Now we show that x_d satisfies eq. (43). Indeed,

$$x_d(0) = \left(e^{\lambda t} x_0 + \int_0^t e^{\lambda(t-\tau)} bu(\tau) \, d\tau \right) \Big|_{t=0} = \underbrace{e^{\lambda \cdot 0}}_{=1} x_0 + \underbrace{\int_0^0 e^{\lambda(0-\tau)} bu(\tau) \, d\tau}_{=0} = x_0. \tag{56}$$

Thus x_d satisfies eq. (43).

(b) Now, we will show that x_d is the unique solution to eqs. (42) and (43).

Suppose that $y \colon \mathbb{R} \to \mathbb{R}$ also satisfies eqs. (42) and (43). **Show that $y(t) = x_d(t)$ for all t.**

(HINT: This time, show that $z(t) := y(t) - x_d(t) = 0$ for all t. Do this by showing that $z(0) = 0$ and $\frac{d}{dt} z(t) = \lambda z(t)$, then use the uniqueness theorem for homogeneous first-order linear differential equations from the last homework. Note that the specific form of $x_d(t)$ in eq. (44) is irrelevant for the solution and should not be used.)

Solution: Again, the solution is in some parts.

Step 1. We show that $z(0) = 0$. Indeed,

$$z(0) = y(0) - x_d(0) = x_0 - x_0 = 0. \tag{57}$$

Step 2. We show that $\frac{d}{dt} z(t) = \lambda z(t)$. Indeed,

$$\frac{d}{dt} z(t) = \frac{d}{dt}(y(t) - x_d(t)) \tag{58}$$

$$= \frac{d}{dt} y(t) - \frac{d}{dt} x_d(t) \tag{59}$$

$$= (\lambda y(t) + bu(t)) - (\lambda x_d(t) + bu(t)) \tag{60}$$

$$= \lambda y(t) - \lambda x_d(t) \tag{61}$$

$$= \lambda z(t). \tag{62}$$

Step 3. We show that $z(t) = 0$ for all t. Indeed, we know that $z(t)$ satisfies the differential equation

$$\frac{d}{dt} z(t) = \lambda z(t) \tag{63}$$

$$z(0) = 0. \tag{64}$$

This is a first-order linear differential equation, so we know from the previous homework that its unique solution is

$$z(t) = z(0) \cdot e^{\lambda t} = 0 \cdot e^{\lambda t} = 0. \tag{65}$$

This is what was claimed, so we are done.

(c) In this part, we will calculate some values of x_d for common values of u.

i. If $u(t) := u$ is a constant function, **what is $x_d(t)$?**

ii. If $u(t) := e^{\alpha t}$ for some real number $\alpha \neq \lambda$, **what is $x_d(t)$?**

iii. If $u(t) := \mathrm{e}^{\lambda t}$, **what is $x_d(t)$?**

NOTE: Assume for simplicity that $\lambda \neq 0$.

Solution:

i. We calculate

$$x_d(t) = \mathrm{e}^{\lambda t} x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)} b u(\tau)\,\mathrm{d}\tau \tag{66}$$
$$= \mathrm{e}^{\lambda t} x_0 + bu \int_0^t \mathrm{e}^{\lambda(t-\tau)}\,\mathrm{d}\tau \tag{67}$$
$$= \mathrm{e}^{\lambda t} x_0 + \frac{\mathrm{e}^{\lambda t} - 1}{\lambda} bu. \tag{68}$$

ii. We calculate

$$x_d(t) = \mathrm{e}^{\lambda t} x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)} b u(\tau)\,\mathrm{d}\tau \tag{69}$$
$$= \mathrm{e}^{\lambda t} x_0 + b \int_0^t \mathrm{e}^{\lambda(t-\tau)} \mathrm{e}^{\alpha\tau}\,\mathrm{d}\tau \tag{70}$$
$$= \mathrm{e}^{\lambda t} x_0 + b\mathrm{e}^{\lambda t} \int_0^t \mathrm{e}^{-\lambda\tau} \mathrm{e}^{\alpha\tau}\,\mathrm{d}\tau \tag{71}$$
$$= \mathrm{e}^{\lambda t} x_0 + b\mathrm{e}^{\lambda t} \int_0^t \mathrm{e}^{(\alpha-\lambda)\tau}\,\mathrm{d}\tau \tag{72}$$
$$= \mathrm{e}^{\lambda t} x_0 + b\frac{\mathrm{e}^{\alpha t} - \mathrm{e}^{\lambda t}}{\alpha - \lambda} \tag{73}$$
$$= \left(x_0 - \frac{b}{\alpha - \lambda}\right)\mathrm{e}^{\lambda t} + b\frac{\mathrm{e}^{\alpha t}}{\alpha - \lambda}. \tag{74}$$

iii. We calculate

$$x_d(t) = \mathrm{e}^{\lambda t} x_0 + \int_0^t \mathrm{e}^{\lambda(t-\tau)} b u(\tau)\,\mathrm{d}\tau \tag{75}$$
$$= \mathrm{e}^{\lambda t} x_0 + b \int_0^t \mathrm{e}^{\lambda(t-\tau)} \mathrm{e}^{\lambda\tau}\,\mathrm{d}\tau \tag{76}$$
$$= \mathrm{e}^{\lambda t} x_0 + b \int_0^t \mathrm{e}^{\lambda t} \mathrm{e}^{-\lambda\tau} \mathrm{e}^{\lambda\tau}\,\mathrm{d}\tau \tag{77}$$
$$= \mathrm{e}^{\lambda t} x_0 + b \int_0^t \mathrm{e}^{\lambda t}\,\mathrm{d}\tau \tag{78}$$
$$= \mathrm{e}^{\lambda t} x_0 + b\mathrm{e}^{\lambda t} \int_0^t \mathrm{d}\tau \tag{79}$$
$$= \mathrm{e}^{\lambda t} x_0 + bt\mathrm{e}^{\lambda t}. \tag{80}$$

3. **Tracking Terry**

Unlocked by Lecture 5

Terry is a mischievous child, and his mother is interested in tracking him.

(a) Terry texts his current location as a vector $\vec{x}_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, but there is a problem! These coordinates are *not* in the standard basis, but rather in the basis $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. That is to say that the first number 2 above is how many multiples of $\vec{v}_1$ to use and the second number 3 is how many multiples of $\vec{v}_2$ to use in computing his actual location. Here, both $\vec{v}_1$ and $\vec{v}_2$ are vectors in the standard basis.

Let Terry's location in the standard basis be $\vec{x}$. Write $\vec{x}$ in terms of $\vec{v}_1$ and $\vec{v}_2$.

Solution: By definition, the first coordinate in the V basis is the coefficient of $\vec{v}_1$ and second coordinate in the V basis is the coefficient for $\vec{v}_2$. Hence

$$\vec{x} = V\vec{x}_v = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\vec{v}_1 + 3\vec{v}_2. \tag{81}$$

(b) Terry's friend tells you that Terry's location in the standard basis is $\vec{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Using this along with the previous info that Terry's location in the V basis is $\vec{x}_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, **is it possible to determine the basis vectors $\vec{v}_1, \vec{v}_2$ Terry is using. If it is impossible to do so, explain why.**

(HINT: How many unknowns do you have? How many equations?)

Solution: Solving for the basis vectors Terry is using (or in other words the axes in his coordinate space) is the same as solving for V in the change of basis equation:

$$V\vec{x}_v = \vec{x} \tag{82}$$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \tag{83}$$

There are four unknowns and only two equations, so this task is impossible.

(c) Terry's basis vectors $\vec{v}_1, \vec{v}_2$ get leaked to his mom on accident, so she knows they are

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}. \tag{84}$$

To hide his location, Terry wants to switch to a new coordinate system with the basis vectors

$$\vec{p}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{85}$$

In order to do this, he needs a way to convert coordinates from the V basis to the P basis. Thus, **find the matrix T such that if $\vec{x}_v$ is a location expressed in V coordinates and $\vec{x}_p$ is the same location expressed in P coordinates, then $\vec{x}_p = T\vec{x}_v$.**

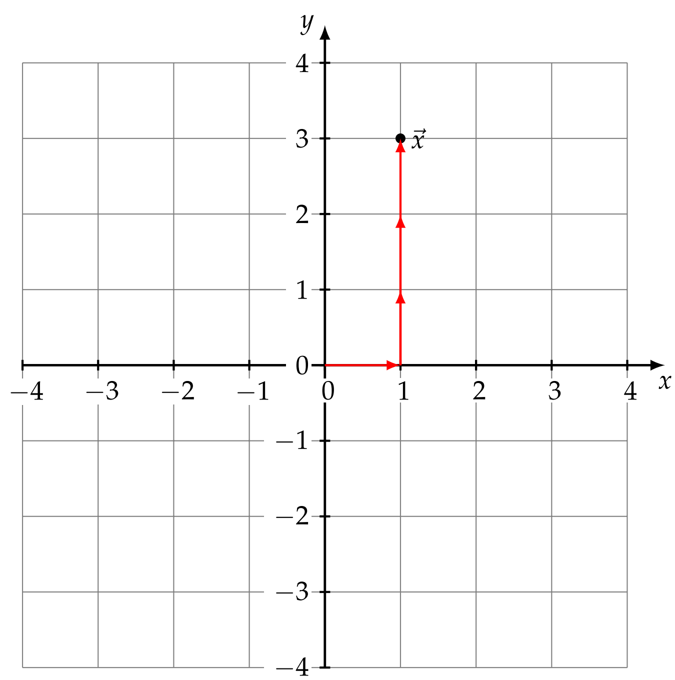
Solution: The problem can be formulated as a change of basis problem. Since both $\vec{x}_v$ and $\vec{x}_p$ correspond to the same point, converting them to the standard basis gives us

$$V\vec{x}_v = P\vec{x}_p \tag{86}$$

Since we want to find T such that $\vec{x}_p = T\vec{x}_v$, we have:

$$\vec{x}_p = P^{-1}V\vec{x}_v \tag{87}$$
$$T = P^{-1}V \tag{88}$$
$$= \begin{bmatrix}1 & 0\\4 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\1 & 2\end{bmatrix} \tag{89}$$
$$= \begin{bmatrix}1 & 0\\-4 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 2\end{bmatrix} \tag{90}$$
$$= \begin{bmatrix}1 & 0\\-3 & 2\end{bmatrix} \tag{91}$$

(d) Terry now wants to make a map and route to where he currently is, $\vec{x} = \begin{bmatrix}1\\3\end{bmatrix}$. **For both the P and V bases from part 3.c, illustrate the sum of scaled basis vectors that are necessary to go from the origin to $\vec{x}$.** An example is shown below when using the standard basis. This illustrates that the same location can be represented by many different coordinate systems/bases.



Solution:

Since we know $\vec{x} = \begin{bmatrix}1\\3\end{bmatrix}$, $V = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\1 & 2\end{bmatrix}$, and $\vec{x} = V\vec{x}_v$, we can derive:

$$\vec{x}_v = V^{-1}\vec{x} \tag{92}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{93}$$

$$= \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{94}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{95}$$

Similarly, in order to compute $\vec{x}_p$, we have:

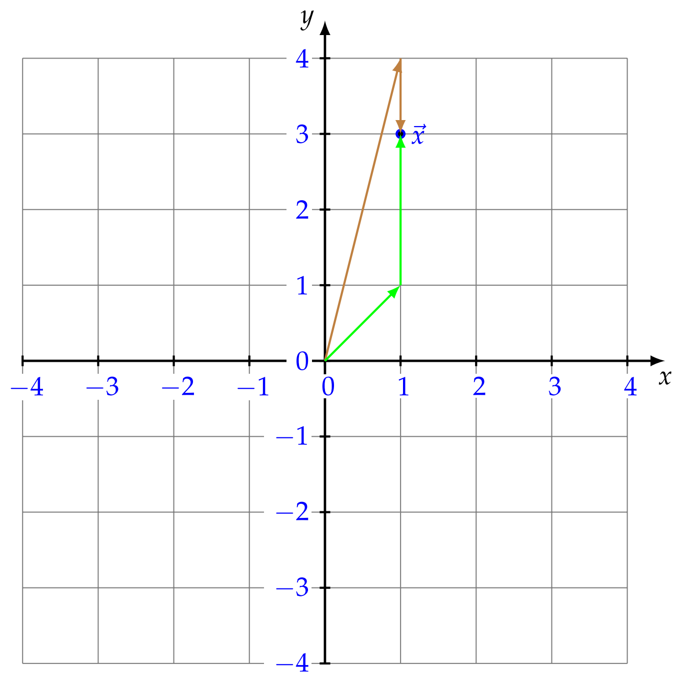
$$\vec{x}_p = P^{-1}\vec{x} \tag{96}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{97}$$

$$= \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{98}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{99}$$

Therefore, we can illustrate the sum of scaled basis vectors according to $\vec{x}_v$ (green path), and $\vec{x}_p$ (brown path).



4. Eigenvectors and Diagonalization

Unlocked by Lecture 6

(a) Let A be an $n \times n$ matrix with n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define V to be a matrix with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ as its columns, $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$.

Show that $AV = V\Lambda$, where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a diagonal matrix with the eigenvalues of A as its diagonal entries.

Solution:

$$AV = A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \tag{100}$$

$$= \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots, A\vec{v}_n \end{bmatrix} \tag{101}$$

$$= \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \end{bmatrix} \tag{102}$$

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \tag{103}$$

$$= V\Lambda \tag{104}$$

(b) **Argue that V is invertible, and therefore**

$$A = V\Lambda V^{-1}. \tag{105}$$

<u>(Hint: Why is V invertible? It is fine to cite the appropriate result from 16A.)</u>

Solution: Columns of V are eigenvectors of A which are known to be linearly independent. Since V has linearly independent columns, it has full column rank, and therefore, is invertible.

$$AV = V\Lambda \tag{106}$$

$$AVV^{-1} = V\Lambda V^{-1} \tag{107}$$

$$A = V\Lambda V^{-1} \tag{108}$$

(c) **Write Λ in terms of the matrices A, V, and V^{-1}.**

Solution: We take $A = V\Lambda V^{-1}$ and apply invertible operations to both sides of the equality:

$$A = V\Lambda V^{-1} \tag{109}$$

$$V^{-1}A = V^{-1}V\Lambda V^{-1} \tag{110}$$

$$V^{-1}AV = V^{-1}V\Lambda V^{-1}V \tag{111}$$

$$V^{-1}AV = I\Lambda I \tag{112}$$

$$V^{-1}AV = \Lambda. \tag{113}$$

(d) A matrix A is deemed diagonalizable if there exists a square matrix U so that A can be written in the form $A = UDU^{-1}$ for the choice of an appropriate diagonal matrix D.

Show that the columns of U must be eigenvectors of the matrix A, and that the entries of D must be eigenvalues of A.

(HINT: What does it mean to be an eigenvector? What is $U^{-1}U$? How does matrix multiplication work column-wise?)

Solution: We start with a calculation which is essentially the reverse of the calculation in part (b):

$$A = UDU^{-1} \tag{114}$$
$$AU = UDU^{-1}U \tag{115}$$
$$AU = UD. \tag{116}$$

Now let's expand the definitions of U as a square matrix and D as a diagonal matrix:

$$AU = A\begin{bmatrix}\vec{u}_1 & \dots & \vec{u}_n\end{bmatrix} \tag{117}$$
$$= \begin{bmatrix}A\vec{u}_1 & \dots & A\vec{u}_n\end{bmatrix} \tag{118}$$
$$UD = \begin{bmatrix}\vec{u}_1 & \dots & \vec{u}_n\end{bmatrix}\begin{bmatrix}d_1 & & \\ & \ddots & \\ & & d_n\end{bmatrix} \tag{119}$$
$$= \begin{bmatrix}d_1\vec{u}_1 & \dots & d_n\vec{u}_n\end{bmatrix}. \tag{120}$$

Comparing columns, we see that $A\vec{u}_i = d_i\vec{u}_i$. This is exactly the eigenvector-eigenvalue equation! In particular, this says that $\vec{u}_i$ is an eigenvector of A, with eigenvalue d_i.

The previous part shows that the *only* way to diagonalize A is using its eigenvalues/eigenvectors. Now we will explore a payoff for diagonalizing A – an operation that diagonalization makes *much* simpler.

(e) For a matrix A and a positive integer k, we define the exponent to be

$$A^k = \underbrace{A \cdot A \cdot \dots \cdot A \cdot A}_{k \text{ times}} \tag{121}$$

Let's assume that matrix A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ (i.e. the n eigenvectors are all linearly independent).

Show that A^k has eigenvalues $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k$ and eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Conclude that A^k is diagonalizable.

Solution: Consider the i^{th} eigenvector of A, $\vec{v}_i$ and the corresponding eigenvalue λ_i.

$$A^k\vec{v}_i = A^{k-1} \cdot A\vec{v}_i \tag{122}$$
$$= A^{k-1}\lambda_i\vec{v}_i \tag{123}$$
$$= \lambda_i A^{k-2} \cdot A\vec{v}_i \tag{124}$$
$$= \lambda_i^2 A^{k-3} \cdot A\vec{v}_i \tag{125}$$

$$\vdots \tag{126}$$

$$= \lambda_i^k \vec{v}_i \tag{127}$$

Thus by definition, v_i is an eigenvector of A^k with corresponding eigenvalue λ_i^k.

Alternate solution: Since A is diagonalizable, we can express A as

$$A = V \Lambda V^{-1} \tag{128}$$

Substituting A as shown in Equation 128 in 121, we get

$$A^k = \underbrace{A \cdot A \cdot \dots \cdot A \cdot A}_{k \text{ times}} \tag{129}$$

$$= \underbrace{V\Lambda V^{-1} \cdot V\Lambda V^{-1} \cdot \dots \cdot V\Lambda V^{-1} \cdot V\Lambda V^{-1}}_{k \text{ times}} \tag{130}$$

$$= \underbrace{V\Lambda \left(V^{-1} \cdot V\right) \Lambda V^{-1} \cdot \dots \cdot V\Lambda \left(V^{-1} \cdot V\right) \Lambda V^{-1}}_{k \text{ times}} \tag{131}$$

$$= V \underbrace{\Lambda \cdot \Lambda \cdot \dots \cdot \Lambda \cdot \Lambda}_{k \text{ times}} V^{-1} \tag{132}$$

$$= V \Lambda^k V^{-1} \tag{133}$$

Since Λ is a diagonal matrix,

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix} \tag{134}$$

Thus, A^k is clearly diagonalizable, where the eigenvectors of A^k are just the eigenvectors of A, and the eigenvalues of A^k are $\lambda_1^k, \dots, \lambda_n^k$.

5. Vector Differential Equations

Unlocked by Lecture 6

Note: it's recommended to finish Question 4 (Eigenvectors and Diagonalization) before this problem.

Consider a system of ordinary differential equations that can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) := \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2(t) \end{bmatrix} = A\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\vec{x}(t) \tag{135}$$

where $x_1, x_2 : \mathbb{R} \to \mathbb{R}$ are scalar functions of time t, and $A \in \mathbb{R}^{2\times 2}$ is a 2×2 matrix with constant coefficients. We call eq. (135) a vector differential equation.

(a) Suppose we have a system of ordinary differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 7x_1 - 8x_2, \tag{136}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 4x_1 - 5x_2, \tag{137}$$

Here, we denote $x_1(t), x_2(t)$ as x_1, x_2 for notational simplicity.

Find an appropriate matrix A to write this system in the form of eq. (135). Compute the eigenvalues of A. Denote the smaller and larger eigenvalues by λ_1 and λ_2 respectively, so that $\lambda_1 \leq \lambda_2$.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 & -8 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{138}$$

The characteristic polynomial of A is

$$\det\left(\begin{bmatrix} 7-\lambda & -8 \\ 4 & -5-\lambda \end{bmatrix}\right) = (7-\lambda)(-5-\lambda) + 32 \tag{139}$$

$$= \lambda^2 - 7\lambda + 5\lambda - 35 + 32 \tag{140}$$

$$= \lambda^2 - 2\lambda - 3 \tag{141}$$

$$= (\lambda+1)(\lambda-3). \tag{142}$$

Thus the eigenvalues of A are $\lambda_1 = -1, \lambda_2 = 3$.

(b) **Compute the eigenvectors of A. Is A diagonalizable? Why?**

Solution: We will use the standard null space approach.

$$(A - \lambda_1 I)\vec{v}_1 = (A + I)\vec{v}_1 = \begin{bmatrix} 8 & -8 \\ 4 & -4 \end{bmatrix}\vec{v}_1 = 0 \tag{143}$$

$$(A - \lambda_2 I)\vec{v}_2 = (A - 3I)\vec{v}_2 = \begin{bmatrix} 4 & -8 \\ 4 & -8 \end{bmatrix}\vec{v}_2 = 0 \tag{144}$$

$$\tag{145}$$

From the first equation, we see that any vector $\vec{v}_1 = (v_{11}, v_{12})^\top$ satisfies $v_{11} = v_{12}$. From the second equation, we see that any vector $\vec{v}_2 = (v_{21}, v_{22})^\top$ satisfies $v_{21} = 2v_{22}$. Thus we have eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{146}$$

Since these are linearly independent, the columns of $V = [\vec{v}_1, \vec{v}_2]$ are linearly independent. Thus V^{-1} exists and is well-defined, so A can be written $A = V\Lambda V^{-1}$, where Λ is the diagonal matrix of eigenvalues. Thus A is diagonalizable. This can also be seen from the fact that the eigenvalues were calculated to be distinct.

(c) We now transform our system (eq. (136) and eq. (137)) in x coordinates, to new coordinates z to simplify our system of differential equations. **What basis V should we use so that in the new coordinates $\vec{z} = V^{-1}\vec{x}$, the Λ matrix in the equation $\frac{\mathrm{d}\vec{z}(t)}{\mathrm{d}t} = \Lambda\vec{z}(t)$ is diagonal? Write out this new system in the $\vec{z}$ coordinates.**

Solution: Let our basis V be the eigenvector matrix $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. Then, since A can be diagonalized, we have $A = V\Lambda V^{-1}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = A\vec{x} = V\Lambda V^{-1}\vec{x} = V\Lambda\vec{z} \tag{147}$$

$$V^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = V^{-1}V\Lambda\vec{z} \tag{148}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V^{-1}\vec{x} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{z} = \Lambda\vec{z} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\vec{z} \tag{149}$$

In the last line, we have $V^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \frac{\mathrm{d}}{\mathrm{d}t}V^{-1}\vec{x}$ because for

$$V^{-1} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

we have

$$V^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \begin{bmatrix} v_{11}\frac{\mathrm{d}}{\mathrm{d}t}x_1 + v_{12}\frac{\mathrm{d}}{\mathrm{d}t}x_2 \\ v_{21}\frac{\mathrm{d}}{\mathrm{d}t}x_1 + v_{22}\frac{\mathrm{d}}{\mathrm{d}t}x_2 \end{bmatrix} \tag{150}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V^{-1}\vec{x} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_{11}x_1 + v_{12}x_2 \\ v_{21}x_2 + v_{22}x_2 \end{bmatrix} \tag{151}$$

$$= \begin{bmatrix} v_{11}\frac{\mathrm{d}}{\mathrm{d}t}x_1 + v_{12}\frac{\mathrm{d}}{\mathrm{d}t}x_2 \\ v_{21}\frac{\mathrm{d}}{\mathrm{d}t}x_1 + v_{22}\frac{\mathrm{d}}{\mathrm{d}t}x_2 \end{bmatrix}. \tag{152}$$

(d) **Solve the new system in the $\vec{z}$ coordinates, using the initial conditions that $x_1(0) = 1, x_2(0) = -1$.**

Solution: Now z_1 and z_2 are their own separated differential equations so $\frac{\mathrm{d}}{\mathrm{d}t}z_1 = -z_1$ and $\frac{\mathrm{d}}{\mathrm{d}t}z_2 = 3z_2$. We know the general form solution of these differential equations:

$$z_1(t) = k_1 \mathrm{e}^{-t} \tag{153}$$

$$z_2(t) = k_2 \mathrm{e}^{3t} \tag{154}$$

We now find the initial conditions in the $\vec{z}$ coordinates with

$$\vec{z}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \tag{155}$$

This means that $k_1 = z_1(0) = -3$ and $k_2 = z_2(0) = 2$. Thus, our final solutions for $z(t)$ are

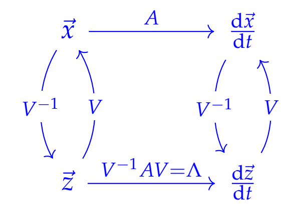
$$z_1(t) = -3e^{-t} \tag{156}$$
$$z_2(t) = 2e^{3t} \tag{157}$$

(e) Now **convert your solution from the $\vec{z}$ coordinates back to the original $\vec{x}$ coordinates.** In other words, give us the functions $x_1(t)$ and $x_2(t)$.

Solution: We can just use the coordinate transformation we defined in the beginning, that $\vec{x} = B\vec{z}$. Then,

$$\vec{x}(t) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3e^{-t} \\ 2e^{3t} \end{bmatrix} = \begin{bmatrix} -3e^{-t} + 4e^{3t} \\ -3e^{-t} + 2e^{3t} \end{bmatrix} \tag{158}$$

At this point we can summarize the picture with the following diagram:



(f) It turns out that we can actually turn all higher-order differential equations with constant coefficients into vector differential equations of the style of eq. (135).

Consider a second-order ordinary differential equation

$$\frac{d^2y(t)}{dt^2} + a\frac{dy(t)}{dt} + by(t) = 0, \tag{159}$$

where $a, b \in \mathbb{R}$.

Write this differential equation in the form of (eq. (135)), by choosing appropriate variables $x_1(t)$ and $x_2(t)$.

(HINT: Your original unknown function $y(t)$ has to be one of those variables. The heart of the question is to figure out what additional variable can you use so that you can express eq. (159) without having to take a second derivative, and instead just taking the first derivative of something. This is another manifestation of the larger thought pattern of "lifting.")

Solution:

If we set $x_1(t) = y(t)$, $x_2(t) = \frac{dy(t)}{dt}$, then we have

$$\frac{dx_1(t)}{dt} = \frac{dy(t)}{dt} = x_2(t) \tag{160}$$
$$\frac{dx_2(t)}{dt} = \frac{d^2y(t)}{dt^2} = -a\frac{dy(t)}{dt} - by(t) = -ax_2(t) - bx_1(t) \tag{161}$$

We can write this in the form of eq. (135) as follows

$$\frac{d}{dt}\vec{x} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{162}$$

(g) It turns out that all two-dimensional vector linear differential equations with distinct eigenvalues have a solution in the general form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_1 t} + c_1 e^{\lambda_2 t} \\ c_2 e^{\lambda_1 t} + c_3 e^{\lambda_2 t} \end{bmatrix} \tag{163}$$

where c_0, c_1, c_2, c_3 are constants, and λ_1, λ_2 are the eigenvalues of A (this can be proven by just repeating the same steps in the previous parts and using the fact that distinct eigenvalues implies linearly independent eigenvectors). Thus, an alternate way of solving this type of differential equation in the future is to now use your knowledge that the solution is of this form and just solve for the constants c_i.

Now let $a = -1$ and $b = -2$ in eq. (159), i.e.

$$\frac{d^2 y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = 0, \tag{164}$$

Verify that eq. (164) has a solution in the general form eq. (163). Solve eq. (164) with the initial conditions $y(0) = 1$, $\frac{dy}{dt}(0) = 1$, using this method.

(HINT: You get two equations using the initial conditions above. How many unknowns are here?) (SECOND HINT: Given your specific choice of x_1 and x_2 in part (f), how many unknowns are there really?)

Solution: We have

$$\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \tag{165}$$

First, we calculate the eigenvalues of this matrix. The characteristic polynomial is

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1). \tag{166}$$

Thus the eigenvalues are $\lambda_1 = -1, \lambda_2 = 2$. Since they are distinct, we can proceed with this method.

We know the solution for $x_1(t), x_2(t)$ is of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{-t} + c_1 e^{2t} \\ c_2 e^{-t} + c_3 e^{2t} \end{bmatrix}. \tag{167}$$

At $t = 0$, we have $y(0) = 1$ and $\frac{dy}{dt}(0) = 1$. Using our differential equation (eq. (164)), we can get $\frac{d^2y}{dt^2}(0) = \frac{dy}{dt}(0) + 2y(0) = 3$. Plugging these in,

$$x_1(0) = y(0) = 1 = c_0 + c_1 \tag{168}$$

$$x_2(0) = \frac{dy}{dt}(0) = 1 = c_2 + c_3 \tag{169}$$

$$\frac{dx_1}{dt}(0) = \frac{dy}{dt}(0) = 1 = -c_0 + 2c_1 \tag{170}$$

$$\frac{dx_2}{dt}(0) = \frac{d^2y}{dt^2}(0) = 3 = -c_2 + 2c_3 \tag{171}$$

This gives $c_0 = \frac{1}{3}, c_1 = \frac{2}{3}, c_2 = -\frac{1}{3}, c_3 = \frac{4}{3}$. Alternatively, you could've seen that $c_2 = -c_0$ and $c_3 = 2c_1$ since $x_2(t)$ is the derivative of $x_1(t)$ which makes it solvable with just the first 2 equations. Thus we have

$$x_1(t) = y(t) = \frac{1}{3}e^{-t} + \frac{2}{3}e^{2t} \tag{172}$$

$$x_2(t) = \frac{dy(t)}{dt} = -\frac{1}{3}e^{-t} + \frac{4}{3}e^{2t} \tag{173}$$

6. **Uniqueness justification for solutions to matrix/vector differential equations**

Unlocked by Lectures 6 and 7

In general, we have seen that we need to justify our methods of solving differential equations with a uniqueness proof. This is important as it allows us to trust our solution as being the only one for the problem at hand.

Consider matrix-vector differential equations of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{174}$$

with some initial condition $\vec{x}(0) = \vec{x}_0$.

All the uniqueness proofs that you have done for yourself have been concerned with scalar differential equations, and scalar differential equations driven by inputs. So, why can we trust the solutions that we are getting for such matrix-vector differential equations?

This question takes us part of the way to the answer.

(a) Suppose that the $n \times n$ matrix A has n distinct eigenvalues and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, so that the matrix $V = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n\end{bmatrix}$ has linearly independent columns.
Find the diagonalized system corresponding to (174). Show that if you are given any valid solution for the original system (174), you can change coordinates to the eigenbasis and also get a valid solution for the diagonalized system. List the new initial conditions that are satisfied in the diagonalized system.

Solution: Because the eigenvector matrix V has linearly independent columns, we know it is invertible. We can therefore change the coordinates into the eigenbasis coordinates $\vec{y}$ with the following conversions:

$$\vec{y} = V^{-1}\vec{x} \qquad \vec{x} = V\vec{y} \tag{175}$$

We transform the original differential equation by substituting $\vec{x} = V\vec{y}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}V\vec{y}(t) = AV\vec{y}(t) + \vec{b}u(t) \tag{176}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{y}(t) = V^{-1}AV\vec{y}(t) + V^{-1}\vec{b}u(t) \tag{177}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{y}(t) = \Lambda\vec{y}(t) + \vec{W}u(t) \tag{178}$$

where $\vec{W} = V^{-1}\vec{b}$ and $\Lambda = V^{-1}AV$.
We also transform the initial conditions into the eigenbasis:

$$\vec{y}(0) = \vec{y}_0 = V^{-1}\vec{x}_0 \tag{179}$$

Because the matrix V is composed of the eigenvectors of A, and is invertible, we know that $\Lambda = V^{-1}AV$ is the diagonal eigenvalue matrix, with each diagonal entry $\Lambda_{i,i}$ equal to the eigenvalue associated with the i^{th} eigenvector of V.
Because the Λ matrix is diagonal, the above matrix equation yields a collection of uncoupled scalar differential equations with initial conditions:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{y}_k = \lambda_k\vec{y}_k + \vec{W}_k u(t), \qquad \vec{y}_k(0) = \vec{y}_0[k] \tag{180}$$

for $k \in [1, 2, \ldots, n]$ where the subscript indicates indexing into the vector.

Finally, we must show that if we are given a valid solution for (174), then it remains a valid solution for the collection of differential equations in the transformed coordinates (180). Let $\vec{x}_{\text{sol}}(t)$ satisfy (174) and the initial condition $\vec{x}_{\text{sol}}(0) = \vec{x}_0$.

We can transform this solution into the eigenbasis: $\vec{y}_{\text{sol}}(t) = V^{-1}\vec{x}_{\text{sol}}(t)$.

At $t = 0$, $\vec{y}_{\text{sol}}(0) = V^{-1}\vec{x}_{\text{sol}}(0) = V^{-1}\vec{x}_0$, so we see that the initial condition is satisfied in the eigenbasis.

We also check if the transformed solution $\vec{y}_{\text{sol}}(t)$ satisfies the transformed differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{y}_{\text{sol}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}V^{-1}\vec{x}_{\text{sol}}(t) = V^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}_{\text{sol}}(t) \tag{181}$$

$$= V^{-1}\left(A\vec{x}_{\text{sol}}(t) + \vec{b}u(t)\right) \tag{182}$$

$$= V^{-1}\left(AV\vec{y}_{\text{sol}}(t) + \vec{b}u(t)\right) \tag{183}$$

$$= V^{-1}AV\vec{y}_{\text{sol}}(t) + V^{-1}\vec{b}u(t) \tag{184}$$

$$= \Lambda\vec{y}_{\text{sol}}(t) + \vec{W}u(t) \tag{185}$$

We see that indeed $y_{\text{sol}}(t)$ satisfied the transformed differential equations. Thus a valid solution in the original basis produces a valid solution in the changed basis.

(b) You have already proved the uniqueness of solutions for any scalar differential equation of the form $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t)$ with specified initial condition $x(0) = x_0$. How can you use this fact and the result of the previous part to argue that the solution must be unique for the matrix/vector differential equation?

(HINT: (Start by assuming that you have two solutions to the original problem. Use the result of part a) to formulate solutions to the transformed, diagonal problem. Apply the uniqueness results for scalar ODEs to the solutions of the diagonalized problem. Finally conclude by considering what the invertibility of the transformation matrix V implies about the two solutions of the original problem?))

Solution: Two common approaches to show uniqueness (in general) are directly, and via contradiction. To show uniqueness directly here, we would consider two (not necessarily distinct) solutions that satisfy (174), and show that they are equal. This means that any two solutions of the differential equation are equal to each other, so there is only one possible solution. To show uniqueness by contradiction, we would consider two *distinct* solutions to (174), and show that they are equal, which is a contradiction. The end result is the same, but the overall logic is slightly different.

If we wanted to show uniqueness by contradiction, we could do the following: let us assume that the solution is not unique, so we have two distinct solutions in the original basis $\vec{x}_1(t)$ and $\vec{x}_2(t)$ which satisfy (174) and the initial condition. Because V has linearly independent columns, the matrix V is invertible. Thus, we can transform the two distinct original solutions into two distinct solutions in the eigenbasis:

$$\vec{y}_1(t) = V^{-1}\vec{x}_1(t) \qquad \vec{y}_2(t) = V^{-1}\vec{x}_2(t) \qquad \vec{y}_1(t) \neq \vec{y}_2(t) \tag{186}$$

We have already proved that the solutions for scalar equations of the form $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t)$ with specified initial condition $x(0) = x_0$ are unique. In the previous part of the problem, we saw that we could transform the original matrix-vector differential equations into a collection of scalar equations of this form. Therefore, the solutions to these scalar equations must be unique.

This now leads to the contradiction: We know there can only be a single unique solution in the eigenbasis, but our initial assumption of two unique solutions in the original basis leads to two distinct solutions in the eigenbasis. Thus our assumption is not valid, and so there must be a unique solution.

If we wanted to show uniqueness directly, we would begin by assuming that we have two, not necessarily distinct, solutions in the original basis $\vec{x}_1(t)$ and $\vec{x}_2(t)$. We can again transform them into two solutions $\vec{y}_1(t) = V^{-1}\vec{x}_1(t)$ and $\vec{y}_2(t) = V^{-1}\vec{x}_2(t)$. We know that $\vec{y}_1(t)$ and $\vec{y}_2(t)$ satisfy the same diagonalized differential equation, which means each of their elements satisfy the same collection of scalar equations of the form $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t)$. We know that the solutions to these differential equations are unique, so each element of $\vec{y}_1(t)$ and $\vec{y}_2(t)$ are the same. Then $\vec{y}_1(t) = \vec{y}_2(t)$ as desired, so $\vec{x}_1(t) = V\vec{y}_1(t) = V\vec{y}_2(t) = \vec{x}_2(t)$. We conclude that the matrix differential equation (174) admits a unique solution.

Side note: what both of these approaches are getting at is that since V is invertible, the mapping from y to x via $x = Vy$ is bijective. That is, each x in the standard basis corresponds to exactly one y in the new basis. Then, since there is only one solution in the eigenbasis, there must be also exactly one solution in the standard basis.

We will see later in the course how the assumption we made on the eigenvectors of A is not actually needed for this proof to hold. But for now, it is important to understand this case first.

7. (PRACTICE) Solving the Differential Equation with Input

Unlocked by Lectures 3, 4, and Discussion 2A

Recall that in Discussion 2A we tried to solve the differential equation with input:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu_c(t) \tag{187}$$

$$x(0) = x_0. \tag{188}$$

for some continuous input $u_c(t)$.

The general strategy we employ is:

- First we replace our continuous input $u_c(t)$ with an input $u(t)$ which is piecewise constant on the intervals $[i\Delta, (i+1)\Delta)$, that is,

$$u(t) = u(i\Delta) = u[i] \qquad t \in [i\Delta, (i+1)\Delta) \qquad i \in \{0, 1, 2, \dots\} := \mathbb{N}. \tag{189}$$

  Using this assumption, in discussion we:

  – solved the differential equation on each interval $[i\Delta, (i+1)\Delta)$ and got a solution expressing $x(t)$ in terms of $x_d[i] := x(i\Delta)$ and $u[i]$, for $t \in [i\Delta, (i+1)\Delta)$;
  – arrived at a formula for $x_d[i+1]$ in terms of $x_d[i]$ and $u[i]$;
  – used this to get a formula for $x_d[i]$ in terms of x_0 and the inputs $u[0], u[1], \dots, u[i-1]$;
  – approximated $x(t) \approx x_d[\lfloor \frac{t}{\Delta} \rfloor]$ to recover an approximate value for $x(t)$, that is,

$$x(t) \approx \left(\mathrm{e}^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} x_0 + b\frac{\mathrm{e}^{\lambda\Delta} - 1}{\lambda} \sum_{k=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} \left(\mathrm{e}^{\lambda\Delta}\right)^{(\lfloor \frac{t}{\Delta} \rfloor - 1) - k} u[k]. \tag{190}$$

- In this homework, we will take the limit $\Delta \to 0$. This transfers back from u to u_c – we saw in discussion that piecewise constant functions on very small intervals, i.e., our u, approximate general continuous functions u_c arbitrarily well. Using Riemann sums and calculus, we will turn the sum into an integral and show that, if u approximates u_c as $\Delta \to 0$, then

$$x(t) = \mathrm{e}^{\lambda t} x_0 + b \int_0^t \mathrm{e}^{\lambda(t-\tau)} u_c(\tau)\, \mathrm{d}\tau\,. \tag{191}$$

(a) We first need to relate $u[i]$ to u_c. Suppose that the $u[i]$ is a sample of $u_c(t)$, namely,

$$u[i] = u_c(i\Delta). \tag{192}$$

To clarify where this fits in with the earlier notation:

- $u(t)$ is a piecewise constant function;
- $u[i]$ is the discrete input that constructs $u(t)$ based on eq. (189);
- and $u_c(t)$ is the underlying input $u[i]$ is sampled from based on eq. (192).

This is one good way to get a piecewise constant approximator of a continuous function.

Substitute an appropriate value of u_c for $u[k]$ in eq. (190) from the discussion.

NOTE: Don't take any limits in this part of the problem; just do the substitution.

Solution: Using the substitution $u_c(j\Delta)$ for $u[j]$, we get

$$x(t) \approx \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} x_0 + b\frac{e^{\lambda\Delta}-1}{\lambda}\sum_{k=0}^{\lfloor \frac{t}{\Delta} \rfloor -1}(e^{\lambda\Delta})^{(\lfloor \frac{t}{\Delta} \rfloor -1)-k}u[k] \quad (193)$$

$$= \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} x_0 + b\frac{e^{\lambda\Delta}-1}{\lambda}\sum_{k=0}^{\lfloor \frac{t}{\Delta} \rfloor -1}(e^{\lambda\Delta})^{(\lfloor \frac{t}{\Delta} \rfloor -1)-k}u_c(k\Delta). \quad (194)$$

(b) To simplify our (discrete-time) eq. (190) so we can take $\Delta \to 0$, we would like to make some approximations which are valid for small Δ.

By using the following two estimates for small Δ:[1]

i. $\lfloor \frac{t}{\Delta} \rfloor \approx \frac{t}{\Delta}$;
ii. $\frac{e^{\lambda\Delta}-1}{\lambda} \approx \Delta$;[2]

show that

$$x(t) \approx e^{\lambda t}x_0 + be^{-\lambda\Delta}\sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)}u_c(k\Delta)\Delta. \quad (195)$$

Solution: The first estimate justifies getting rid of the "floor" terms. We have a lot of those terms, so it's good to use it here.

Plugging in $\lfloor \frac{t}{\Delta} \rfloor \approx \frac{t}{\Delta}$ gives

$$x(t) \approx \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} x_0 + b\frac{e^{\lambda\Delta}-1}{\lambda}\sum_{k=0}^{\lfloor \frac{t}{\Delta} \rfloor -1}(e^{\lambda\Delta})^{(\lfloor \frac{t}{\Delta} \rfloor -1)-k}u_c(k\Delta) \quad (196)$$

$$\approx \left(e^{\lambda\Delta}\right)^{\frac{t}{\Delta}} x_0 + b\frac{e^{\lambda\Delta}-1}{\lambda}\sum_{k=0}^{\frac{t}{\Delta}-1}(e^{\lambda\Delta})^{(\frac{t}{\Delta}-1)-k}u_c(k\Delta) \quad (197)$$

$$\approx e^{\lambda t}x_0 + b\frac{e^{\lambda\Delta}-1}{\lambda}\sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda t-\lambda\Delta-\lambda\Delta k}u_c(k\Delta) \quad (198)$$

$$\approx e^{\lambda t}x_0 + b\frac{e^{\lambda\Delta}-1}{\lambda}e^{-\lambda\Delta}\sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)}u_c(k\Delta). \quad (199)$$

Then plugging in $\frac{e^{\lambda\Delta}-1}{\lambda} \approx \Delta$ gives

$$x(t) \approx e^{\lambda t}x_0 + b\frac{e^{\lambda\Delta}-1}{\lambda}e^{-\lambda\Delta}\sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)}u_c(k\Delta) \quad (200)$$

$$\approx e^{\lambda t}x_0 + be^{-\lambda\Delta}\sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)}u_c(k\Delta)\Delta. \quad (201)$$

Note that the dependence of $x(t)$ on both x_0 and the input u_c is the same; it's been preserved, and perhaps made more clear, through our approximations.

NOTE: This may seem like a long solution, but the main idea is to just use the estimates one by one, and simplify as much as possible.

[1] Both these approximations become equalities in the limit $\Delta \to 0$.
[2] We can see this approximation using Taylor's theorem from calculus.

(c) **Take the limit of $x(t)$ as $\Delta \to 0$, and show that $x(t)$ is given by eq. (191).**

Recall that the definite integral is defined from Riemann sums as

$$\int_0^t f(\tau)\,\mathrm{d}\tau = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(\tau_k^*)\Delta_k \tag{202}$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_n = t$, $\tau_k^* \in [\tau_k, \tau_{k+1}]$, and $\Delta_k = \tau_{k+1} - \tau_k$. The Δ_k is the length of the base of the rectangles and the $f(\tau_k^*)$ are the heights. As n goes to infinity, the rectangles get skinnier and skinnier, but there are more and more of them.

(HINT: Start with eq. (195) and take limits on both sides. What is n? What is τ_k and τ_k^*? What is Δ_k? What is f?)

(HINT: We chose the form of eq. (195) carefully; it turns out that Δ_k is one particular term involving Δ that goes to 0 as $\Delta \to 0$, and also that it is independent of k.)

Solution: We are evaluating

$$\lim_{\Delta\to 0} x(t) = \lim_{\Delta\to 0}\left[\mathrm{e}^{\lambda t}x_0 + b\mathrm{e}^{-\lambda\Delta}\sum_{k=0}^{\frac{t}{\Delta}-1}\mathrm{e}^{\lambda(t-k\Delta)}u_c(k\Delta)\Delta\right] \tag{203}$$

$$= \mathrm{e}^{\lambda t}x_0 + b\lim_{\Delta\to 0}\left(\mathrm{e}^{-\lambda\Delta}\sum_{k=0}^{\frac{t}{\Delta}-1}\mathrm{e}^{\lambda(t-k\Delta)}u_c(k\Delta)\Delta\right) \tag{204}$$

$$= \mathrm{e}^{\lambda t}x_0 + b\left(\lim_{\Delta\to 0}\mathrm{e}^{-\lambda\Delta}\right)\left(\lim_{\Delta\to 0}\sum_{k=0}^{\frac{t}{\Delta}-1}\mathrm{e}^{\lambda(t-k\Delta)}u_c(k\Delta)\Delta\right) \tag{205}$$

$$= \mathrm{e}^{\lambda t}x_0 + b\lim_{\Delta\to 0}\sum_{k=0}^{\frac{t}{\Delta}-1}\mathrm{e}^{\lambda(t-k\Delta)}u_c(k\Delta)\Delta. \tag{206}$$

Here, we want to evaluate the sum on the right side; by pattern matching with the Riemann integration template and the fact that Δ_k should shrink to 0 in the limit, we have

$$n = \frac{t}{\Delta} \qquad \Delta_k = \Delta. \tag{207}$$

This implies that

$$\tau_k = k\Delta. \tag{208}$$

To recover τ_k^* and f from what we already have, one notes that $\tau_k^* \in [k\Delta, (k+1)\Delta]$ and that we must have

$$f(\tau_k^*) = \mathrm{e}^{\lambda(t-k\Delta)}u_c(k\Delta). \tag{209}$$

From here we see that

$$\tau_k^* = k\Delta \qquad f(\tau) = \mathrm{e}^{\lambda(t-\tau)}u_c(\tau). \tag{210}$$

We have

$$\lim_{\Delta\to 0} x(t) = \mathrm{e}^{\lambda t}x_0 + b\lim_{\Delta\to 0}\sum_{k=0}^{\frac{t}{\Delta}-1}\mathrm{e}^{\lambda(t-k\Delta)}u_c(k\Delta)\Delta \tag{211}$$

$$= \mathrm{e}^{\lambda t}x_0 + b\lim_{n\to\infty}\sum_{k=0}^{n-1}\mathrm{e}^{\lambda(t-\tau_k^*)}u_c(\tau_k^*)\Delta_k \tag{212}$$

$$= e^{\lambda t} x_0 + b \lim_{n \to \infty} \sum_{k=0}^{n-1} f(\tau_k^*) \Delta_k \tag{213}$$

$$= e^{\lambda t} x_0 + b \int_0^t f(\tau) \, d\tau \tag{214}$$

$$= e^{\lambda t} x_0 + b \int_0^t e^{\lambda(t-\tau)} u_c(\tau) \, d\tau \tag{215}$$

which is our final answer. We can't simplify further because we don't know the form of $u_c(\tau)$.

Note that the dependence of $x(t)$ on both x_0 and the input u_c is the same. This is a special case of a crucial point: *sums of small quantities behave roughly the same as integrals*. This is one of the main ways to fluently transfer between discrete and continuous time.

This problem brings together many different concepts and uses a lot of notation. As such, it may be difficult to fully comprehend everything the first time. Being able to grind through complex mathematical problems like this is part of the vaunted "mathematical maturity" that this class helps you foster. As the semester continues, you will find that these kinds of problems will seem progressively easier, both to understand quickly and to solve. But it won't happen without practice.

8. Homework Process, Study Group, and Course Weekly Survey

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

At the same time, we want to check-in weekly regarding Discussions, Lectures, Lab, and Office Hours and see how effective they have all been for you as students.

Please fill out this survey link. For your submission, please attach a screenshot indicating that you have completed the survey this week.

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